BANACH ALGEBRAS HOMEWORK, WINTER 2017
April 12, 2017

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1. Due January 10

1. Let Ω be a topological space. Prove that $C_0(\Omega)$ is a subalgebra of $C(\Omega)$ and that $(C_0(\Omega), \|\cdot\|_\infty)$ is a Banaach algebra.
2. Prove that $(C_b(\Omega), \|\cdot\|_\infty)$ is also a Banach algebra.

2. Due January 12

1. Let $(\mathcal{A}, \|\cdot\|)$ be a Banach algebra with unit. Set

$$|||a||| = sup\{||ab|| : \|b\| \leq 1\}.$$

Prove that $|||\cdot|||$ is an equivalent norm and that $(\mathcal{A}, |||\cdot|||)$ is a Banach algebra with a unit of norm 1, i.e., a UBA.

3. Due January 24

1. Let X be a compact topological space and let $C(X)$ denote the Banach algebra of continuous complex-valued functions on X with the supremum norm, $\|f\| = \sup\{|f(x)| : x \in X\}$. It is easy to see that setting $x \sim y \iff f(x) = f(y), \forall f \in C(X)$, defines an equivalence relation on X. Let $[x] = \{y \in X : y \sim x\}$ denote the equivalence class of x, let $[X]$ denote the set of equivalence classes and let $q : X \to [X]$ be defined by $q(x) = [X]$. Define a subset $U \subseteq [X]$ to be open iff $q^{-1}(U)$ is open in X and let $\mathcal{T}$ denote the collection of open subsets of $[X]$. Prove that:

- $\mathcal{T}$ is a topology on $[X]$ and that q is a continuous function.
- $([X], \mathcal{T})$ is a compact, Hausdorff topological space.
- for each $x \in X$, $[x] \subseteq X$ is a closed subset.
- for each $f \in C([X])$ set $q^*(f) = f \circ q$. Prove that $q^* : C([X]) \to C(X)$ is an isometric, isomorphism.

The space $[X]$ is sometimes referred to as the Hausdorffisation of X.

2. Let X and Y be compact, Hausdorff spaces, $C(X), C(Y)$ the Banach algebras of continuous complex-valued functions with the supremum norms and let $\pi : C(X) \to C(Y)$ be a unital algebra homomorphism. Prove that π is a bounded linear map, in fact, $\|\pi\| = 1$.

Remark: In finite dimensions all norms are equivalent, so there are many equivalent norms on the $n \times n$ complex matrices M_n that makes them into a

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unital Banach algebra. In particular, if we let X be an n dimensional Banach space, then as algebras, $B(X) = M_n$. You may use this identification, without proof below.

3. Let X be a finite dimensional Banach space. Prove that there does not exist a unital homomorphism from $B(X)$ to $\mathbb{C}$.

4. Due January 26

1. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Prove that there does not exist a unital, bounded homomorphism from the Banach algebra $B(\mathcal{H})$ to $\mathbb{C}$.

Remark: There is an infinite dimensional Banach space X for which there is a unital bounded homomorphism from $B(X)$ to $\mathbb{C}$. So the proof of the above requires you to use some results from your functional analysis course.

2. Prove that the group of invertibles in M_n is connected.

3. Prove that every invertible matrix is an exponential.

4. Let $\mathcal{A}$ be an UBA, $a \in GL(\mathcal{A})$ and U an open subset. Show that aU is open.

5. Due February 7

An element of a Banach algebra is called an **idempotent** if $p^2 = p$. Note that if p is an idempotent then so is $(1 - p)$.

A closed subspace V of a Banach space X is called **complemented** if there exists a closed subspace W so that $V \cap W = (0)$ and $V + W = X$, W is called a *complement* of V. Unlike Hilbert spaces not every closed subspace is complemented.

If $T \in B(X)$ then a closed subspace $V \subseteq X$ is said to be *invariant* for T if $T(V) \subseteq V$. In this case we let $T_V \in B(V)$ defined by $T_V(v) = T(v)$.

1. Let $\mathcal{A}$ be a Banach algebra and let $p \in \mathcal{A}$ be a non-zero idempotent. Prove that $\{pap : a \in \mathcal{A}\}$ is closed and is a Banach algebra with unit p(not necessarily of norm 1). We denote this Banach algebra by $\mathcal{A}_p$.

