SYDE Advanced Math 2, Solutions for Practice Problem Set 2

1: (a) The substitution $u(x) = y'(x)$ and $u'(x) = y''(x)$ transforms a second order DE of the form $y'' = F(y', x)$ for $y = y(x)$ to the first order DE $u' = F(u, x)$ for $u = u(x)$. Use this substitution to solve the IVP $xy'' + y' = 1$ with $y(1) = 2$ and $y'(1) = 3$.

Solution: Make the substitution $y' = u$, $y'' = u'$. The DE becomes $xu' + u = 1$. This is linear as we can write it as $u' + \frac{1}{x}u = \frac{1}{x}$. An integrating factor is $\lambda = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$, and the solution is given by $u = \frac{1}{x}\int 1\,dx = \frac{1}{x}(x+a) = 1 + \frac{a}{x}$. Put in $x = 1$ and $u = y' = 3$ to get $3 = 1 + a$, so $a = 2$. Thus the solution is given by $u = 1 + \frac{2}{x}$, that is $y' = 1 + \frac{2}{x}$. Integrate to get $y = \int 1 + \frac{2}{x}\,dx = x + 2\ln x + b$. Put in $x = 1$ and $y = 2$ to get $2 = 1 + b$, so $b = 1$ and the solution to the given IVP is $y = 1 + x + 2\ln x$.

(b) The substitution $u(y(x)) = y'(x)$ and $u'(y(x))y'(x) = y''(x)$ transforms a second order DE of the form $y'' = F(y', y)$ for $y = y(x)$ to the first order DE $u\,u' = F(u, y)$ for $u = u(y)$. Use this substitution to solve $y\,y'' + (y')^2 = 0$ with $y(1) = 2$ and $y'(1) = 3$.

Solution: Make the substitution $y' = u$, $y'' = u\,u'$. The DE becomes $yu\,u' + u^2 = 0$. This is linear since we can write it as $u' + \frac{1}{y}u = 0$. An integrating factor is $\lambda = e^{\int \frac{1}{y}\,dy} = e^{\ln y} = y$ and the solution is $u = \frac{1}{y}\int 0\,dy = \frac{a}{y}$. Put in $x = 1$, $y = 2$, $u = y' = 3$ to get $3 = \frac{a}{2}$ so $a = 6$ and the solution is $u = \frac{6}{y}$, that is $y' = \frac{6}{y}$. This DE is separable since we can write it as $y\,y' = 6$. Integrate both sides (with respect to x) to get $\frac{1}{2}y^2 = 6x + c$. Put in $x = 1$, $y = 2$ to get $2 = 6 + x$ so $c = -4$ and the solution is $\frac{1}{2}y^2 = 6x - 4$, that is $y = \pm\sqrt{12x - 8}$. Since $y(1) = 2$, we must use the $+$ sign, so $y = \sqrt{12x - 8}$.

2: Consider the IVP $y'' = y\,y'$ with $y(0) = 1$ and $y'(0) = 1$.

(a) Find the exact solution $y = f(x)$ to the given IVP.

Solution: Make the substitution $y' = u$, $y'' = u\,u'$, where $u = u(y)$. The DE becomes $u\,u' = yu$, that is $u' = y$. Integrate both sides (with respect to y) to get $u = \frac{1}{2}y^2 + a$. Put in $x = 0$, $y = 1$, $u = y' = 1$ to get $1 = \frac{1}{2} + a$ so $a = \frac{1}{2}$ and the solution is $u = \frac{1}{2}y^2 + \frac{1}{2} = \frac{y^2+1}{2}$, that is $y' = \frac{y^2+1}{2}$. This is separable as we can write it as $\frac{y'}{y^2+1} = \frac{1}{2}$. Integrate (with respect to x) to get $\tan^{-1} y = \frac{1}{2}x + b$. Put in $x = 0$, $y = 1$ to get $\frac{\pi}{4} = b$ and so the solution is $\tan^{-1} y = \frac{1}{2}x + \frac{\pi}{4}$, that is $y = f(x) = \tan\left(\frac{1}{2}x + \frac{\pi}{4}\right)$.

