Part 2. Systems of First Order ODEs

2.1 Definition: A **system** of first order ODEs for the functions $x_1(t), x_2(t), \cdots, x_n(t)$, is a list of equations of the form

$$\begin{gathered}G_1(t, x_1, x_2, \cdots, x_n, x_1', x_2', \cdots, x_n') = 0\\ G_2(t, x_1, x_2, \cdots, x_n, x_1', x_2', \cdots, x_n') = 0\\ \vdots\\ G_m(t, x_1, x_2, \cdots, x_n, x_1', x_2', \cdots, x_n') = 0\end{gathered}$$

We can write the above system in vector form as

$$G(t, x, x') = 0$$

where $x = (x_1, x_2, \cdots, x_n)^T$, $x' = (x_,', x_2', \cdots, x_n')^T$ and $G = (G_1, G_2, \cdots, G_m)^T$. Here, G is a function $G : U \subseteq \mathbb{R}^{2n+1} \to \mathbb{R}^m$, and a **solution** is a function $x : I \subseteq \mathbb{R} \to \mathbb{R}^n$, where I is an interval in $\mathbb{R}$, for which $(t, x(t), x'(t)) \in U$ with $G(t, x(t), x'(t)) = 0 \in \mathbb{R}^m$ for all $t \in I$. We shall usually consider systems which can be written in the form

$$\begin{aligned}x_1' &= F_1(t, x_1, x_2, \cdots, x_n)\\ x_2' &= F_2(t, x_1, x_2, \cdots, x_n)\\ &\vdots\\ x_n' &= F_n(t, x_1, x_2, \cdots, x_n)\end{aligned}$$

or, more simply, in vector form as

$$x' = F(t, x)$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$. A **solution** is a function $x : I \subseteq \mathbb{R} \to \mathbb{R}^n$, where $I \subseteq$ is an interval, such that $x'(t) = F(t, x(t))$ for all $t \in I$. An **autonomous system** is a system which can be written in vector form as

$$x' = F(x).$$

2.2 Note: Higher order ODEs and systems of higher order ODEs can be converted to systems of first order ODEs. For example, given the n^{th} order ODE

$$y^{(n)} = F(t, y, y', y'', \cdots, y^{(n-1)}),$$

we can let $x_1 = y, x_2 = y', x_3 = y'', \cdots, x_n = y^{(n-1)}$ and then the given n^{th} order ODE is equivalent to the system of first order ODEs

$$x_1' = x_2\ ,\ x_2' = x_3\ ,\ x_{n-1}' = x_n\ ,\ x_n' = F(t, x_1, x_2, \cdots, x_n).$$

As another example, given the pair of second order ODEs

$$\begin{gathered}G_1(t, y_1, y_2, y_1', y_2') = 0\\ G_2(t, y_1, y_2, y_1' y_2') = 0\end{gathered}$$

we can let $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_1'$ and $x_4 = y_2'$, and then the given pair of second order ODEs is equivalent to the system of first order ODEs

$$\begin{gathered}x_1' = x_3\\ x_2' = x_4\\ G_1(t, x_1, x_2, x_3, x_4) = 0\\ G_2(t, x_1, x_2, x_3, x_4) = 0.\end{gathered}$$

2.3 Exercise: An object of mass m hangs at the end of a string of length ℓ and swings back and forth under the influence of the downward force of gravitation. Show that the angle $\theta = \theta(t)$ between the string and the vertical at time t satisfies the second order ODE

$$\theta'' = -\frac{g}{\ell}\sin\theta.$$

Letting $\omega(t) = \theta'(t)$ (ω is called the **angular velocity**), this second order DE is equivalent to the autonomous pair of first order ODEs given by

$$\theta' = \omega \quad \omega' = -\tfrac{g}{\ell}\sin\theta.$$

2.4 Exercise: A large object of mass M is fixed at the origin. A small object of mass m begins at the point $(x, y) = (x_0, y_0)$ with initial velocity $(u, v) = (u_0, v_0)$, and it moves in the xy-plane under the influence of gravity. Show that the position $(x, y) = (x(t), y(t))$ of the small object satisfies the pair of second order ODEs given by

$$x'' = \frac{-GMx}{(x^2+y^2)^{3/2}} \;,\; y'' = \frac{-GMy}{(x^2+y^2)^{3/2}}.$$

Letting $u = x'$ and $v = y'$, this is equivalent to the autonomous system of first order ODEs

$$x' = u \;,\; y' = v \;,\; u' = \frac{-GMx}{(x^2+y^2)^{3/2}} \;,\; v' = \frac{-GMy}{(x^2+y^2)^{3/2}}.$$

