Numerical Ranges and Spectral Sets

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- Numerical Range and Radius
- Crouzeix's Conjecture
- Drury's Result
- Some 70's Hits
- A new/old proof of Drury's result
- Generalizations

Notations

- $\mathcal{H}$-HIlbert space, $B(\mathcal{H})$-bounded operators on $\mathcal{H}$,
- $T \in B(\mathcal{H})$, $\|T\| = \sup\{\|Th\| : \|h\| = 1\}$,
- $W(T) = \{\langle Th, h\rangle : \|h\| \le 1\}^-$
- $w(T) = \sup\{|z|; z \in W(T)\}$
- $w(T) \le \|T\| \le 2w(T)$,
- $\mathcal{H}^n = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$(n copies),
- given a $n \times n$ matrix of polynomials $P = (p_{i,j})$ we set $P(T) = (p_{i,j}(T)) \in B(\mathcal{H}^n)$
- given a set $K \subseteq \mathbb{C}$ we set $\|P\|_K = \sup\{\|(p_{i,j}(z))\|_{B(\mathbb{C}^n)} : z \in K\}$

Crouzeix's Conjecture

Crouzeix: There exists a universal constant $2 \leq K_C \leq 11.03$, so that for $T \in B(\mathcal{H})$ and $W = W(T)$, and $P = (p_{i,j})$,

$$\|P(T)\| \leq K_C \|P\|_W.$$

Crouzeix's Conjecture: $K_C = 2$.
Rumor: Palencia has shown $2 \leq K_C \leq 1 + \sqrt{2}$.
Drury: What if we replace $\| \cdot \|$ in LHS by $w(\cdot)$?

Drury's Result

Berger-Stamplfi: Let $w(T) \leq 1$ and let $p(0) = 0$, then

$$w(p(T)) \leq \|p\|_{\mathbb{D}^-}.$$

Drury: Let $w(T) \leq 1$, then

$$w(p(T)) \leq \frac{5}{4}\|p\|_{\mathbb{D}^-}.$$

DPW: Give a new/old proof and extend to matrices of polynomials.

Two 70's Hits

Ando: $w(T) = \frac{1}{2}\min\{\|A+B\| : \begin{pmatrix} A & T \\ T^* & B \end{pmatrix} \geq 0\}$.

Ando-Okubo: If $w(T) \leq 1$, then there exists $\|C\| \leq 1$ such that $T = S^{-1}CS$ with $\|S\|\|S^{-1}\| \leq 2$.

DPW: If $T = S^{-1}CS$, with $\|C\| \leq 1$ and $\|S\|\|S^{-1}\| \leq r$, then

$$w(P(T)) \leq \frac{r + r^{-1}}{2}\|P\|_{\mathbb{D}^-}.$$

The Proof

WLOG $r^{-1/2}I \leq S \leq r^{1/2}I$ and $\|p\| = 1$.
By von Neumann's inequality $\|p(C)\| \leq 1$.
Hence, $\begin{pmatrix} I & p(C) \\ p(C)^* & I \end{pmatrix} \geq 0$.
So $\begin{pmatrix} S^{-2} & S^{-1}p(C)S \\ Sp(C)^*S^{-1} & S^2 \end{pmatrix} \geq 0$
Since $S^{-1}p(C)S = p(T) \implies w(p(T)) \leq \frac{1}{2}\|S^{-2} + S^2\| \leq \frac{r+r^{-1}}{2}$.
Proof for matrix-valued similar, uses matrix-valued version of von Neumann's inequality.

Recall that an operator T on $\mathcal{H}$is said to be of class $\mathcal{C}_\rho$ if there exists a unitary U on a larger Hilbert space such that

$$T^n = \rho P_{\mathcal{H}} U^n|_{\mathcal{H}}, \forall n \in \mathbb{N}.$$

DPW: Let T be of class $\mathcal{C}_\rho, \rho \geq 1$ then

$$w(P(T)) \leq \frac{\rho + \rho^{-1}}{2} \|P\|_{\mathbb{D}^-}.$$