(b) With a calculator, use Euler's method with step size $\Delta x = 0.2$ to approximate $f(1)$.

Solution: The DE can be written as $y'' = F(x, y, y')$ where $F(x, y, z) = yz$. We let $\Delta x = 0.2$ and start with $x_0 = 0$, $y_0 = 1$ and $z_0 = 1$, and then for $k \geq 0$ we let $x_{k+1} = x_k + \Delta x$, $y_{k+1} = y_k + z_k\Delta x$ and $z_{k+1} = z_k + F(x_k, y_x, z_k)\Delta x = z_k + y_k z_k \Delta x$. We make a table showing the values of x_k, y_k, z_k and $F(x_k, y_k, z_k) = y_k z_k$.

k	x_k	y_k	z_k	$y_k z_k$
0	0.0	1	1	1
1	0.2	1.2	1.2	1.44
2	0.4	1.44	1.488	2.14272
3	0.6	1.7376	1.916544	3.3301869
4	0.8	2.1209088	2.5825814	5.4774196
5	1.0	2.6374251		

Thus $f(1) \cong y^5 \cong 2.6$ (this is not a very good approximation, as you can check using part (a)).

3: Solve the following IVPs.

(a) $y'' + 3y' + 2y = 0$ with $y(0) = 1$, $y'(0) = 0$

Solution: The characteristic equation is $r^2 + 3r + 2 = 0$. We solve this to get $r = -1, -2$ so the general solution to the DE is $y = Ae^{-x} + Be^{-2x}$, and then we have $y' = -Ae^{-x} - 2Be^{-2x}$. To get $y(0) = 1$ we need $A + B = 1$ (1), and to get $y'(0) = 0$ we need $-A - 2B = 0$ (2). Solve these two equations to get $A = 2$, $B = -1$, so the solution to the IVP is $y = 2e^{-x} - e^{-2x}$.

(b) $y'' + 4y' + 5y = 0$ with $y(0) = 3$, $y'(0) = 1$

Solution: The characteristic equation is $r^2 + 4r + 5 = 0$. Solve this to get $r = -2 \pm i$, so the general solution is $y = Ae^{-2x}\sin x + Be^{-2x}\cos x$, and we have $y' = -2Ae^{-2x}\sin x + Ae^{-2x}\cos x - 2Be^{-2x}\cos x - Be^{-2x}\sin x$. To get $y(0) = 3$ we need $B = 3$, and to get $y'(0) = 1$ we need $A - 2B = 1$ o $A = 7$. Thus the solution to the IVP is $y = 7e^{-2x}\sin x + 3e^{-2x}\cos x$.

(c) $4y'' - 4y' + y = 0$ with $y(1) = 1$, $y'(1) = 2$

Solution: The characteristic equation is $4r^2 - 4r + 1 = 0$, that is $(2r - 1)^2 = 0$, so $r = \frac{1}{2}$ and the general solution to the DE is $y = Ae^{x/2} + Bxe^{x/2}$, and then $y' = \frac{1}{2}Ae^{x/2} + Be^{x/2} + \frac{1}{2}Bxe^{x/2}$. To get $y(1) = 1$ we need $Ae^{1/2} + Be^{1/2} = 1$ (1), and to get $y'(1) = 2$ we need $\frac{1}{2}Ae^{1/2} + \frac{3}{2}Be^{1/2} = 2$ (2). Multiply equation (1) by $\frac{3}{2}$ and subtract equation (2) to get $Ae^{1/2} = -\frac{1}{2}$ so $A = -\frac{1}{2}e^{-1/2}$, and then multiply equation (2) by 2 and subtract equation (1) to get $2Be^{1/2} = 3$ so $B = \frac{3}{2}e^{-1/2}$. Thus the solution to the IVP is $y = -\frac{1}{2}e^{-1/2}e^{x/2} + \frac{3}{2}e^{-1/2}xe^{x/2} = \frac{1}{2}(3x - 1)e^{(x-1)/2}$.