The appropriate initial condition is given by $(x(0), y(0), u(0), v(0)) = (x_0, y_0, u_0, v_0)$.

Euler's Method

2.5 Definition: For a system of first order ODEs, an **initial value problem**, or IVP, consists of a system of first order ODEs of the form

$$\begin{aligned} x_1' &= F_1(t, x_1, \cdots, x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, \cdots, x_n) \end{aligned}$$

together with **initial conditions** of the form $x_1(s) = c_1$, $x_2(s) = c_2, \cdots, x_n(s) = c_n$ where $s \in \mathbb{R}$ and each $c_k \in \mathbb{R}$ with $(s, c_1, \cdots, c_n)$ in the domain of F. In vector form, an IVP consists of a system of the form

$$x' = F(t, x)$$

together with initial conditions of the form $x(s) = c$ with $s \in \mathbb{R}$ and $c \in \mathbb{R}^n$. We can approximate the solution to the above IVP numerically using the following method, known as **Euler's method**: We choose a small value of Δt, called the **step size**. We start at the point $(t_0, x_{1,0}, x_{2,0}, \cdots, x_{n,0}) = (s, c_1, c_2, \cdots, c_n) \in \mathbb{R}^{n+1}$, and then for $k \geq 0$ we let

$$t_{k+1} = t_k + \Delta t \;\text{ and }\; x_{j,k+1} = x_{j,k} + F_j(t_k, x_{1,k}, x_{2,k}, \cdots, x_{n,k})\,\Delta t.$$

In the case of the IVP given by the pair of ODEs

$$\begin{aligned} x' &= F(t, x, y) \\ y' &= G(t, x, y) \end{aligned}$$

with the initial condition $x(s) = b$, $y(s) = c$, we start at $(t_0, x_0, y_0) = (s, b, c)$ then let

$$t_{k+1} = t_k + \Delta t \;,\; x_{k+1} = x_k + F(t_k, x_k, y_k)\Delta t \;, \text{ and } y_{k+1} = y_k + G(t_k, x_k, y_k)\Delta t.$$

2.6 Exercise: Approximate the solution to $x' = \frac{1}{y}$, $y' = 2xy$ with $x(0) = 0$, $y(0) = 1$.

The Vector Field and the Direction Field of an Autonomous System

2.7 Definition: A **vector field** on $U \subseteq \mathbb{R}^n$ is a function $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$, in other words, a vector field on U consists of a vector $F(x) \in \mathbb{R}^n$ at each point $x \in U \subseteq \mathbb{R}^n$. For the autonomous system of ODEs given in vector form by $x' = F(x)$, where $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$, the set U (that is the domain of F) is called the **phase space** and the function F is called the **associated vector field** of the system. When $x = x(t)$ is a solution to the system of ODEs, it determines a parametric curve, also denoted by $x = x(t)$, and the tangent vector (or the velocity in the case that t represents time) at the point $x = x(t)$ on this curve is equal to $x'(t) = F(x)$, so the solution curves follow the vectors in the vector field. The **direction field** for an autonomous system consists of a short line segment at each point in U in the direction of the vector field. The difference between the direction field and the vector field is simply that the direction field indicates the direction of the solution curves, while the vector field indicates not only the direction but also the velocity (if we think of $x(t)$ as the position of a moving object at time t).

In the case of a pair of autonomous equations given by $x' = F(x, y)$ and $y' = G(x, y)$, the phase space is also called the **phase plane** and the direction field is also called the **slope field**, and it consists of a short line segment of slope $m = \frac{G(x,y)}{F(x,y)}$ at each point (x, y). A picture which shows the direction field (or the vector field), along with a representative set of solution curves, is called a **phase portrait**.

