Applications of Injective Envelopes

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<u>Idea</u>: Using inj. env. + algebraic tricks yields direct proofs of some previously "harder" theorems

- Bl-Ru-Si rep'n thm for abstract operator alg's via Bl-Ef-Za $\mathfrak{X}$-trick

- Chr-Sin rep'n thm for bilinear cb maps

- New approach to WEP via inj. env.

2

Thm 1 (CS + Pi) A, B unital C^*-alg's
$\varphi : A \times B \to B(\mathcal{H})$ c.b., then
$\exists\, \mathcal{K}$; $V, W : \mathcal{H} \to \mathcal{K}$, $\pi : A \to B(\mathcal{K})$
$\rho : B \to B(\mathcal{K})$ $*$-homo. $\ni$

$$\varphi(a,b) = V^* \pi(a) \rho(b) W$$

$$\|\varphi\|_{cb} = \|V\| \|W\|$$

Thm 2 (CES + Pi) The map

$$A \otimes_h B \longrightarrow A \underset{\mathbb{C}}{*} B$$

$$a \otimes b \to a * b$$

is a comp. isometry

Rem: i) Thm 1 proved first
Thm 2 later as a Corollary
ii) Thm 1 $\Longleftrightarrow$ Thm 2 and equivalenc easy
iii) I'll prove Thm 2

<u>Thm 3 (BRS)</u> A - alg. and an operator space, $1 \in A$, $\|1\| = 1$

$$\|(a_{ij})\cdot(b_{ij})\| = \|(\sum_k a_{ik} b_{kj})\| \le \|(a_{ij})\|\,\|(b_{ij})\|$$

<u>Then</u> $\exists\ \mathcal{H}$ and

$\pi : A \to B(\mathcal{H})$ comp. iso. homomorphism

Inj. Envelopes - Quick Review

1) $B(\mathcal{H})$ is injective (in lots of senses)

2) If $\phi : B(\mathcal{H}) \to B(\mathcal{H})$ comp. contr. $\phi \circ \phi = \phi$ then range(ϕ) inj.

3) If $V \subseteq B(\mathcal{H})$ and among all ϕ as in 2) <u>and</u> $\phi(v) = v\ \forall v \in V$

can pick "minimal" such projection

$$(\phi < \psi \iff \phi \circ \psi = \psi \circ \phi = \phi)$$

4) Ranges of all minimal are comp. isom. isomorphic, called $I(V)$ – "injective envelope"

5) $\phi : I(V) \to I(V)$, $\|\phi\|_{cb} \le 1$

$\phi(v) = v \ \forall v \in V \Rightarrow \phi = id_{I(V)}$

– rigidity

6) $\mathcal{S} \subseteq B(\mathcal{H})$, $\mathcal{S} = \mathcal{S}^*$, $1 \in \mathcal{S}$

then $\exists$ product $\circ$ $\ni$

$I(\mathcal{S})$ becomes a C^*-alg.

($\circ \neq$ product from $B(\mathcal{H})$, generally)

7) If $\underset{\hookrightarrow C^*\text{-alg}}{A} \subseteq \text{range}(\phi) \subseteq B(\mathcal{H})$

$\Rightarrow \phi(a_1 x a_2) = a_1 \phi(x) a_2$.

8) (Choi-Effros) $\exists$ abstract char. of op. systems.