4: Solve the following linear ODEs.

(a) $y'' - 2y' + 5y = 10x^2 - 3x$

Solution: The characteristic equation is $r^2 - 2r + 5 = 0$. Solve this to get $r = 1 \pm 2i$ so the general solution to the associated homogeneous DE is $y = Ae^x \sin 2x + Be^x \cos 2x$. To find a particular solution to the given (non-homogeneous) DE, we try $y = y_p = ax^2 + bx + c$. Then $y' = 2ax + b$ and $y'' = 2a$. Put these in the DE to get

$$\begin{aligned} 10x^2 - 3x &= y'' - 2y' + 5y \\ &= 2a - 4ax - 2b + 5ax^2 + 5bx + 5c \\ &= 5a\,x^2 + (5b - 4a)x + (2a - 2b + 5c)\,. \end{aligned}$$

Equating coefficients gives $5a = 10$, $5b - 4a = -3$ and $2a - 2b + 5c = 0$. Solve these three equations to get $a = 2$, $b = 1$ and $c = -\frac{2}{5}$, so we obtain the particular solution $y_p = 2x^2 + x - \frac{2}{5}$. The general solution to the given DE is $y = Ae^x \sin 2x + Be^x \cos 2x + 2x^2 + x - \frac{2}{5}$.

(b) $y'' + 2y' - 2y = 3xe^{2x}$

Solution: The characteristic equation is $r^2 + 2r - 2 = 0$. Solve this to get $r = -1 \pm \sqrt{3}$, so the general solution to the associated homogeneous DE is $y = Ae^{(-1+\sqrt{3})x} + Be^{(-1-\sqrt{3})x}$. To find a particular solution to the given DE, we try $y = y_p = (ax + b)e^{2x}$. Then $y' = (2ax + a + 2b)e^{2x}$ and $y'' = (4ax + 4a + 4b)e^{2x}$. Put these into the DE to get

$$\begin{aligned} 3xe^{2x} &= y'' + 2y' - 2y \\ &= (4ax + 4a + 4b)e^{2x} + 2(2ax + a + 2b)e^{2x} - 2(ax + b)e^{2x} \\ &= 6axe^{2x} + (6a + 6b)e^{2x}\,. \end{aligned}$$

Divide both sides by e^{2x} to get $3x = 6ax + (6a + 6b)$. Equating coefficients gives $6a = 3$ and $6a + 6b = 0$. Solving these two equations gives $a = \frac{1}{2}$ and $b = -\frac{1}{2}$, so we obtain the particular solution $y_p = \left(\frac{1}{2}x - \frac{1}{2}\right)e^{2x}$. Thus the general solution to the given DE is $y = Ae^{(-1+\sqrt{3})x} + Be^{-(1+\sqrt{3})x} + \frac{1}{2}(x - 1)e^{2x}$.

5: Solve the following linear ODEs.

(a) $2y'' + y' - y + x + e^{-x} = 0$

Solution: The characteristic equation is $2r^2 + r - 1 = 0$. Solve this to get $r = \frac{1}{2}, -1$, and so the general solution to the associated homogeneous DE is $y = Ae^{x/2} + B^{-x}$. To find a particular solution to the DE $2y'' + y' - y = -x$, we try $y = ax + b$. Then $y' = a$ and $y'' = 0$. We put these in the DE to get

$$-x = 2y'' + y' - y = a - ax - b = -ax + (a - b)\,.$$

Equating coefficients gives $-a = -1$ and $a - b = 0$, so we get $a = 1$ and $b = 1$, and we obtain the particular solution $y = x + 1$. To find a particular solution to the DE $2y'' + y' - y = -e^{-x}$, we try $y = axe^{-x}$. Then $y' = ae^{-x} - axe^{-x}$ and $y'' = -2ae^{-x} + axe^{-x}$. Put these in the DE to get

$$-e^{-x} = 2y'' + y' - y = -4ae^{-x} + 2axe^{-x} + ae^{-x} - axe^{-x} - axe^{-x} = -3ae^{-x}\,.$$