2.8 Exercise: Sketch the direction field, and a few solution curves for the autonomous pair of ODEs given by $x' = \frac{1}{y}$ and $y' = 2xy$.

Two Solution Methods for Pairs of Autonomous ODEs

2.9 Note: Sometimes we can solve an autonomous pair of ODEs given by $x' = F(x, y)$ and $y' = G(x, y)$ by first solving the DE $\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)}$ for $y = y(x)$, then putting this back into either one of the two given DEs. Alternatively, sometimes we can first solve the DE $\frac{dx}{dy} = \frac{F(x,y)}{G(x,y)}$ for $x = x(y)$.

2.10 Exercise: Solve the IVP given by $x' = \frac{1}{y}$ and $y' = 2xy$ with $x(0) = 0$ and $y(0) = 1$.

2.11 Note: Sometimes we can solve an autonomous pair of ODEs given by $x' = F(x, y)$ and $y' = G(x, y)$ by using the two equations $x' = F(x, y)$ and $y' = G(x, y)$ to eliminate y and y' from the equation $x'' = \frac{\partial F}{\partial x}x' + \frac{\partial F}{\partial y}y'$. Alternatively, sometimes we can eliminate x and x' from the equation $y'' = \frac{\partial G}{\partial x}x' + \frac{\partial G}{\partial y}y'$.

2.12 Exercise: Solve the IVP given by $x' = xy$ and $y' = x$ with $x(0) = 1$ and $y(0) = -1$.

Linear Systems of First Order ODEs

2.13 Definition: A **linear system** of first order ODEs is a system which can be written in the form

$$\begin{aligned} x_1' &= a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ x_2' &= a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ &\vdots \\ x_n' &= a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,1}x_n = b_n \end{aligned}$$

where each of the terms $a_{k,\ell}$ and b_k is a continuous function of t (defined for t in some interval I). This system can be written in matrix form as

$$x' = Ax + b.$$

A **homogeneous** linear system of first order ODEs can be written in the form

$$x' = Ax.$$

2.14 Theorem: *Given an interval $I \subseteq \mathbb{R}$, given continuous functions $a_{k,\ell}, b_k : I \subseteq \mathbb{R} \to \mathbb{R}$, and given $s \in I$ and $c \in \mathbb{R}^n$, there exists a unique solution $x = x(t)$ to the IVP given by $x' = Ax + b$ with $x(s) = c$.*

2.15 Note: Using the above theorem, one can show that given an interval $I \subseteq \mathbb{R}^n$ and continuous functions $a_{k,\ell}, b_k : I \subseteq \mathbb{R} \to \mathbb{R}$ for $1 \leq k, \ell \leq n$, the set of all solutions to the homogeneous linear system $x' = Ax$ is an n-dimensional vector space and the set of solutions to the linear system $x' = Ax + b$ is an n-dimensional plane: indeed there exist independent solutions $x_1, x_2, \cdots, x_n$ to the system $x' = Ax$ and the general solution is given by $x = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ where each $c_k \in \mathbb{R}$, and there exists a particular solution x_p to the system $x' = Ax+b$ and the general solution is given by $x = x_p + c_1x_1 + \cdots + c_nx_n$ where each $c_k \in \mathbb{R}$.

Reduction of Order for a Homogeneous Linear Pair of ODEs

2.16 Note: Given one solution $\binom{x_1}{y_1}$ to a homogeneous linear pair of ODEs $\binom{x'}{y'} = A\binom{x}{y}$, where the entries of A are continuous functions of t, we can find a second independent solution $\binom{x_2}{y_2}$ using the following method, which we call **reduction of order**. For $u = u(t)$ and $v = v(t)$ we let

$$\binom{x}{y} = \binom{x_2}{y_2} = \begin{pmatrix} 1 & x_1 \\ 0 & y_1 \end{pmatrix}\binom{u}{v} = \binom{1}{0}u + \binom{x_1}{y_1}v.$$

