Part 1. Review of ODEs

Introduction

1.1 Definition: When A and B are sets, we write $f : A \to B$ to indicate that f is a **function** from A to B, which means that for every $x \in A$ there is a unique corresponding element $y = f(x) \in B$. The set A is called the **domain** of the function. A **differential equation**, or **DE**, is an equation which involves a function $f : A \subseteq \mathbb{R}^n \to \mathbb{R}$ and some of its derivatives. The **order** of a DE is the highest of the orders of the derivatives which occur in the equation. A **solution** to a DE is a function $f : U \subseteq A \to \mathbb{R}$, whose domain is a connected set U, such that the DE holds for all $x \in U$. The **general solution** to a DE is the set of all possible solutions. Often, a DE has infinitely many solutions and we impose some additional constraints, so the a DE has a unique solution which satisfies the constraints. Sometimes the constraints are **initial conditions**, and sometimes they are **boundary conditions**, as we shall see later. When $n = 1$, so the DE involves a function $y = f(x)$ of a single real variable $x \in \mathbb{R}$, the DE is called an **ordinary differential equation**, or ODE. When $n > 1$ so that the DE involves a function $y = f(x_1, x_2, \cdots, x_n)$ of two or more variables, the DE is called a **partial differential equation**, or PDE.

1.2 Example: The equation $y''' + 2y^3 y' = \sin x$ is a third order ODE for $y = y(x)$. The equation $2xy \frac{\partial^2 u}{\partial x \partial y} + x \frac{\partial u}{\partial y} + y \frac{\partial u}{\partial x} = 0$ is a second order PDE for $u = u(x, y)$.

1.3 Exercise: Find a solution of the form $y = ax^2 + bx + c$ to the DE $y''y' + x^2 = y$.

1.4 Exercise: Find two distinct constants r_1 and r_2 such that $y = e^{r_1 x}$ and $e^{r_2 x}$ are both solutions to the DE $y'' + 3y' + 2y = 0$, show that $y = a\, e^{r_1 x} + b\, e^{r_2 x}$ is a solution for any constants a and b, and then find a solution to the DE with $y(0) = 1$ and $y'(0) = 0$.

First Order ODEs

1.5 Definition: In general, a first order ODE can be written in the form

$$G(x,y,y')=0$$

for some function $G:A\subseteq\mathbb{R}^3\to\mathbb{R}$. The graph of a solution $y=y(x)$ to a first order ODE is called a **solution curve**. Given a first order ODE there is often a unique solution curve $y=y(x)$ which passes through any given point (a,b) in the domain of F, equivalently there is often a unique solution $y=y(x)$ to the DE with $y(a)=b$. A first order ODE for a function $y=y(x)$, together with an additional constraint of the form $y(a)=b$, is called an **initial value problem**, of IVP, and the constraint is called the **initial condition**.

Direction Fields

1.6 Note: A first order ODE can often be written in the form

$$y'=F(x,y)$$

It is easy to sketch the solution curves to any DE of the form $y'=F(x,y)$ in the following way. First choose many points (x,y), and for each point (x,y) find the value of $F(x,y)$. If $y=y(x)$ is any solution to the DE, so that $y'(x)=F(x,y)$, then $F(x,y)$ is the slope of the solution curve at the point (x,y). At each point (x,y), draw a short line segment with slope $F(x,y)$. The resulting picture is called the **slope field** or the **direction field** of the DE. If we choose enough points (x,y) it should be possible to visualize the solution curves; they follow the direction of the short line segments.

To draw the direction field of the DE $y'=F(x,y)$ by hand, it helps to first lightly draw several **isoclines**; these are the curves $F(x,y)=m$, where m is a constant. Along the isocline $F(x,y)=m$ we then draw many short line segments of slope m.

To draw the graph of the solution to the IVP $y'=F(x,y)$, with $y(a)=b$, sketch the direction field for the DE $y'=F(x,y)$ and then draw the solution curve which passes through the point (a,b).

1.7 Exercise: Sketch the direction field for the DE $y'=x-y$, then sketch the solution curves through each of the points $(0,-2)$, $(0,-1)$, $(0,0)$ and $(0,1)$.