Divide both sides by e^{-x} to get $-1 = -3a$ so $a = \frac{1}{3}$ and we obtain the particular solution $y = \frac{1}{3}xe^{-x}$. A particular solution to the given DE is obtained by adding together these two particular solutions to get $y_p = x + 1 + \frac{1}{3}xe^{-x}$. The general solution to the given DE is $y = Ae^{x/2} + Be^{-x} + x + 1 + \frac{1}{3}xe^{-x}$.

(b) $y'' - 6y' + 10y = e^{3x}\sin x$

Solution: The characteristic equation is $r^2 - 6r + 10 = 0$. Solve this to get $r = 3 \pm i$, so two linearly independent solutions to the associated homogeneous DE are $y_1 = e^{3x}\sin x$ and $y_2 = e^{3x}\cos x$. Note that ${y_1}' = 3e^{3x}\sin x + e^{3x}\cos x = 3y_1 + y_2$, and ${y_2}' = 3e^{3x}\cos x - e^{3x}\sin x = -y_1 + 3y_2$. To find a particular solution to the given DE, we try $y = y_p = Axy_1 + Bxy_2$. Then $y' = Ay_1 + Ax{y_1}' + By_2 + Bx{y_2}'$ and $y'' = 2A{y_1}' + Ax{y_1}'' + 2B{y_2}' + Bx{y_2}''$. Put these in the DE to get

$$\begin{aligned}
y_1 = e^{3x}\sin x &= y'' - 6y' + 10y \\
&= 2A{y_1}' + Ax{y_1}'' + 2B{y_2}' + Bx{y_2}'' - 6Ay_1 - 6Ax{y_1}' - 6By_2 - 6Bx{y_2}' + 10Axy_1 + 10Bxy_2 \\
&= 2A{y_1}' + 2B{y_2}' - 6Ay_1 - 6By_2 + Ax({y_1}'' - 6{y_1}' + 10y) + Bx({y_2}'' - 6{y_2}' + 10y_2) \\
&= 2A{y_1}' + 2B{y_2}' - 6Ay_1 - 6By_2 \\
&= 2A(3y_1 + y_2) + 2B(-y_1 + 3y_2) - 6Ay_1 - 6By_2 \\
&= -2By_1 + 2Ay_2\,,
\end{aligned}$$

where we used the fact that y_1 and y_2 are solutions to the associated homogeneous DE, and we used the earlier-mentioned identities ${y_1}' = 3y_1 + y_2$ and ${y_2}' = -y_1 + 3y_2$. Since y_1 and y_2 are independent, we can equate coefficients to get $-2B = 1$ and $2A = 0$, that is $A = 0$ and $B = -\frac{1}{2}$, and so we obtain the particular solution $y_p = -\frac{1}{2}xe^{3x}\cos x$. Thus the general solution to the given DE is $y = Ae^{3x}\sin x + Be^{3x}\cos x - \frac{1}{2}xe^{3x}\cos x$.

6: Solve the following IVPs.

(a) $4y'' - y = x$ with $y(0) = 2$, $y'(0) = 1$

Solution: The characteristic equation is $4r^2 - 1 = 0$. Solve this to get $r = \pm\frac{1}{2}$, so the general equation to the associated homogeneous DE is $y = Ae^{x/2} + Be^{-x/2}$. To find a particular solution to the given DE, we try $y = y_p = ax + b$. Then $y' = a$ and $y'' = 0$. Put these in the DE to get $x = 4y'' - y = -ax - b$. Equating coefficients gives $a = -1$ and $b = 0$, so we obtain the particular solution $y_p = -x$. The general solution to the given DE is $y = Ae^{x/2} + Be^{-x/2} - x$, and then $y' = \frac{1}{2}Ae^{x/2} - \frac{1}{2}Be^{-x/2} - 1$. To get $y(0) = 2$ we need $A + B = 2$ (1), and to get $y'(0) = 1$ we need $\frac{1}{2}A - \frac{1}{2}B - 1 = 1$, that is $A - B = 4$ (2). Solving these two equations gives $A = 3$ and $B = -1$, so the solution to the given IVP is $y = 3e^{x/2} - e^{-x/2} - x$.