Then $\binom{x'}{y'} = \binom{1}{0}u' + \binom{x_1}{y_1}v' + \binom{x_1'}{y_1'}v = \begin{pmatrix} 1 & x_1 \\ 0 & y_1 \end{pmatrix}\binom{u'}{v'} + A\binom{x_1}{y_1}v$ and $A\binom{x}{y} = A\binom{1}{0}u + A\binom{x_1}{y_1}v$, so we have $\binom{x'}{y'} = A\binom{x}{y}$ when $\begin{pmatrix} 1 & x_1 \\ 0 & y_1 \end{pmatrix}\binom{u'}{v'} = A\binom{1}{0}u$, that is when

$$\binom{u'}{v'} = \begin{pmatrix} 1 & x_1 \\ 0 & y_1 \end{pmatrix}^{-1} A\binom{1}{0}\, u.$$

The first entry is a linear first order ODE for $u = u(t)$ and, after solving for $u = u(t)$, the second entry becomes a simple ODE for $v = v(t)$. Once we have found $u = u(t)$ and $v = v(t)$, we have obtained our second independent solution $\binom{x_1}{y_2} = \begin{pmatrix} 1 & x_1 \\ 0 & y_1 \end{pmatrix}\binom{u}{v}$.

2.17 Exercise: Solve $\binom{x'}{y'} = \begin{pmatrix} t^{-1} & t^{-2} \\ 2 & t^{-1} \end{pmatrix}\binom{x}{y}$ given that $\binom{x_1}{y_1} = \binom{t^2}{t^3}$ is one solution.

Variation of Parameters for a Linear System of ODEs

2.18 Note: Let A be an $n\times n$ matrix whose entries are continuous functions of t, and let b be an $n\times 1$ column vector whose entries are continuous functions of t. Given n independent solutions $x_1, x_2, \cdots, x_n$ to the homogeneous system of n linear ODEs given by $x' = Ax$, we can find a particular solution x_p the the non-homogeneous system $x' = Ax + b$, using the following method, which we call **variation of parameters**: For functions $u_k = u_k(t)$, we let

$$x = x_p = x_1u_1 + x_2u_2 + \cdots x_nu_n.$$

We can write this in matrix form as

$$x = x_p = Xu,$$

where $X = (x_1, x_2 \cdots, x_n)$ (this is the $n\times n$ matrix whose columns are the functions $x_k(t)$) and $u = (u_1, \cdots, u_n)^T$ (this is an $n\times 1$ column vector whose entries are the functions $u_k(t)$). Since each x_k satisfies $x_k' = Ax_k$, it follows that $X' = AX$. We have $x' = X'u + Xu' = AXu + Xu'$ and we have $Ax = AXu$, and so in order to obtain $x' = Ax + b$ we need $Xu' = b$. Thus we need to choose u such that

$$u' = X^{-1}b.$$

The k^{th} entry is a simple ODE which we can solve for $u_k = u_k(t)$.

2.19 Exercise: Solve $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} t^{-1} & t^{-2} \\ 2 & t^{-1} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 2t\ln t \end{pmatrix}$.

Systems of First Order ODEs with Constant Coefficients

2.20 Note: When A is an $n\times n$ matrix with constant real entries, we can solve the homogeneos system of first order ODEs $x' = Ax$ as follows: we try $x = e^{rt}u$ where $r \in \mathbb{R}$ and $u \in \mathbb{R}^n$. and note that $x' = re^{rt}u$. We have $x' = Ax$ when $re^{rt}u = Ae^{rt}u$, that is when $Au = ru$, or equivalently when $(A - rI)u = 0$. Recall that this occurs when r is an eigenvalue of A and u is a corresponding eigenvector. The polynomial $g(r) = \det(A - rI)$ of the matrix A is also called the **characteristic polynomial** of the system of ODEs, and its roots are the eigenvalues of A.

When r is a real eigenvalue of A and u is a corresponding eigenvector, we obtain the real-valued solution $x = e^{rt}u$.