5)

$V \subseteq B(\mathcal{H})$

$$\mathcal{S}_V = \left\{ \begin{pmatrix} \lambda \cdot I & v_1 \\ v_2^* & \mu \cdot I \end{pmatrix} : \lambda, \mu \in \mathbb{C},\ v_1, v_2 \in V \right\}$$

$$= \begin{pmatrix} \mathbb{C} & V \\ V^* & \mathbb{C} \end{pmatrix} \subseteq B(\mathcal{H} \oplus \mathcal{H})$$

<u>PROP'N</u> $I(\mathcal{S}_V) = \begin{pmatrix} I_{11} & I(V) \\ I(V)^* & I_{22} \end{pmatrix}$

where I_{11}, I_{22} inj. C^*-alg's

$I_{11} \circ I(V) \circ I_{22} \subseteq I(V)$

<u>Thm</u> **(τ-trick)** Let $\varphi : V \to V$

If $\tau : C_2(V) \to C_2(V)$, $\tau \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \varphi(v_1) \\ v_2 \end{pmatrix}$

is comp. contr., then

$\exists !\ b \in I_{11} \ni \varphi(v_1) = b \circ v_1$

(60)

pf: $\mathcal{B} = \left(\begin{array}{c:cc} I_{11} & I_{11} & I(V) \\ \hdashline I_{11} & I_{11} & I(V) \\ I(V)^* & I(V)^* & I_{22} \end{array}\right)$ - inj. C^*-alg

$\mathcal{S} = \begin{pmatrix} \mathbb{C} & 0 & V \\ 0 & \mathbb{C} & V \\ V^* & V^* & \mathbb{C} \end{pmatrix}$, $I(\mathcal{S}_V)$

$\Phi : \mathcal{S} \to \mathcal{S}$ $\quad \left(\begin{array}{c:cc} \lambda & 0 & v_1 \\ \hdashline 0 & \mu & v_2 \\ v_3^* & v_4^* & \alpha \end{array}\right) \to \left(\begin{array}{c:cc} \lambda & 0 & \varphi(v_1) \\ \hdashline 0 & \mu & v_2 \\ \varphi(v_3)^* & v_4^* & \alpha \end{array}\right)$

then Φ c.p. $\qquad \mathcal{S}_V$

extend to $\tilde{\Phi} : \mathcal{B} \to \mathcal{B}$ c.p.

$\Longrightarrow$

fixes $\mathcal{S}_V$ $\Longrightarrow$ $\tilde{\Phi}$ fixes $I(\mathcal{S}_V)$

$\Rightarrow \tilde{\Phi}$ $\left(\begin{array}{c:cc} \mathbb{C} & 0 & 0 \\ \hdashline 0 & \multicolumn{2}{c}{I(\mathcal{S}_V)} \\ 0 & & \end{array}\right)$ - bimodule map

$\Rightarrow \begin{pmatrix} 0 & 0 & \varphi(v) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \tilde{\Phi}\left(\begin{pmatrix} 0 & 0 & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) =$

$$= \tilde{\Phi}\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix}\right)$$

$$= \tilde{\Phi}\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore \ \varphi(v) = b \circ v$

<u>COR (BRS)</u>

<u>Pf</u>: $a \in A = V$, $\varphi = L_a$ get

$b_a \in I_{11} \ni L_a(a_2) = a \cdot a_2 = b \circ a_2$

<u>Check</u>: $A \hookrightarrow I_{11}$ $\quad a \hookrightarrow b$

comp. contr. homo into a C^*-alg.

8] Chr.-Sin. / C-E-S + Pi

$A \otimes_h B \hookrightarrow A \underset{\mathbb{C}}{*} B$ comp. isom.

<u>Thm:</u> $\mathcal{S}$ - abs. op. system

$1 \in \underset{C^*\text{-alg.}}{\mathcal{A}} \subseteq \mathcal{S}$, $\mathcal{S}$ $\mathcal{A}$-bimodule

<u>and</u> $(\cdot\cdot)$ $(a_{ij})^* M_n(\mathcal{S})^+ (a_{ij}) \subseteq M_m(\mathcal{S})^+$

$\forall (a_{ij}) \in M_{n,m}(\mathcal{A})$

then $\mathcal{A}$ is a C^*-subalg. of $(I(\mathcal{S}), \circ)$

<u>Rem:</u> Proof uses M. Walter's 3x3 completion trick:

u, v unitary, $\begin{pmatrix} 1 & u & x \\ u^* & 1 & v \\ x^* & v^* & 1 \end{pmatrix} \geq 0$

$\Longrightarrow x = u \cdot v$

Pa-Smith $\exists\ \mathcal{S}$ · abstract op. system

$$\mathcal{S} = \begin{pmatrix} A & A \otimes_h B \\ B \otimes_h A & B \end{pmatrix}$$

Pf: Let $\mathcal{Q} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ C^*-subalg.