Euler's Method

1.8 Note: We can approximate the solution to the IVP $y'=F\big(x,y\big)$ with $y(a)=b$ using the following method, which is known as **Euler's Method**. Pick a small value Δx, which we call the **step size**. Let $x_0=a$ and $y_0=b$. Having found x_k and y_k, we let

$$\begin{aligned} x_{k+1}&=x_k+\Delta x\\ y_{k+1}&=y_k+F(x_k,y_k)\,\Delta x\,. \end{aligned}$$

The solution curve $y=y(x)$ is then approximated for values $x\geq a$ by the piecewise linear curve whose graph has vertices at the points (x_k,y_k). If we also wish to approximate the solution for values $x\leq a$, we can construct points (x_k,y_k) with $k<0$ by letting $x_{k-1}=x_k-\Delta x$ and $y_{k-1}=y_k-F(x_k,y_k)\,\Delta x$.

1.9 Exercise: Consider the IVP $y'=x-y$ with $y(0)=0$. Apply Euler's method with step size $\Delta x=\frac{1}{2}$ to approximate the value of $y(2)$.

Separable First Order ODEs

1.10 Definition: A **separable** first order ODE is a DE which can be written in the form

$$y' = f(x)g(y)$$

or to be more precise, $y'(x) = f(x)g(y(x))$, for some continuous functions $f(x)$ and $g(y)$.

1.11 Note: We can solve the separable DE $y' = f(x)g(y)$ by dividing by $g(y)$ then integrating both sides, using the Change of Variables Theorem to get

$$\int \frac{dy}{g(y)} = \int \frac{y'(x)\,dx}{g(y(x))} = \int f(x)\,dx.$$

Equivalently, we can write the DE in the differential form $\frac{dy}{g(y)} = f(x)\,dx$ and integrate both sides.

1.12 Exercise: Solve the DE $y' = x^2y$.

1.13 Exercise: Solve the IVP $y' = \frac{y}{\sqrt{x+1}}$ with $y(3) = 1$.

Linear First Order ODEs

1.14 Definition: A **linear** first order ODE is a DE which can be written in the form

$$y' + py = q,$$

that is $y'(x) + p(x)\,y(x) = q(x)$, for some continuous functions $p(x)$ and $q(x)$.

1.15 Note: We can solve the linear DE $y' + py = q$ as follows. If we can find a function $\lambda = \lambda(x)$ such that $\lambda' = \lambda\,p$, then we have $(\lambda y)' = \lambda\,y' + \lambda' y = \lambda y' + \lambda\,py$, and so multiplying both sides of the DE by λ gives $(\lambda y)' = \lambda q$, which we can solve by integrating to get $\lambda y = \int \lambda q$ so that $y = \frac{1}{\lambda}\int \lambda q$.. And indeed we can find such a function $\lambda = \lambda(x)$ because the DE $\lambda' = \lambda p$ is separable: we write the DE as $\frac{d\lambda}{\lambda} = p\,dx$ and integrate both sides to get $\ln\lambda = \int p$, that is $\lambda = e^{\int p}$. To summarize, in order to solve the DE $y' + py = q$, we let $\lambda = e^{\int p}$ and the solution is given by $y = \frac{1}{\lambda}\int \lambda q$. To be more precise, the solution is

$$y(x) = \frac{1}{\lambda(x)}\int \lambda(x)q(x)\,dx \quad \text{where} \quad \lambda(x) = e^{\int p(x)\,dx}.$$

The function $\lambda(x)$ is called an **integrating factor** for the linear DE.

1.16 Exercise: Solve the DE $y' - x^2y = 0$ (by treating it as a linear DE).

1.17 Exercise: Find the solution to the IVP $y' + 2y = e^{-5x}$, $y(0) = 1$.

1.18 Exercise: Find the solution to the IVP $y' - 2xy = x$, $y(0) = 0$.

1.19 Definition: A **Bernoulli** first order ODE is a DE which can be written in the form $y' + py = qy^r$ for some $1 \neq r \in \mathbb{R}$ and some continuous functions $p(x)$ and $q(x)$.

1.20 Note: We can often solve the Bernoulli DE $y' + py = qy^r$ by letting $u = y^{1-r}$ so that $u' = (1-r)y^{-r}y'$. If w multiply both sides of the DE by $(1-r)y^{-r}$, it becomes $(1-r)y^{-r}y' + (1-r)y^{1-r} = (1-r)q$, that is $u' + (1-r)u = (1-r)q$, which is a linear DE for the function $u = u(x)$.