When $r + is$ is a complex eigenvalue and $u + iv$ is an associated complex eigenvector, then $r - is$ is another eigenvalue with eigenvector $u - iv$, so we obtain two complex-valued solutions $z_1 = e^{(r+is)t}(u + iv) = e^{rt}(\cos st + i\sin st)(u + iv)$ and $z_2 = e^{(r+is)t}(u - iv)$, and these give the two real-valued solutions $x_1 = \frac{z_1+z_2}{2} = 2\,\mathrm{Re}\,(z_1) = e^{rt}\big(\cos(st)u - \sin(st)v\big)$ and $x_2 = \frac{z_1-z_2}{2i} = 2\,\mathrm{Im}\,(z_1) = e^{rt}\big(\cos(st)v + \sin(st)u\big)$.

In the case that A is diagonalizable, it has n linearly independent eigenvectors and we obtain n linearly independent solutions. In the case that A is not diagonalizable, there is still a routine procedure for obtaining n linearly independent solutions. In general, the procedure is a bit complicated, and we omit the details, but let us explain what to do in the case that a real eigenvalue $r \in \mathbb{R}$ has algebraic multiplicity 2 but it only has one independent eigenvector $u \in \mathbb{R}^n$. In this case, we have one solution $x_1 = e^{rt}u$ and we can find a second solution as follows: We let $x = x_2 = te^{rt}u + e^{rt}v$ with $v \in \mathbb{R}^n$, and we note that $x' = (1 + rt)e^{rt}u + re^{rt}v$ and $Ax = rte^{rt}u + e^{rt}Av$, so that to get $x' = Ax$ we need $Av = u + rv$, that is

$$(A - rI)v = u.$$

We solve this equation for v, then we have two solutions $x_1 = e^{rt}u$ and $x_2 = te^{rt}u + e^{rt}v$.

2.21 Exercise: Solve each of the following linear homogeneous pairs of first order ODEs with constant coefficients. In each case, sketch a phase portrait.

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ (b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

(c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ (d) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

2.22 Exercise: Solve each of the following linear homogeneous systems of first order ODEs with constant coefficients.

(a) $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & -2 & -6 \\ 0 & 2 & 3 \\ -2 & -4 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (b) $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(c) $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (d) $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 8 & -5 & 5 \\ 5 & -2 & 5 \\ -6 & 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

2.23 Note: Let A be an $n \times n$ matrix with constant real entries, and let b be an $n \times 1$ column vector whose entries are continuous functions of t. Given n independent solutions to the linear homogeneous system $x' = Ax$, we can find a particular solution to the non-homogeneous system $x' = Ax + b$ using the method of variation of parameters (which was discussed earlier in Note 2.18). Alternatively, it is often easier to find a particular solution using trial and error, looking for a solution $x = x_p = x_p(t)$ of an appropriate form, depending on the form of b. When b is of the form $b = \sum_{k=0}^m u_k t^k$, where each $u_k \in \mathbb{R}^n$, we can try $x_p = \sum_{k=0}^m v_k t^k$, where each $v_k \in \mathbb{R}^m$ (but if, for example, $r = 0$ is a root of the characteristic polynomial of multiplicity 1 then we would try $x_p = \sum_{k=0}^{m+1} v_k t^k$). When b is of the form $b = \sum_{k=0}^m u_k t^k e^{rt}$, we can try $x_p = \sum_{k=0}^m v_k t^k e^{rt}$ (but if, for example, r is a root of the characteristic polynomial of multiplicity 1 then we would try $x_p = \sum_{k=0}^{m+1} v_k t^k e^{rt}$). When b is of the form $b = \sum_{k=0}^m u_k t^k e^{rt} \sin(st)$, we can try $x_p = \sum_{k=0}^m v_k t^k e^{rt} \sin st + \sum_{k=0}^m w_k t^k e^{rt} \cos st$. The method of trial and error, using an educated guess, is called the method of **undetermined coefficients**.

