SYDE Advanced Math 2, Practice Problem Set 4

1: The ODE $(1-x^2)y'' - 2xy' + k(k+1)y = 0$ is called **Legendre's Equation**. For each integer $k \geq 0$, Legendre's equation has a unique polynomial solution $y = P_k(x)$ with $P_k(1) = 1$. These are called the **Legendre polynomials**. Use power series, centred at 0, to solve the ODE, and find $P_k(x)$ for $k = 0, 1, 2, 3, 4$.

2: The ODE $x^2y'' + kxy' + \ell y = 0$ is called the **Cauchy-Euler Equation**. We can solve the Cauchy-Euler equation by letting $y = x^r$ so that $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Putting these in the DE gives $0 = r(r-1)x^r + krx^r + \ell x^r = \big(r(r-1) + kr + \ell\big)x^r$, so we see that $y = x^r$ is a solution when r is a root of the polynomial $g(r) = r(r-1) + kr + \ell$.

(a) When $g(r)$ has two real roots r_1 and r_2, we obtain two independent solutions $y_1(x) = x^{r_1}$ and $y_2(x) = x^{r_2}$. Use this to solve the ODE $x^2y'' - 2xy + 2y = 0$.

(b) When $g(r)$ has complex roots $r \pm is$, we obtain the complex solutions $z_1(x) = x^{r+is} = e^{(r+is)\ln x} = e^{r\ln x}e^{is\ln x} = x^r\big(\cos(s\ln x) + i\sin(s\ln x)\big)$ and $z_2(x) = x^{r-is} = x^r\big(\cos(s\ln x) - i\sin(s\ln x)\big)$, and hence we obtain the two independent real solutions given by $y_1(x) = \frac{z_1(x)+z_2(x)}{2} = \mathrm{Re}(z_1(x)) = x^r\cos(s\ln x)$ and $y_2(x) = \frac{z_1(x)-z_2(x)}{2i} = \mathrm{Im}(z_1(x)) = x^r\sin(s\ln x)$. Use this to solve the ODE $x^2y'' + 3xy' + 5y = 0$.

(c) When $g(x)$ has a repeated real root r we only obtain one solution $y_1(x) = x^r$. Use reduction of order to find a formula for a second independent solution $y = y_2(x)$ of the form $y_2(x) = y_1(x)u(x)$.

(d) Use the formula found in Part (c) to solve the ODE $x^2y'' + 5xy + 4y = 0$.

3: The ODE $x^2y'' + xy' + (x^2 - k^2)y = 0$ is called **Bessel's Equation**. Use Frobenius' method to show that for all $k \geq 0$ there is a nonzero solution of the form $y = J_k(x) = x^k\sum_{n\geq 0} c_{2n}x^{2n}$ and, if k is not an integer, there is a second independent solution of the form $y = J_{-k}(x) = x^{-k}\sum_{n\geq 0} c_{2n}x^{2n}$. These solutions (multiplied by various constants) are called the **Bessel functions of the first kind**.