Preservation of the joint essential matricial range

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Background and Motivation

Let $\mathbf{A} = (A_1, \ldots, A_m)$ be an m-tuple of elements of a unital C*-algebra $\mathcal{A}$ and let M_q denote the $q \times q$ matrices.
The **joint q-matricial range** $W^q(\mathbf{A})$ is the set of $(B_1, \ldots, B_m) \in M_q^m$ such that $B_j = \Phi(A_j)$ for some unital completely positive linear map $\Phi : \mathcal{A} \to M_q$.

When $\mathcal{A} = \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators on the Hilbert space $\mathcal{H}$, $\mathcal{K}(\mathcal{H})$ is the set of compact operators in $\mathcal{B}(\mathcal{H})$, and π is the canonical surjection from $\mathcal{B}(\mathcal{H})$ to the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, let $\pi(\mathbf{A}) = (\pi(A_1), \ldots, \pi(A_m))$.

Results of Narcowich-Ward and Smith-Ward show that for a single operator:

$$W^q(\pi(A)) = \cap\{W^q(A+K) : K \in \mathcal{K}(\mathcal{H})\}$$

and for each fixed N there exists $K_N \in \mathcal{K}(\mathcal{H})$ such that

$$W^q(\pi(A)) = W^q(A+K_N), \forall 1 \leq q \leq N.$$

We extend both of these results and some related results to m-tuples using some recent results of Kavruk about operator systems.

The still open Smith-Ward problem asks: Does there always exist $K \in \mathcal{K}(\mathcal{H})$ such that

$$W^q(\pi(A)) = W^q(A+K), \forall q?$$

In 1981 I showed that the 2-tuple version of this problem is false.

Results and Methods

Let $\mathbf{A} = (A_1, \ldots, A_m) \in \mathcal{B}(\mathcal{H})^m$, then the **joint spatial q-matricial range** $W_s^q(\mathbf{A})$ of $\mathbf{A}$ is the set of $(V^* A_1 V, \ldots, V^* A_m V) \in \mathbf{M}_q^m$ such that $V : \mathbb{C}^q \to \mathcal{H}$ is an isometry.
We define the **joint essential spatial q-matricial range** as

$$W_{ess}^q(\mathbf{A}) = \cap \{ W_s^q(A_1 + K_1, \ldots, A_m + K_m))^- : K_1, \ldots, K_m \text{ are compact } \}.$$

Theorem (LPP)

$W_{ess}^q(\mathbf{A}) = \cap \{ W^q(A_1 + K_1, \ldots, A_m + K_m)) : K_i \text{ compact } \} = W^q(\pi(\mathbf{A}))$.

The proof uses results of Bunce-Salinas, the connections between UCP and UCC maps, operator system methods and a lifting theorem of Kavruk.

The second result only uses operator system methods and the result of Kavruk, and we sketch this proof.

Let $\mathcal{S}_{\mathbf{A}}$ be the operator system spanned by $I, A_1, ..., A_m, A_1^*, ..., A_m^*$.

By Arveson's extension theorem:

$$W^q(\mathbf{A}) = \{(\Phi(A_1), \ldots, \Phi(A_m)) : \Phi : \mathcal{S}_{\mathbf{A}} \to M_q \text{ is ucp } \}.$$

Theorem (Kavruk)

Let $\mathcal{A}$ be a C-algebra, $\mathcal{I}$ an ideal in $\mathcal{A}$, $\mathcal{S}$ a finite dimensional operator system and $\psi : \mathcal{S} \to \mathcal{A}/\mathcal{I}$ a unital k-positive map. Then ψ has a unital k-positive lifting, $\Psi : \mathcal{S} \to \mathcal{A}$.*

In 1981, using an examples of Choi, I showed that the corresponding statement with "completely positive" replacing "k-positive" is not true.
"Naughty" and "nice" reversed!

Theorem (LPP)

Let $\mathbf{A} = (A_1, ..., A_m) \in \mathcal{B}(\mathcal{H})^m$. Then for each k there exist $(K_1, ..., K_m) \in \mathcal{K}(\mathcal{H})^m$ such that

$$W^q(\pi(\mathbf{A})) = W^q(A_1 + K_1, ..., A_m + K_m), \forall 1 \leq q \leq k.$$

Proof: Apply Kavruk's result to $\psi = id : \mathcal{S}_{\pi(\mathbf{A})} \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

Remarks on Kavruk's proof

B. Xhabli(2009) introduced and studied $OMAX_k(\mathcal{S})$ and $OMIN_k(\mathcal{S})$ of an operator system $\mathcal{S}$. They have the following universal properties:

$$\phi : \mathcal{T} \to \mathcal{S} \text{ is k-positive } \iff \phi : \mathcal{T} \to OMIN_k(\mathcal{S}) \text{ is cp },$$

and

$$\phi : \mathcal{T} \to \mathcal{S} \text{ is k-positive } \iff \phi : OMAX_k(\mathcal{T}) \to \mathcal{S} \text{ is cp }.$$

Also for finite dimensional operator systems, $OMIN_k(\mathcal{S})^d = OMAX_k(\mathcal{S}^d)$, $OMAX_k(\mathcal{S})^d = OMIN_k(\mathcal{S})^d$, where d denotes "operator system dual", i.e. a dual in the ordered sense.

Kavruk proves that for finite dimensional operator systems, the dual of a 1-exact operator system has the lifting property for cp maps(LP). This result uses the tensor product characterizations of these properties found in KPTT:
1-exact iff (min, el)-nuclear,
LP iff $\mathcal{S} \otimes_{min} \mathcal{B}(\mathcal{H}) = \mathcal{S} \otimes_{max} \mathcal{B}(\mathcal{H}), \forall \mathcal{H}$.
Kavruk then proves that $OMIN_k(\mathcal{S})$ is 1-exact.
Hence, $OMAX_k(\mathcal{S})$ has LP from which the k-positive lifting theorem follows.
The Smith-Ward problem is equivalent to asking if every 3 dimensional operator system has the lifting property.

Theorem (Kavruk)

The Smith-Ward problem is equivalent to the condition that every 3-dimensional operator system is both 1-exact and has the LP.

Conjecture: The Smith-Ward problem and Connes' embedding problem cannot both be true.

Thanks!