2.24 Exercise: Solve each of the following linear non-homogeneous systems of ODEs.

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5t \\ 3e^t \end{pmatrix}$ (b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + e^{5t} \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

(c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + e^{4t} \begin{pmatrix} -6 \sin t \\ 8 \cos t \end{pmatrix}$

Equilibrium Points of Autonomous Systems

2.25 Definition: Consider the autonomous system of ODEs given by $x' = F(x)$ where $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is continuous. When $p \in U$ and the constant function $x : \mathbb{R} \to U \subseteq \mathbb{R}^n$ given by $x(t) = p$ for all t is a solution to the system, we say that p is an **equilibrium point** and that x is an **equilibrium solution** of the system. Note that p is an equilibrium point when $F(p) = 0$ (since when $x(t) = p$ for all t we have $x' = 0$). Given an equilibrium point $p \in U$, we say that p is **stable** when it has the property that for all $\epsilon > 0$ there exists $\delta > 0$ such that if $\|x(s) - p\| < \delta$ for some $s \in \mathbb{R}$ then $\|x(t) - p\| < \epsilon$ for all $t \geq s$, and otherwise we say that p is **unstable**. We say that p is **asymptotically stable**, or that p is **attracting**, when it is stable and there exists $\delta > 0$ such that if $\|x(s) - p\| < \delta$ for some $s \in \mathbb{R}$ then $\lim\limits_{t\to\infty} x(t) = p$.

2.26 Example: When A is an invertible $n \times n$ matrix with constant real entries, the system $x' = Ax$ has a unique stable point, namely the origin 0. When all the eigenvalues of A have strictly negative real parts, the point 0 is asymptotically stable (because all the independent solutions include an exponential decay term e^{rt} with $r < 0$). When all eigenvalues have real parts which are less than or equal to zero, the point 0 is stable. When at least one eigenvalue has a positive real part, the point 0 is unstable (because there is a solution of the form $e^{rt}u$ with $r > 0$). When all the eigenvalues have positive real part, we can say that the point 0 is **repelling**.

2.27 Note: Let $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be twice continuously differentiable with $F(p) = 0$ so that p is a stable point for the system $x' = F(x)$. The function F is approximated near p by its **linearization** $L : \mathbb{R}^n \to \mathbb{R}^n$ given by $L(x) = DF(p)(x - p)$, where $DF(p)$ is the **derivative matrix**, also called the **Jacobian matrix**, of F (the k, ℓ entry of $DF(p)$ is $\frac{\partial F_k}{\partial x_\ell}$). The autonomous system $x' = F(x)$ is approximated by the system $x' = L(x)$ which can be written as $y' = Ay$ where $y = x - p$ and $A = DF(p)$. Each solution $x = x(t)$ to the original system $x' = F(x)$ is approximated by $x(t) \cong y(t) + p$ where $y = y(t)$ is a solution to the system $y' + Ay$, which is a linear homogeneous system with constant coefficients. When all the eigenvalues of $A = DF(p)$ have negative real parts, the point p is asymptotically stable.

2.28 Exercise: Recall that a population $x = x(t)$ grows exponentially when it satisfies the ODE $x' = rx$ for some constant $r > 0$ (which determines the rate of growth). For a species which lives in an environment with limited resources, the population growth can be modelled slightly more realistically using the logistic equation, which is an ODE of the form $x' = r(1 - ax)x$ for some constant $a > 0$. The populations $x = x(t)$ and $y = y(t)$ of two **competing species** which live in the same environment and compete for resources can be modelled using a pair of first order ODEs of the form

$$\begin{aligned} x' &= r(1 - ax - by)x \\ y' &= s(1 - cx - dy)y \end{aligned}$$

for some constants $r, s, a, b, c, d > 0$. Consider the competing species model

$$\begin{aligned} x' &= \tfrac{1}{2}\big(1 - \tfrac{1}{2}x - \tfrac{1}{2}y\big)x \\ y' &= \tfrac{1}{4}\big(1 - \tfrac{1}{3}x - \tfrac{2}{3}y\big)y. \end{aligned}$$

Find the equilibrium solutions, and at each equilibrium point, linearize the system and determine the behaviour of the solution curves near the equilibrium points.

