SYDE Advanced Math 2, Solutions for Practice Problem Set 4

1: The ODE $(1-x^2)y''-2xy'+k(k+1)y=0$ is called **Legendre's Equation**. For each integer $k\geq 0$, Legendre's equation has a unique polynomial solution $y=P_k(x)$ with $P_k(1)=1$. These are called the **Legendre polynomials**. Use power series, centred at 0, to solve the ODE, and find $P_k(x)$ for $k=0,1,2,3,4$.

Solution: Let $y=\sum\limits_{n\geq 0}c_nx^n$ so $y'=\sum\limits_{n\geq 1}nc_nx^{n-1}$ and $y''=\sum\limits_{n\geq 2}n(n-1)c_nx^{n-2}$. Put these in the DE to get

$$\begin{aligned}0&=y''-x^2y''-2xy'+k(k+1)y\\&=\sum_{n\geq 2}n(n-1)c_nx^{n-2}-\sum_{n\geq 2}n(n-1)c_nx^n-\sum_{n\geq 1}2nc_nx^n+\sum_{n\geq 0}k(k+1)c_nx^n\\&=\sum_{m\geq 0}(m+2)(m+1)c_{m+2}x^m-\sum_{m\geq 2}m(m-1)c_mx^m-\sum_{m\geq 1}2mc_mx^m+\sum_{m\geq 0}k(k+1)c_mx^m\end{aligned}$$

We equate the coefficients of x^m: When $m=0$ we obtain $2\cdot 1\,c_2+k(k+1)\,c_0=0$ so that $c_2=-\frac{k(k+1)}{2}\,c_0$. When $m=1$ we obtain $3\cdot 2\,c_3-2c_1+k(k+1)\,c_1$ so that $c_3=\frac{2-k(k+1)}{6}\,c_1$. When $m\geq 2$ we obtain $(m+2)(m+1)c_{m+2}-\big(m(m-1)+2m-k(k+1)\big)c_m$ so that

$$c_{m+2}=\tfrac{m(m+1)-k(k+1)}{(m+1)(m+2)}c_m.$$

We can choose $c_0,c_1\in\mathbb{R}$ to be arbitrary, then c_n is determined from c_{n-2} for all $n\geq 2$ by the recursion formulas. When $c_0=1$ and $c_1=0$, the recursion formulas imply that $c_n=0$ for all odd values of n, and the solution is given by $y=y_1(x)=c_0+c_2x^2+c_4x^4+\cdots$ with $c_0=1$ and $c_{m+2}=\frac{m(m+1)-k(k+1)}{(m+1)(m+2)}c_m$. When $c_0=0$ and $c_1=1$ we get $y=y_2(x)=c_1+c_3x^3+c_5x^5+\cdots$ with $c_1=1$ and $c_{m+2}=\frac{m(m+1)-k(k+1)}{(m+1)(m+2)}c_m$. Notice that when $0\leq k\in\mathbb{Z}$, the recursion formula gives $c_{k+2}=0$ and hence $0=c_{k+2}=c_{k+4}=c_{k+6}=\cdots$. Thus when k is even the solution $y=y_1(x)$ is a polynomial and when k is odd the solution $y=y_2(x)$ is a polynomial. When $k=0$, we have $y_1(x)=1$ and so $P_0(x)=1$. When $k=1$, we have $y_2(x)=x$ and so $P_1(x)=x$. When $k=2$, we have $y_1(x)=c_0+c_2x^2$ with $c_0=1$ and $c_2=-\frac{k(k+1)}{2}c_0=-\frac{2\cdot 3}{2}=-3$ so that $y_1(x)=1-3x^2$. Since $y_1(1)=-2$ we have $P_2(x)=-\frac{1}{2}y_1(x)=\frac{1}{2}(3x^2-1)$. When $k=3$, we have $y_2(x)=c_1+c_3x^3$ with $c_1=1$ and $c_3=\frac{2-k(k+1)}{6}c_1=\frac{2-3\cdot 4}{6}=-\frac{5}{3}$ so that $y_2(x)=x-\frac{5}{3}x^3$. Since $y_2(1)=-\frac{2}{3}$ we have $P_3(x)=-\frac{3}{2}y_2(x)=\frac{1}{2}(5x^3-3x)$. When $k=4$, we have $y_1(x)=c_0+c_2x^2+c_4x^4$ with $c_0=1$. $c_2=-\frac{k(k+1)}{2}c_0=-\frac{4\cdot 5}{2}=-10$ and $c_4=\frac{2\cdot 3-k\cdot(k+1)}{3\cdot 4}\cdot c_2=\frac{2\cdot 3-4\cdot 5}{3\cdot 4}\cdot(-10)=\frac{35}{3}$ so that $y_1(x)=1-10x^2+\frac{35}{3}x^4$. Since $y_1(1)=\frac{8}{3}$ we have $P_4(x)=\frac{3}{8}y_1(x)=\frac{1}{8}(35x^4-30x^2+3)$.