2. Let $\mathcal{A}$ be a UBA, p be an idempotent $p \neq 0$ and $p \neq 1$ and let $a \in \mathcal{A}$ be an element such that $0 = pa(1 - p) = (1 - p)ap$. Prove that $\sigma(a) = \sigma_{\mathcal{A}_p}(pap) \cup \sigma_{\mathcal{A}_{1-p}}((1 - p)a(1 - p))$, where the subscripts indicate spectrum with respect to those Banach algebras.

3. Let X be a Banach space and let V be a closed subspace. Prove that V is complemented if and only if there is an idempotent $P \in B(X)$ whose range is equal to V.

4. Let X be a Banach space, $T \in B(X)$, let $U_1, U_2 \subseteq \mathbb{C}$ be disjoint non-empty open sets such that $\sigma(T) \cap U_i = C_i$ is non-empty for $i = 1, 2$ AND $\sigma(T) = C_1 \cup C_2$. Prove that there is a subspace V with complement W such that both V and W are invariant for T with $\sigma(T_V) = C_1,\ \sigma(T_W) = C_2$.

6. Due February 16

1. Let X and Y be compact Hausdorff spaces. Let $\gamma : X \to Y$ be continuous and let $\gamma^* : C(Y) \to C(X)$ be the homomorphism induced by composition. Prove that γ is onto iff γ^* is one-to-one and that γ is one-to-one iff γ^* is onto. (You will need to recall some results from topology like Tietze's extension theorem.)

2. Let $\mathcal{A}$ be a CUBA. Prove that $x \in rad(\mathcal{A})$ iff $(1 + xy) \in GL(\mathcal{A})$ for every $y \in \mathcal{A}$.

3. Let X be a compact Hausdorff space, let $\mathcal{A}$ be a UBA and define $C(X : \mathcal{A}) = \{f : X \to \mathcal{A} | f \text{ is continuous }\}$. Show that $C(X; \mathcal{A})$ endowed with the sup norm and pointwise addition and multiplication of functions is a UBA.

4. Let X and Y be compact Hausdorff spaces. Prove that $C(X; C(Y))$ is isometrically isomorphic to $C(X \times Y)$.

5. Let $\mathcal{A}$ be a CUBA. We call $a \in A$ a *topological divisor of 0* if there exists $x_n \in \mathcal{A} \|x_n\| = 1, \forall n$, such that $\lim_n \|ax_n\| = 0$. Prove that if a is a topological divisor of 0, then $0 \in \sigma(a)$.

6. Let $\mathcal{A}$ be a CUBA. Prove that $a \in \mathcal{A}$ is a topological divisor of 0 iff for every CUBA $\mathcal{B}$ and every isometric, unital homomorphism $\pi : \mathcal{A} \to \mathcal{B}$, we have that $0 \in \sigma(\pi(a))$.

7. Due March 2

1. Let $\mathcal{A}$ be a C*-algebra that does not have a unit and let $\pi : \mathcal{A} \to B(\mathcal{A})$ be the map that sends $a \in \mathcal{A}$ to the linear operator $\pi(a) = L_a$ of left multiplication by a on the Banach space $\mathcal{A}$. Prove that π is an isometric homomorphism. Let I denote the identity operator on the Banach space $\mathcal{A}$. Prove that

$$\tilde{\mathcal{A}} = \{\pi(a) + \lambda I : \lambda \in \mathbb{C}\}$$

is a closed subalgebra of $B(\mathcal{A})$ and that if we define $\big(\pi(a)+\lambda I\big)^* = \pi(a^*)+\overline{\lambda} I$ then $\tilde{\mathcal{A}}$ endowed with the operator norm is a unital C*-algebra.

2. Let $\mathcal{A}$ be a non-unital C*-algebra and set $\mathcal{A}^+ = \tilde{\mathcal{A}}^+ \cap \mathcal{A}$. You may use that $\Lambda = \{p \in \mathcal{A}^+ : p \leq 1\}$ is a directed set under $\leq$. Prove that for every $a \in \mathcal{A}$, we have that $\lim_p \|a - ap\| = \lim_p \|a - pa\| = \lim_p \|a - pap\| = 0$.

