# Part 3. Power Series and Fourier Series for ODEs

**3.1 Definition:** A function $f : U \subseteq \mathbb{R} \to \mathbb{R}$ is **analytic** at $a \in U$ when there exists $R > 0$ with $(a-R, a+R) \subseteq U$ such that $f(x)$ is equal to the sum of its Taylor series centred at $a$ for all $|x-a| < R$, that is when

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for all $x$ with $|x-a| < R$, where $c_n = \frac{f^{(n)}(a)}{n!}$.

**3.2 Example:** The functions $e^x$, $\sin x$, $\cos x$, $\sinh x$, $\cosh x$, are all analytic at 0 with

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \tfrac{1}{n!} x^n = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \cdots \\ \sin x &= \sum_{n=0}^{\infty} \tfrac{(-1)^n}{(2n+1)!} x^{2n+1} = x = \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots \\ \cos x &= \sum_{n=0}^{\infty} \tfrac{(-1)^n}{(2n)!} x^{2n} = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 = \tfrac{1}{6!}x^6 + \cdots \\ \sinh x &= \sum_{n=0}^{\infty} \tfrac{1}{(2n+1)!} x^{2n+1} = x + \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 + \cdots \\ \cosh x &= \sum_{n=0}^{\infty} \tfrac{1}{(2n)!} x^{2n} = 1 + \tfrac{1}{2}x^2 + \tfrac{1}{4!}x^4 + \cdots \end{aligned}$$

for all $x \in \mathbb{R}$, and the functions $\frac{1}{1-x}$ and $(1+x)^p$ where $p \in \mathbb{R}$, are analytic at 0 with

$$\begin{aligned} \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \\ (1+x)^p &= \sum_{n=0}^{\infty} \tbinom{p}{n} x^n = 1 + px + \tfrac{p(p-1)}{2!}x^2 + \tfrac{p(p-1)(p-2)}{3!}x^3 + \cdots \end{aligned}$$

for all $x \in \mathbb{R}$ with $|x| < 1$, where $\binom{p}{0} = 1$ and $\binom{p}{n} = \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}$.

**3.3 Example:** Analytic functions can be added, subtracted, multiplied, divided, composed, differentiated and integrated as if they were polynomials. For example, for $|x| < 1$ we have

$$\begin{aligned} \frac{1}{1+x^2} &= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots \\ \arctan x &= \sum_{n=-0}^{\infty} \tfrac{(-1)^n}{(2n+1)} x^{2n+1} = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots \end{aligned}$$

**3.4 Theorem:** *If the functions $r(x), p_0(x), p_1(x), \cdots, p_{n-1}(x)$ are all analytic at $a \in U$ and are all equal to the sum of their Taylor series for $|x-a| < R$ where $R > 0$, then for all $b_0, b_1, b_2, \cdots, b_{n-1} \in \mathbb{R}$, the unique solution $y = y(x)$ to the IVP given by*

$$\begin{aligned} &y^{(n)} = p_{n-1} y^{(n-1)} + \cdots + p_1 y' + p_0 = r \ \ , \textit{ with} \\ &y(a) = b_0, y'(a) = b_1, y''(a) = b_2, \cdots, y^{(n-1)}(a) = b_{n-1} \end{aligned}$$

*is also analytic at $a$ and equal to the sum of its Taylor series converges for $|x-a| < R$.*

Proof: We omit the proof.

**3.5 Exercise:** Solve the first order ODE $y' = 2y$ using power series (centred at 0).

**3.6 Exercise:** Find the Taylor polynomial of degree 5, centred at 0, for the solution to the IVP given by $y' + e^{2x}y = 3x$ with $y(0) = 1$.

**3.7 Exercise:** Use power series (centred at 0) to solve the ODE $(1+x^2)y''+3xy'+y = 0$. Find an explicit, closed form, formula for one solution. For an optional challenge, find an explicit, closed form, formula for two independent solutions.

**3.8 Exercise:** A number of differential equations, which are named after various mathematician's, involve a parameter $k \in \mathbb{R}$, and admit polynomial solutions when the parameter is a positive integer. Solve some of the following ODEs and determine the polynomial solutions (which are named after the same mathematician).

**Hermite' Equation:** $y'' - 2xy' + 2ky = 0$
**Chebyshev's Equation:** $(1 - x^2)y'' - xy' + k^2y = 0$
**Legendre's Equation:** $(1 - x^2)y'' - 2xy' + k(k + 1)y = 0$

## Frobenius' Method

**3.9 Exercise:** Solve the **Cauchy-Euler Equation**, which is given by $x^2y''+kxy'+\ell y = 0$ for $x > 0$, where $k, \ell \in \mathbb{R}$, by looking for a solution of the form $y(x) = x^r$ or, alternatively, by making the substitution $t = \ln x$.

**3.10 Definition:** For the second order homogeneous linear ODE $y''+p(x)y'+q(x)y = 0$, we say that the point $a \in \mathbb{R}$ is an **ordinary point** of the ODE when $p(x)$ and $q(x)$ are both analytic at $a$, and otherwise we say that $a$ is a **singular point** of the ODE. For a singular point $a \in \mathbb{R}$, we say that $a$ is a **regular singular point** of the ODE when $(x - a)p(x)$ and $(x - a)^2q(x)$ are both analytic at $a$, and otherwise we say that $a$ is an **irregular singular point** of the ODE.

**3.11 Theorem:** *(Frobenius) If $(x - a)p(x)$ and $(x - a)^2q(x)$ are both analytic at $a$, then the homogeneous linear ODE $y'' + p(x)y' + q(x) = 0$ has at least one solution of the form $y = y(x) = x^r f(x)$ for some $r \in \mathbb{R}$ and some function $f(x)$ which is analytic at $a$.*

Proof: We omit the proof.

**3.12 Note:** To solve an ODE $y''+p(x)y'+q(x) = 0$, as in the above theorem, we can try $y = x^r f(x)$ with $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$. This method is known as **Frobenius' method**.

**3.13 Exercise:** Use Frobenius' method to solve the ODE $2x^2y'' - xy' + (1 + x)y = 0$. Find explicit, closed form formulas for two independent solutions.

**3.14 Exercise:** Use Frobenius' method to solve the ODE $xy'' + 2y' + xy = 0$. Find explicit, closed form formulas for two independent solutions.

**3.15 Exercise:** Solve **Laguerre's Equation**, given by $xy''+(1-x)y'+ky = 0$ with $k\in\mathbb{R}$, and find the polynomial solutions when $k\in\mathbb{Z}^+$ (which are called Laguerre polynomials).

**3.16 Exercise:** Solve **Bessel's Equation** $x^2y'' + xy' + (x^2 - k^2) = 0$, where $k\in\mathbb{R}$.