1.21 Exercise: Solve the IVP $y' + y = x\sqrt{y}$ with $y(0) = 4$.

1.22 Definition: A **homogeneous** first order ODE is a DE which can be written in the form $y' = f\left(\frac{y}{x}\right)$ for some continuous function $f(x)$.

1.23 Note: We can solve the homogeneous DE $y' = f\left(\frac{y}{x}\right)$ by letting $u = \frac{y}{x}$, that is $y = xu$, so that $y' = u + xu'$. The DE becomes $u + xu' = f(u)$, that is $xu' = f(u) - u$, which is a separable DE for the function $u = u(x)$.

1.24 Exercise: Solve the IVP $y' = \frac{x^3+y^3}{xy^2}$ with $y(1) = 2$.

1.25 Remark: There are various other kinds of first order ODEs which can be solved. For example , an **exact** first order ODE can be written in differential form as $\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = 0$, that is as $du = 0$, for some differentiable function $u = u(x,y)$. The solution is then given by $u(x,y) = c$. Note that when $a = a(x,y)$ and $b = b(x,y)$ are continuously differentiable with $\frac{\partial a}{\partial y} = \frac{\partial b}{\partial y}$, we can find $u = u(x,y)$ such that $\frac{\partial u}{\partial x} = a$ and $\frac{\partial u}{\partial y} = b$, and so the DE $a(x,y) + b(x,y)y' = 0$ is exact. Also, sometimes a DE can be made exact by multiplying both sides by a suitable function $\mu = \mu(x,y)$, called an **integrating factor**.

1.26 Exercise: Solve the IVP $(2y + 2xy)y' + y^2 + 1 = 0$ with $y(0) = 1$ by recognizing that the DE is exact.

1.27 Exercise: Solve the IVP $(x^2y^2 + 1)y' + xy^3 = 0$ with $y(0) = 1$ by first finding an integrating factor $\mu = \mu(y)$.

Second Order ODEs

1.28 Definition: In general, a second order ODE can be written in the form

$$G(x,y,y',y'')=0$$

for some function $G:A\subseteq\mathbb{R}^4\to\mathbb{R}$. The graph of a solution $y=y(x)$ is called a **solution curve**. Given a second order ODE, there is often a unique solution curve $y=y(x)$ which passes through a given point (a,b) with a given slope $c=y'(a)$, equivalently, there is often a unique solution $y=y(x)$ to the DE with $y(a)=b$ and $y'(a)=c$. A second order ODE for a function $y=y(x)$ together with two additional constraints of the form $y(a)=b$ and $y'(a)=c$, is called an **initial value problem**, or IVP, and the two constraints are called the **initial conditions**. Sometimes, a given second order ODE has a unique solution $y=y(x)$, defined for all x in the interval $[a,b]$, which satisfies additional constraints of the form $y(a)=c$ and $y(b)=d$. A second order ODE for $y=y(x)$, defined for $x\in[a,b]$, together with constraints of the form $y(a)=c$ and $y(b)=d$, is called a **boundary value problem**, or BVP, and the constraints are called **boundary conditions**.

Direction Fields and Euler's Method

1.29 Note: The graph of a differentiable function $y=f(x)$ can be given parametrically by $(x,y)=(x(t),y(t))=(t,f(t))$. The **lift** of this curve to $\mathbb{R}^3$ is given by $(x,y,z)=(x(t),y(t),z(t))=(t,f(t),f'(t))$. Consider a second order ODE of the form

$$y''=F(x,y,y').$$

A solution curve $y=f(x)$ can be given parametrically by $(x,y)=(t,f(t))$, and its lift is the curve in $\mathbb{R}^3$ given by $(x,y,z)=(t,f(t),f'(t))$. At the point $(x,y,z)=(t,f(t),f'(t))$, the tangent vector is $(x',y',z')=(1,f'(t),f''(t))=(1,z,F(x,y,z))$. The **direction field** of the ODE $y''=F(x,y,y')$ is constructed as follows: at each point (x,y,z) in the domain of F, we place a short line segment in the direction of the vector $(1,z,F(x,y,z))$. As explained above, for any solution $y=f(x)$, the lift of the solution curve is a curve in $\mathbb{R}^3$ which follows the direction indicated by direction field.