Recall that a subset C of a vector space is **convex** if $a, b \in C \implies ta + (1-t)b \in C, \forall 0 \leq t \leq 1$. A point x in a convex set C is called **extreme** if $x = (a + b)/2$ with $a, b \in C$ implies that $x = a = b$.

3. Let $\mathcal{A}$ be a unital C*-algebra and let $B = \{a \in \mathcal{A} : \|a\| \leq 1\}$ which is convex. Prove that every unitary is an extreme point.

4. Let $S \in B(\ell^2_{\mathbb{N}})$ denote the unilateral shift, defined by $Se_n = e_{n+1}$. Prove that S is an extreme point of the unit ball that is not a unitary.

5. Let $\mathcal{A}$ be a unital C*-algebra and let $x \in \mathcal{A}$ with $\|x\| < 1$. Prove that

- the function $u(z) = (1 - xx^*)^{-1/2}(z1 + x)(1 + zx^*)^{-1}(1 - x^*x)^{1/2}$ is holomorphic in a neighborhood of the closed unit disk,

- $u(z)$ is unitary for $|z| = 1$,
- $x = \frac{1}{2\pi}\int_0^{2\pi} u(e^{it})dt$, where the integration is in the Riemann sense,
- deduce that the unit ball of $\mathcal{A}$ is the closed convex hull of its unitary elements.

8. Due March 14

These exercises look at the order on self-adjoint operators induced by the positive cone.

1. Prove that a 2×2 matrix $A = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$ is positive semidefinite iff $a \geq 0, c \geq 0$ and $ac - |b|^2 \geq 0$.

2. Consider the following 2×2 self-adjoint matrices: $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} 1.1 & 0.5 \\ 0.5 & 3.6 \end{pmatrix}, B_2 = \begin{pmatrix} 3.6 & 0.5 \\ 0.5 & 1.1 \end{pmatrix}$. Prove that:

- $A_i \leq B_j$ for $1 \leq i, j \leq 2$.
- there is no 2×2 matrix C so that $A_i \leq C \leq B_j$ for $1 \leq i, j \leq 2$.

3. Let $\mathcal{A}$ be a unital C*-algebra, $p, q \in \mathcal{A}^+ \cap GL(\mathcal{A})$. Prove that $p \leq q \iff q^{-1} \leq p^{-1}$.

4. Let $\mathcal{A}$ be a unital C*-algebra and let $0 \leq p \leq q$. Prove that $p^{1/2} \leq q^{1/2}$. [Hint: First do the case that q is invertible.]

5. Give an example of 2×2 matrices with $0 \leq P \leq Q$ but such that P^2 is NOT less than Q^2, i.e., $P^2 - Q^2$ is NOT positive semidefinite.

6. Let $\mathcal{A}$ be a unital C*-algebra and let $\Lambda = \{p \in \mathcal{A}^+ : ||p|| < 1\}$. Let $p, q \in \Lambda$. Prove that

$$r = 1 - \left(1 + (1-p)^{-1}p + (1-q)^{-1}q\right)^{-1} \in \Lambda$$

and that $r \geq p$, $r \geq q$. [Hint: $1 - (1+t)^{-1} = (1+t)^{-1}t$.]

9. Due March 23

These exercises assume that you are familiar with the *polar decomposition* of an operator, i.e., that every T can be written uniquely as $T = W|T|$ where W is a partial isometry from the closure of the range of $|T|$, $\mathcal{R}(|T|)^-$ to the closure of the range of T, $\mathcal{R}(T)^-$.

1. Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a VNA and let $T \in \mathcal{M}$. Prove that $|T|$ and W are in $\mathcal{M}$.

2. Let $\mathcal{M}$ be a VNA, $T \in \mathcal{M}$ and let E and F be the projections onto the closure of $\mathcal{R}(T)$ and $\mathcal{R}(T^*)$, respectively. Prove that $E, F \in \mathcal{M}$ and that $E \sim_{MVN} F$.

3. Let S be the unilateral shift on ℓ^2. Prove that $\{I, S, S^*\}'' = B(\ell^2)$.

10. Due April 4

1. Let X be a compact Hausdorff space, let $\pi : C(X) \to B(\mathcal{H})$ be a unital *-homomorphism and let $\tilde{\pi} : B^{\infty}(X) \to B(\mathcal{H})$ be its extension to the bounded Borel functions. Prove that the range of $\tilde{\pi}$ is contained in $\pi(C(X))''$.

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