Conservative Systems and Hamiltonian Systems

2.29 Definition: Consider a second order ODE $y'' = F(t, y, y')$. A **conserved quantity** for this ODE is a function $H(y, v)$ with the property that for any solution $y = y(t)$ to the ODE, the value of $H(y(t), y'(t))$ is constant. In this case, the fact that $H(y(t), y'(t))$ is constant for solutions $y = y(t)$ is called a **conservation law** for the ODE. More generally, given a system $x' = F(t, x)$ of first order ODEs, a **conserved quantity** of the system is a function $H(x)$ with the proerty that $H(x(t))$ is constant for every solution $x = x(t)$ and, in this case, the fact that $H(x(t))$ is constant is called a **conservation law** for the system.

2.30 Exercise: Show that when an object at position $x = x(t)$ experiences a force $F = F(x)$, which depends on position but not time, if we define the **potential energy** to be given by $U = U(x) = -\int F(x)\,dx$, so that $U'(x) = -F(x)$, and the **kinetic energy** to be given by $K = K(v) = \frac{1}{2}mv^2$ where $v(t) = x'(t)$, then then the **total energy** $E(x, v) = K(v) + U(x)$ is a conserved quantity for the ODE $mx'' = F(x)$.

2.31 Exercise: Find a conserved quantity for the ODE $\theta'' = -\frac{g}{\ell}\sin\theta$, which models the position $\theta = \theta(t)$ of a pendulum. Sketch some solution curves (in the $\theta\omega$-plane) for the associated pair of first order ODEs given by $\theta' = \omega$ and $\omega' = -\frac{g}{\ell}\sin\theta$, say when $\ell = g$.

2.32 Exercise: Show that for the system given by

$$x' = u\ ,\ y' = v\ ,\ u' = \frac{-GMx}{(x^2+y^2)^{3/2}}\ ,\ v' = \frac{-GMy}{(x^2+y^2)^{3/2}}$$

which models the position $(x, y) = (x(t), y(t))$ and velocity $(u, v) = (x'(t), y'(t))$ of a small object, at time t, orbiting a large object of mass M fixed at the origin, if we define the **potential energy** to be given by $U(x, y) = -\frac{GMm}{(x^2+y^2)^{1/2}}$ and the **kinetic energy** to be given by $K(u, v) = \frac{1}{2}m(u^2 + v^2)$, then the total energy $E(x, y, u, v) = K(u, v) + U(x, y)$ is a conserved quantity for the system.

2.33 Note: For the autonomous system $x' = F(x)$, where $F : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is differentiable, when $\nabla H(x) \cdot F(x) = 0$ for all $x \in U$, it follows that for every solution $x = x(t)$ to the system $x' = F(x)$, we have

$$\tfrac{d}{dt}H(x(t)) = \nabla H(x(t)) \cdot x'(t) = \nabla H(x(t)) \cdot F(x(t)) = 0$$

so that $H = H(x)$ is a conserved quantity.

2.34 Note: In the case of an autonomous pair of linear ODEs given by

$$x' = f(x, y)\ ,\ y' = g(x, y)$$

where $f, g : U \subseteq \mathbb{R}^2 \to \mathbb{R}$, one can show that (if the domain U has no holes), if

$$\nabla \cdot (f, g) = \tfrac{\partial f}{\partial x} + \tfrac{\partial g}{\partial y} = 0$$

then there exists a function $H : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ such that

$$\nabla H = \left(\tfrac{\partial H}{\partial x}, \tfrac{\partial H}{\partial y}\right) = (-g, f).$$

For such a function H, we have $\nabla H \cdot (f, g) = (-g, f) \cdot (f, g) = 0$, and hence H is a conserved quantity. In this case the pair of ODEs is called a **Hamiltonian system**, and the conserved quantity $H = H(x, y)$ is called a **Hamiltonian** for the system.

2.35 Exercise: Find a Hamiltonian for the pair of ODEs $x' = 2y - x + 3$, $y' = y + 4x^3 - 2x$.

2.36 Exercise: Find a conserved quantity for the 2^{nd} order ODE $x^2x'' = x'$, for $x = x(t)$.