We can approximate the solution to the IVP $y''=F(x,y,y')$ with $y(a)=b$ and $y'(a)=c$, by approximating its lift, using the following method which is called **Euler's method**. We choose a small value Δx, which we call the **step size**, we begin at the point $(x_0,y_0,z_0)=(a,b,c)$, then for each $k\geq 0$ we let

$$(x_{k+1},y_{k+1},z_{k+1})=(x_k,y_k,z_k)+(1,z_k,F(x_k,y_k,z_k)\Delta x$$

that is

$$\begin{aligned}x_{k+1}&=x_k+\Delta z\\ y_{k+1}&=y_k+z_k\Delta x\\ z_{k+1}&=z_k+F(x_k,y_k,z_k)\Delta x\end{aligned}$$

The polygonal path in $\mathbb{R}^3$ with vertices at (x_k,y_k,z_k) approximates the lift of the solution. The solution $y=f(x)$ is approximated by the polygonal path in $\mathbb{R}^2$ with vertices at (x_k,y_k).

1.30 Exercise: Note that $y=\sin x$ is the unique solution to the IVP $y''+y=0$ with $y(0)=0$ and $y'(0)=1$. Apply Euler's Method using step size $\Delta x=\frac{1}{2}$ to approximate the solution (which we know to be $y=\sin x$) for $0\leq x\leq 3$.

Substitutions to Reduce the Order

1.31 Note: For a second order ODE of the form $G(x, y', y'') = 0$ (that is for a DE which does not explicitly involve the dependent variable y), we can let $u = u(x)$ be given by $u(x) = y'(x)$ so that $u'(x) = y'(x)$, so that we have $y' = u$ and $y'' = u'$, and then the DE becomes $G(x, u, u') = 0$ which is a first order DE for $u = u(x)$.

1.32 Exercise: Solve the IVP $y'' + 2y' = 6x$ with $y(0) = 1$ and $y'(0) = 1$.

1.33 Exercise: Solve the BVP $xy'' + y' = 4x$ with $y(1) = y(2) = 1$.

1.34 Note: For a second order ODE of the form $G(y, y', y'') = 0$ (that is for a DE which does not explicitly involve the independent variable x), we can let $u = u(y)$ be given by $u(y(x)) = y'(x)$ so that $u'(y(x))y'(x) = y''(x)$, so that we have $y' = u$ and $y'' = uu'$, and then the DE becomes $G(y, u, uu') = 0$, which is a first order DE for the function $u = u(y)$.

1.35 Exercise: Solve the IVP $y'' + 4(y')^2 = 1$ with $y(0) = 0$ and $y'(0) = 1$.

1.36 Exercise: Solve the IVP $y'' = 4y$ with $y(0) = 1$ and $y'(0) = 1$.

1.37 Exercise: Solve the IVP $y^2y'' = y'$ with $y(0) = 1$ and $y'(0) = 1$.

Second Order Linear ODEs

1.38 Definition: A second order **linear** ODE is a DE which can be written in the form

$$ay'' + by' + cy = d$$

for some continuous functions $a, b, c, d : I \subseteq \mathbb{R} \to \mathbb{R}$ with $a(x) \neq 0$ for all $x \in I$, where I is an interval in $\mathbb{R}$. This linear DE is called **homogeneous** when $d = 0$. Equivalently, a second order **linear** ODE is a DE which can be written in the form

$$y'' + py' + qy = r$$

where $p, q, r : I \subseteq \mathbb{R} \to \mathbb{R}$ are continuous and I is an interval, and this DE is called **homogeneous** when $r = 0$.

1.39 Theorem: *Given an interval $I \subseteq \mathbb{R}$, given continuous maps $p, q, r : I \subseteq \mathbb{R} \to \mathbb{R}$, and given $a \in I$ and $b, c \in \mathbb{R}$, there exists a unique solution $y = y(x)$ to the IVP given by $y'' + py' + q = r$ with $y(a) = b$ and $y'(a) = c$.*

Proof: We omit the proof, which is fairly difficult.