2: The ODE $x^2y''+kxy'+\ell y=0$ is called the **Cauchy-Euler Equation**. We can solve the Cauchy-Euler equation by letting $y=x^r$ so that $y'=rx^{r-1}$ and $y''=r(r-1)x^{r-2}$. Putting these in the DE gives $0=r(r-1)x^r+krx^r+\ell x^r=\big(r(r-1)+kr+\ell\big)x^r$, so we see that $y=x^r$ is a solution when r is a root of the polynomial $g(r)=r(r-1)+kr+\ell$.

(a) When $g(r)$ has two real roots r_1 and r_2, we obtain two independent solutions $y_1(x)=x^{r_1}$ and $y_2(x)=x^{r_2}$. Solve the ODE $x^2y''-2xy+2y=0$.

Solution: We have $g(r)=r(r-1)+kr+\ell=r(r-1)-2r+2=r^2-3r+2=(r-1)(r-2)$. The roots are $r_1=1$ and $r_2=2$, two independent solutions are given by $y_1(x)=x^1$ and $y_2(x)=x^2$, and the general solution is given by $y=ax+bx^2$.

(b) When $g(r)$ has complex roots $r\pm is$, we obtain the complex solutions $z_1(x)=x^{r+is}=e^{(r+is)\ln x}=e^{r\ln x}e^{is\ln x}=x^r\big(\cos(s\ln x)+i\sin(s\ln x)\big)$ and $z_2(x)=x^{r-is}=x^r\big(\cos(s\ln x)-i\sin(s\ln x)\big)$, and hence we obtain the two independent real solutions given by $y_1(x)=\frac{z_1(x)+z_2(x)}{2}=\mathrm{Re}(z_1(x))=x^r\cos(s\ln x)$ and $y_2(x)=\frac{z_1(x)-z_2(x)}{2i}=\mathrm{Im}(z_1(x))=x^r\sin(s\ln x)$. Solve the ODE $x^2y''+3xy'+5y=0$.

Solution: We have $g(r)=r(r-1)+kr+\ell=r(r-1)+3r+5=r^2+2r+5$. The roots are $r=\frac{-2\pm\sqrt{4-20}}{2}=-1\pm 2i$. A complex solution is given by $x^{-1+2i}=e^{(-1+2i)\ln x}=e^{-\ln x}e^{i\,2\ln x}=\frac{1}{x}\big(\cos(2\ln x)+i\sin(2\ln x)\big)$, and two independent real solutions are given by $y_1(x)=\frac{1}{x}\cos(2\ln x)$ and $y_2(x)=\frac{1}{x}\sin(2\ln x)$. The general solution is given by $y=\frac{a}{x}\cos(2\ln x)+\frac{b}{x}\sin(2\ln x)$.

(c) When $g(x)$ has a repeated real root r we only obtain one solution $y_1(x)=x^r$. Use reduction of order to find a formula for a second independent solution $y=y_2(x)$ of the form $y_2=y_1u$.