(b) $y'' - 6y' + 9y = e^{3x}$ with $y(0) = 1$, $y'(0) = 0$

Solution: The characteristic equation is $r^2 - 6r + 9 = 0$. The only solution is $r = 3$ so the general solution to the associated homogeneous DE is $y = Ae^{3x} + Bxe^{3x}$. To find a particular solution to the given DE, we try $y = y_p = ax^2e^{3x}$. We then have $y' = a(2x + 3x^2)e^{3x}$ and $y'' = a(2 + 12x + 9x^2)e^{3x}$. Put these in the DE to get

$$e^{3x} = y'' - 6y' + 9y = a(2 + 12x + 9x^2)e^{3x} - 6a(2x + 3x^2)e^{3x} + 9ax^2e^{3x} = 2ae^{3x}\,.$$

Divide by e^{3x} to get $1 = 2a$, so $a = \frac{1}{2}$ and we obtain the particular solution $y_p = \frac{1}{2}x^2e^{3x}$. The general solution to the given DE is $y = Ae^{3x} + Bxe^{3x} + \frac{1}{2}x^2e^{3x}$, and then $y' = 3Ae^{3x} + Be^{3x} + 3Bxe^{3x} + xe^{3x} + \frac{3}{2}e^{3x}$. To get $y(0) = 1$ we need $A = 1$, and to get $y'(0) = 0$ we need $3A + B = 0$ so $B = -3$. Thus the solution to the given IVP is $y = e^{3x} - 3xe^{3x} + \frac{1}{2}x^2e^{3x}$.

7: Solve the following third-order linear ODEs.

(a) $y''' + 2y'' - 5y' - 6y = 0$

Solution: The characteristic polynomial is $r^3 + 2r^2 - 5r - 6 = (r - 2)(r^2 + 4r + 3) = (r - 2)(r + 1)(r + 3)$, and so the general solution to the DE is $y = Ae^{2x} + Be^{-x} + Ce^{-3x}$.

(b) $y''' - 3y' + 2y = 2\sin x$

Solution: The characteristic polynomial is $r^3 - 3r + 2 = (r - 1)(r^2 + r - 2) = (r - 1)^2(r + 2)$, and so the general solution to the associated homogeneous DE is $y = Ae^x + Bxe^x + Ce^{-2x}$. To find a particular solution to the given DE, we try $y = y_p = a\sin x + b\cos x$. We then have $y' = a\cos x - b\sin x$, $y'' = -a\sin x - b\cos x$ and $y''' = -a\cos x + b\sin x$. Put these in the DE to get

$$\begin{aligned} 2\sin x &= y''' - 3y' + 2y \\ &= -a\cos x + b\sin x - 3a\cos x + 3b\sin x + 2a\sin x + 2b\cos x \\ &= (2a + 4b)\sin x + (-4a + 2b)\cos x\,. \end{aligned}$$

Since $\sin x$ and $\cos x$ are linearly independent, we can equate coefficients to get $2a+4b = 2$ and $-4a+2b = 0$. Solving these two equations gives $a = \frac{1}{5}$ and $b = \frac{2}{5}$, and so we have obtained the particular solution $y_p = \frac{1}{5}\sin x + \frac{2}{5}\cos x$. The general solution to the given DE is $y = Ae^x + Bxe^x + Ce^{-2x} + \frac{1}{5}\sin x + \frac{2}{5}\cos x$.