1.40 Note: As a consequence of the above theorem, it is not hard to show that given an interval $I \subseteq \mathbb{R}$ and given continuous functions $p, q, r : I \subseteq \mathbb{R} \to \mathbb{R}$, the set of solutions to the homogeneous linear DE $y'' + py' + qy = 0$ is a 2-dimensional vector space, and the set of solutions to the DE $y'' + py' + qy = r$ is a 2-dimensional plane. Indeed, there exist two independent solutions $y_1, y_2 : I \to \mathbb{R}$ to the homogeneous DE $y'' + py' + qy = 0$, and the general solution is given by $y = Ay_1 + By_2$ where $A, B \in \mathbb{R}$, and there exists a particular solution $y_p : I \subseteq \mathbb{R} \to \mathbb{R}$ to the DE $y'' + py' + qy = r$, and the general solution is given by $y = y_p + Ay_1 + By_2$ where $A, B \in \mathbb{R}$.

Reduction of Order for Second Order Linear Homogeneous ODEs

1.41 Note: Given one solution $y = y_1(x)$ to the second order linear homogeneous ODE $y'' + p(x) + q(x)y = 0$, we can often find a second independent solution using the following method, which is called **reduction of order**: We let $y_2(x) = y_1(x)u(x)$ for some function $u = u(x)$. For $y = y_2 = y_1 u$, we have $y' = y_1' u + y_1 u'$ and $y'' = y_1'' u + 2y_1' u' + y_1 u''$, so the DE becomes

$$\begin{aligned} 0 = y'' + p' + qy &= (y_1'' u + 2y_1' u' + y_1 u'') + p(y_1' u + y_1 u') + q(y_1 u) \\ &= (y_1'' + py_1' + qy_1)u + (2y_1' + py_1)u' + y_1 u'' \\ &= (2y_1' + py_1)u' + y_1 u'' \end{aligned}$$

where, at the last stage, we used the fact that y_1 satisfies $y_1'' + py_1' + qy_1 = 0$. The resulting DE $y_1 u'' + (2y_1' + py_1)u' = 0$ for $u = u(x)$ does not involve u, so we make the substitution $v(x) = u'(x)$ and the DE becomes $y_1 v' + (2y_1' + py_1)v = 0$, which is a first order linear (and separable) DE for $v = v(x)$. Once we solve for $v = v(x)$, then for $u = u(x)$, we obtain the second independent solution $y_2 = y_1 u$.

1.42 Exercise: Solve the IVP $2x^2 y'' + 3xy' - 6y = 0$ with $y(1) = 3$ and $y'(1) = 1$ given that $= y_1 = \frac{1}{x^2}$ is one solution to the DE.

1.43 Exercise: Solve the ODE $xy'' - y' + (1 - x)y = 0$ for $x > 0$ given that $y = y_1 = e^x$ is one solution.

Variation of Parameters for Second Order Linear Nonhomogeneous ODEs

1.44 Note: Given two independent solutions $y = y_1(x)$ and $y = y_2(x)$ to the linear homogeneous DE $y'' + p(x)y' + q(x)y = 0$ we can often find a particular solution $y = y_p(x)$ to the associated non-homogeneous DE $y'' + p(x)\,y' + q(x)\,y = r(x)$ using the following method, which is known as **variation of parameters**: We let $y_p(x) = y_1(x)u_1(x) + y_2(x)u_2(x)$ for some functions $u_1(x)$ and $u_2(x)$ which satisfy the conditions

$$\begin{aligned} y_1\,{u_1}' + y_2\,{u_2}' &= 0 \;\; (1) \\ {y_1}'{u_1}' + {y_2}'{u_2}' &= r \;\; (2) \end{aligned}$$

Note that these two equations are linear equations which we can easily solve for the unknowns u_1' and u_2', and we can integrate to find such functions $u_1 = u_1(x)$ and $u_2 = u_2(x)$. Note that when $y = y_p = y_1 u_1 + y_2 u_2$ for functions u_1 and u_2 which satisfy the above conditions (1) and (2), we have

$$\begin{aligned} y &= y_1 u_1 + y_2 u_2 \\ y' &= y_1' u_1 + y_1 u_1' + y_2' u_2 + y_2 u_2' = y_1' u_1 + y_2' u_2 \\ y'' &= y_1'' u_1 + y_1' u_1' + y_2'' u_2 + y_2' u_2' = y_1'' u_1 + y_2'' u_2 + r \end{aligned}$$

and hence

$$\begin{aligned} y'' + py' + qy &= (y_1'' u_1 + y_2'' u_2 + r) + p(y_1' u_1 + y_2' u_2) + q(y_1 u_1 + y_2 u_2) \\ &= (y_1'' + py_1' + qy_1)u_1 + (y_2'' + py_2' + qy_2)u_2 + r = r \end{aligned}$$

so that $y = y_p = y_1 u_1 + y_2 u_2$ is indeed a solution to the DE $y'' + py' + qy = r$.