Solution: We have $g(r)=r(r-1)+kr+\ell=r^2+(k-1)r+\ell$. This has a repeated real root when its discriminant is zero, that is when $(k-1)^2=4\ell$, and its root is $r=-\frac{k-1}{2}=\frac{1-k}{2}$. We have one solution $y=y_1(x)=x^r$. To use reduction of order, we let $y=y_2=y_1u$ so we have $y'=y_1'u+y_1u'$ and $y''=y_1''u+2y_1'u'+y_1u''$. Put this in the DE to get

$$\begin{aligned}
0&=x^2y''+kxy'+\ell y\\
&=x^2(y_1''u+2y_1'u'+y_1u'')+kx(y_1'u+y_1u')+\ell(y_1u)\\
&=(x^2y_1''+kxy_1'+\ell y_1)u+(2x^2y_1'+kxy_1)u'+x^2y_1u''\\
&=(2x^2y_1'+kxy_1)u'+x^2y_1u''\ \text{, since } y_1 \text{ is a solution to the DE}\\
&=(2rx^{r+1}+kx^{r+1})u'+x^{r+2}u''\ \text{, since } y_1=x^r \text{ and } y_1'=rx^{r-1}\\
&=(2r+k)u'+x\,u''\ \text{, after dividing both sides by } x^{r+1}\\
&=u'+xu''\ \text{, since } r=\tfrac{1-k}{2}.
\end{aligned}$$

Letting $w=u'$, the DE becomes $xw'+w=0$, that is $w'+\frac{1}{x}w=0$, which is a linear DE for $w=w(x)$. An integrating factor is $\lambda=e^{\int\frac{1}{x}dx}=e^{\ln x}=x$ and the solution is $w=\frac{1}{x}\int 0\,dx=\frac{a}{x}$. which gives $u'=\frac{a}{x}$ and hence $u=a\ln x+b$. We choose $a=1$ and $b=0$ to get $u=\ln x$ giving the second independent solution

$$y=y_2(x)=y_1(x)u(x)=x^r\ln x.$$

(d) Use the formula from Part (c) to solve the ODE $x^2y''+5xy+4y=0$.

Solution: We have $g(r)=r(r-1)+kr+\ell=r(r-1)+5r+4=r^2+4r+4=(r+2)^2$. This has repeated real root $r=-2$. One solution is given by $y_1(x)=x^r=x^{-2}$ and, b Part (c), a second solution is given b $y_2(x)=x^r\ln x=x^{-2}\ln x$. The general solution is $y=ax^{-2}+bx^{-2}\ln x=\frac{a+b\ln x}{x^2}$.

3: The ODE $x^2y'' + xy' + (x^2 - k^2)y = 0$ is called **Bessel's Equation**. Use Frobenius' method to show that for all $k \geq 0$ there is a nonzero solution of the form $y = J_k(x) = x^k \sum_{n\geq 0} c_{2n}x^{2n}$, and, if k is not an integer, there is a second independent solution of the form $y = J_{-k}(x) = x^{-k}\sum_{n\geq 0} c_{2n}x^{2n}$. These solutions (multiplied by a constant) are called the **Bessel functions of the first kind**.

Solution: Let $k \geq 0$. Let $y = \sum\limits_{n\geq 0} c_n x^{n+r}$ so $y' = \sum\limits_{n\geq 0}(n+r)c_n x^{n+r-1}$ and $y'' = \sum\limits_{n\geq 0}(n+r)(n+r-1)c_n x^{n+r-2}$. Put this into the DE to get

$$\begin{aligned}
0 &= \sum_{n\geq 0}(n+r)(n+r-1)c_n x^{n+r} + \sum_{n\geq 0}(n+r)c_n x^{n+r} + \sum_{n\geq 0} c_n x^{n+r+2} - \sum_{n\geq 0} k^2 c_n x^{n+r} \\
&= x^r\Big(\sum_{m\geq 0}\big((m+r)(m+r-1) + (m+r) - k^2\big)c_m x^m + \sum_{m\geq 2} c_{m-2}x^m\Big) \\
&= \sum_{m\geq 0}\big((m+r)^2 - k^2\big)c_m x^m + \sum_{m\geq 2} c_{m-2}x^m
\end{aligned}$$