$\mathcal{Q}$-bimodule, $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \breve{a} & \sum a_i \otimes b_i \\ \sum \tilde{b}_i \otimes \breve{a}_i & \tilde{\tilde{b}} \end{pmatrix}$

$$= \begin{pmatrix} a\breve{a} & \sum a a_i \otimes b_i \\ \sum b \tilde{b}_i \otimes \breve{a}_i & b\tilde{\tilde{b}} \end{pmatrix}$$

check condition $(\cdot\cdot)$ met

Represent $I(\mathcal{S})$ as a C^*-alg. on $B(\mathcal{H})$ with $\mathcal{Q}$ C^*-subalg

result pops out.

Inj. Env. and WEP

Def'n A C^*-alg has WEP

$\Longleftrightarrow$ $A \subseteq A^{**} \subseteq B(\mathcal{H})$ $\exists$ c.p.

$\varphi: B(\mathcal{H}) \to A^{**}$, $\varphi(a) = a$ $\forall a \in A$.

Thm: T.F.A.E. 1) A has WEP

2) $A \otimes_{max} C \hookrightarrow I(A) \otimes_{max} C$ isometry (or 1-1)

$\forall$ C C^*-alg.

3) $A \otimes_{max} C^*(\mathbb{F}_\infty) \hookrightarrow I(A) \otimes_{max} C^*(\mathbb{F}_\infty)$ 1-1

4) $\exists\, \mathcal{S}$, $A \subseteq \mathcal{S} \subseteq A^{**}$

$\mathcal{S} \cong I(A)$ compl. isom.

Basic Q: Every min'l A-proj

$\phi: B(\mathcal{H}) \to B(\mathcal{H})$ has range(ϕ) as a copy of $I(A)$. $\therefore$ many copies of $I(A)$, when $\exists$? a copy in A^{**}.

PROGRAM: "Catalog" copies of $I(A)$

Thm: $A \subseteq B(\mathcal{H})$ C^*-alg, $\{\mathcal{S}_\alpha\}$-all copies of $I(A)$

then $A \subseteq \bigcap_\alpha \mathcal{S}_\alpha = \{x \in B(\mathcal{H}) : \text{c.p.}$

$\varphi(x) = x$ whenever $\varphi(a) = a \ \forall a \in A\} \subseteq A''$

Ex: $A = C[0,1] \subseteq A'' = L^\infty[0,1] \subseteq B(L^2[0,1])$

then $\bigcap \mathcal{S}_\alpha = [R]_{a.e.}$, R: Riemann integrable.

PROP'N: 1) $\exists\, \phi : B(\mathcal{H}) \to B(\mathcal{H})$ min'l A-proj $\ni$ $\phi(A^{**}) = I(A)$

2) If A has WEP, then $\phi(A^{**}) = I(A)$ $\forall\, \phi : B(\mathcal{H}) \to I(A)$ c.p. fix A

Q: Converse?

PROP'N: 1) If $A \cap \mathcal{K}(\mathcal{H}) = (0)$ then $\exists\ \phi$ min'l A-proj $\ni$ $\phi(\mathcal{K}) = 0$. But in general $\exists\ \phi$ min'l A-proj, $\phi(\mathcal{K}) \neq (0)$.

2) If $A \cap \mathcal{K}(\mathcal{H}) = (0)$, A irred. then $\phi(\mathcal{K}) = 0$ $\forall$ min'l A-proj.