1.45 Exercise: Solve the ODE $2x^2 y'' + 3xy' - 6y = \ln x$ for $x > 0$.

Second Order Linear ODEs With Constant Coefficients

1.46 Note: When $a, b, c \in \mathbb{R}$ with $a \neq 0$, it is easy to solve the second order linear homogeneous ODE $ay'' + by' + cy = 0$ as follows: We try $y = e^{rx}$ so that $y' = re^{rx}$ and $y'' = r^2 e^{rx}$. The DE becomes $0 = ay'' + by' + cy = (ar^2 + br + c)e^{rx}$. Since $e^{rx} \neq 0$ for all x, this is equivalent to $0 = g(r) = ar^2 + br + c$. The polynomial $g(r) = ar^2 + br + c$ is called the **characteristic polynomial** for the DE $ay'' + by' + cy = 0$.

When g has two distinct real roots $r = r_1$ and r_2, we have the two independent solutions

$$y_1 = y_1(x) = e^{r_1 x} \text{ and } y_2 = y_2(x) = e^{r_2 x}.$$

When g has one repeated real root r, it is not hard to verify that we have the two independent solutions

$$y_1 = y_1(x) = e^{rx} \text{ and } y_2 = y_2(x) = xe^{rx}.$$

When g has a pair of conjugate complex roots $r = s \pm it$, we have the two independent complex-valued solutions given by $z_1 = z_1(x) = e^{(s+it)x} = e^{sx} e^{itx} = e^{sx}(\cos tx + i \sin tx)$ and $z_2 = z_2(x) = e^{(s-it)x} = e^{sx} e^{-itx} = e^{sx}(\cos tx - i \sin tx)$, and we can take linear combinations to get the two independent real-valued solutions

$$y_1 = y_1(x) = \tfrac{z_1+z_2}{2} = e^{sx} \cos tx \text{ and } y_2 = y_2(x) = \tfrac{z_1-z_2}{2i} = e^{sx} \sin tx.$$

1.47 Exercise: Solve the IVP $6y'' + y' - 2y = 0$ with $y(0) = 2$ and $y'(0) = 1$.

1.48 Exercise: Solve the IVP $9y'' + 12y' + 4y = 0$ with $y(0) = 1$ and $y'(0) = 2$.

1.49 Exercise: Solve the IVP $y'' + 4y' + 5y = 0$ with $y(0) = 2$ and $y'(0) = -1$.

1.50 Note: Given two independent solutions $y_1 = y_1(x)$ and $y_2 = y_2(x)$ to the second order linear homogeneous ODE $ay'' + by' + cy = 0$ with constant coefficients $a, b, c \in \mathbb{R}$ with $a \neq 0$, and given a continuous function $d = d(x)$, we can try to find a particular solution $y_p = y_p(x)$ to the associated non-homogeneous ODE $ay'' + by' + cy = d(x)$ by using the method of **variation of parameters** or, often more easily, simply by using trial and error: when $d(x)$ is a polynomial, we can try letting y_p be a polynomial; when $d(x) = e^{kx}$, we can try letting y_p be a constant multiple of e^{kx} (or sometimes a polynomial multiplied by e^{kx}); and when $d(x) = \cos kx$ or $\sin kx$, we can try letting y_p be a linear combination (or sometimes a polynomial combination) of $\sin kx$ and $\cos kx$. The method of trial and error is sometimes called the method of **undetermined coefficients**.

1.51 Exercise: Find a particular solution to the linear ODE $y'' + y' - 2y = g(x)$ for each of the following functions $g(x)$:

(a) $g(x) = 2x^2$ (b) $g(x) = 5e^{3x}$ (c) $g(x) = 5 \sin x$ (d) $g(x) = x^2 e^{-x}$

1.52 Exercise: Find a particular solution to each of the following linear ODEs:

(a) $y'' + y' - 2y = e^{-2x}$ (b) $y'' + 2y' + y = e^{-x}$ (c) $y'' - 2y' + 10y = 6e^x \cos 3x$

1.53 Exercise: Solve the linear ODE $y'' - 4y' + 13y = 6\, e^{2x} \sin 8x \cos x$.