We equate coefficients: When $m=0$ we get $r^2 - k^2 = 0$ or $c_0 = 0$, that is $r = \pm k$ or c_0. When $m=1$ we get $(r+1)^2 = 0$ or $c_1 = 0$, that is $r = -1 \pm k$, or $c_1 = 0$ (we remark that taking $r = -1 \pm k$, $c_1 = 1$ and $c_0 = 0$ gives the same solution(s) as taking $r = \pm k$, $c_0 = 1$ and $c_1 = 0$, so we shall not consider these cases below). When $m \geq 2$ we get $((m+r)^2 - k^2)c_m + c_{m-2} = 0$ which gives the recursion formula $c_m = \frac{-1}{(m+r)^2-k^2}\,c_{m-2}$.

In the case that $r = k$, the recursion becomes $c_m = \frac{-1}{(m+k)^2-k^2}\,c_{m-2} = \frac{-1}{m(m+2k)}\,c_{m-2}$, so taking $c_0 = 1$ and $c_1 = 0$ gives $c_n = 0$ for all odd values of n and $c_0 = 1$, $c_2 = \frac{-1}{2(2+2k)}$, $c_4 = \frac{1}{4(4+2k)} \cdot \frac{1}{2(2+2k)}$, and $c_6 = \frac{-1}{6(6+2k)} \cdots \frac{1}{4(4+2k)} \cdots \frac{1}{2(2+2k)}$ and so on, so that in general

$$c_{2n} = \frac{-1}{2\cdot 4\cdot 6\cdots(2n))(2+2k)(4+2k)(6+2k)\cdots(2n+2k))} = \frac{(-1)^n}{2^n n!\cdot 2^n(1+k)(2+k)(3+k)\cdots(n+k)} = \frac{(-1)^n}{(2^n n!)^2\binom{n+k}{n}}$$

where we recall that for $p \in \mathbb{R}$ we have $\binom{p}{0} = 1$ and $\binom{p}{n} = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$. This gives the solution

$$y = J_k(x) = x^k \sum_{n\geq 0}\frac{(-1)^n}{(2^n n!)^2\binom{n+k}{n}}\,x^{2n}.$$

In the case that $r = -k$ and k is not an integer, the recursion becomes $c_m = \frac{-1}{(m-k)^2-k^2}\,c_{m-2} = \frac{-1}{m(m-2k)}\,c_{m-2}$ (note that if $k \in \mathbb{Z}$ then c_m is not defined for $m = 2k$), so taking $c_0 = 1$ and $c_1 = 0$ gives $c_n = 0$ for all odd values of n and $c_0 = 1$, $c_2 = \frac{-1}{2(2-2k)}$, $c_4 = \frac{1}{4(4-2k)} \cdot \frac{1}{2(2-2k)}$ and $c_6 = \frac{-1}{6(6-2k)} \cdots \frac{1}{4(4-2k)} \cdots \frac{1}{2(2-2k)}$ and so on, so that in general

$$c_{2n} = \frac{-1}{2\cdot 4\cdot 6\cdots(2n))(2-2k)(4-2k)(6-2k)\cdots(2n-2k))} = \frac{(-1)^n}{2^n n!\cdot 2^n(1-k)(2-k)(3-k)\cdots(n-k)} = \frac{(-1)^n}{(2^n n!)^2\binom{n-k}{n}}$$

(note that if $k \in \mathbb{Z}$ then $\binom{n-k}{n} = 0$ for $k \geq n$). This gives the solution

$$y = J_{-k}(x) = x^{-k}\sum_{n\geq 0}\frac{(-1)^n}{(2^n n!)^2\binom{n-k}{n}}\,x^{2n}.$$

For $k \geq 0$, when k is not an integer, the general solution to Bessel's equation is given by $y = aJ_k(x) + bJ_{-k}(x)$. We remark that when k is an integer there is a (fairly difficult) method which can be used to obtain a second independent solution $y = Y_k(x)$, so that the general solution to Bessel's equation is given by $y = aJ_k(x) + bY_k(x)$. The solutions $y = Y_k(x)$ are called the **Bessel functions of the second kind**.