Perfect Embezzlement, Connes and Tsirelson

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Based on:
Perfect Embezzlement of Entanglement(R. Cleve, L. Liu, V. Paulsen)
A non-commutative unitary analogue of Kirchberg's conjecture(S. Harris)
Unitary correlation sets(S. Harris, V. Paulsen)

Outline

- Overview: Two Models for Joint Quantum Experiments
- Van Dam and Hayden Approximate Embezzlement
- Impossibility of Perfect Embezzlement in Tensor Framework
- Commuting Framework
- The C*-algebra of Non-commuting Unitaries
- Perfect Embezzlement
- New Versions of Tsirelson, Connes, and Kirchberg
- The Coherent Embezzlement Game

Overview

In the math model for QM, a quantum system A is affiliated with a Hilbert space $\mathcal{H}_A$ whose unit vectors represent the possible *states* of the quantum system and that measurements are represented by operators on this space.

Given two separate quantum systems A and B, with state spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, the usual axioms say that their joint state space is given by $\mathcal{H}_A \otimes \mathcal{H}_B$ and that if A has a measurement operator P and B has Q then their joint measurement is represented by $P \otimes Q$. This is called the *tensor product* model.

A second axiomatic model for how two separate quantum systems behave assumes that there is just a single Hilbert space $\mathcal{H}$ and that the measurement operators for A and B must commute. This is called the *commuting* model.

We will show that *catalytic production of entanglement* is possible in the commuting model but not in the tensor product model.... and that this can be related to Connes embedding conjecture.

Tensor products and Schmidt coefficients

Given $\mathcal{H}_A$ with onb $\{e_s : s \in S\}$ and $\mathcal{H}_B$ with onb $\{f_t : t \in T\}$, then $\mathcal{H}_A \otimes \mathcal{H}_B$ has onb $\{e_s \otimes f_t : (s,t) \in S \times T\}$. So every vector $x = \sum_{s,t} x_{s,t} e_s \otimes f_t$. Writing the matrix $(x_{s,t}) = UDV$ in its singular value decomposition leads to

$$x = \sum_i \alpha_i u_i \otimes v_i, \; \{u_i\}, \{v_i\}, orthonormal,$$

with $\alpha_i \geq 0$, $\alpha_i \geq \alpha_{i+1}$ and $\sum_i \alpha_i^2 = \|x\|^2$. The α_i's are called the *Schmidt coefficients* of x.

Note: Independent of the particular bases, uniquely determined by x.

x is called *separable* iff $x = h \otimes k$ iff $(x_{s,t})$ rank one.

x is called *entangled* if it is not separable.

Note that if $x \in \mathcal{H}_A \otimes \mathcal{H}_B$ has Schmidt coefficients as above and we have unitaries $U_A \in B(\mathcal{H}_A)$ and $U_B \in B(\mathcal{H}_B)$, then

$$(U_A \otimes U_B)x = \sum_i \alpha_i (U_a u_i) \otimes (U_B v_i),$$

and so has the same Schmidt coefficients.
Thus we have *preservation of Schmidt coefficients by local unitaries*.

The entangled vector $\mathcal{B} = \frac{1}{\sqrt{2}}(e_0 \otimes e_0 + e_1 \otimes e_1) \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is called the *Bell state*. Suppose that $\psi \in \mathcal{R}_A \otimes \mathcal{R}_B$ is a unit vector with Schmidt coefficients as above. Can we find unitaries $U_A \in B(\mathbb{C}^2 \otimes \mathcal{R}_A)$ and $U_B \in B(\mathcal{R}_B \otimes \mathbb{C}^2)$ such that

$$(U_A \otimes U_B)(e_0 \otimes \psi \otimes e_0) = \frac{1}{\sqrt{2}}\big(e_0 \otimes \psi \otimes e_0 + e_1 \otimes \psi \otimes e_1\big) \sim \mathcal{B} \otimes \psi?$$

NO! Because the Schmidt coefficients of the image vector are

$$\frac{\alpha_1}{\sqrt{2}}, \frac{\alpha_1}{\sqrt{2}}, \frac{\alpha_2}{\sqrt{2}}, ...$$

violates preservation of Schmidt coefficients.
This is summarized by saying that *catalytic production of a Bell state is impossible with local unitaries*.

Embezzlement of A Bell State

Van Dam and Hayden showed that, in a certain sense, one can *appear* to produce entanglement by local methods. They called this *embezzlement*.

They showed that given any $\epsilon > 0$ there are finite dimensional Hilbert spaces, unit vectors $\psi, \psi_\epsilon \in \mathcal{R}_A \otimes \mathcal{R}_B$ and unitaries, U_A on $\mathbb{C}^2 \otimes \mathcal{R}_A$ and U_B on $\mathcal{H}_B \otimes \mathbb{C}^2$ such that,

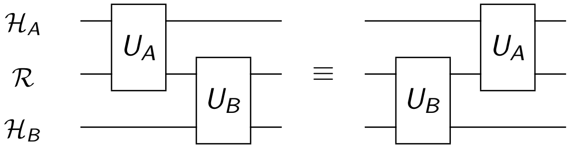
$$(U_A \otimes U_B)(e_0 \otimes \psi \otimes e_0) = \frac{1}{\sqrt{2}}(e_0 \otimes \psi_\epsilon \otimes e_0 + e_1 \otimes \psi_\epsilon \otimes e_1),$$

with $\|\psi - \psi_\epsilon\| < \epsilon$

Van Dam and Hayden even gave lower bounds on the dimensions of the spaces as a function of ϵ, which they showed tend to $+\infty$. This left people puzzled about what does happen in the limit.

The Commuting Operator Framework

We no longer require that the resource space have a bipartite structure.
Instead, we only ask for a resource space $\mathcal{R}$, and unitaries, U_A on $\mathbb{C}^2 \otimes \mathcal{R}$ and U_B on $\mathcal{R} \otimes \mathbb{C}^2$ such that $(U_A \otimes id_2)$ commutes with $(id_2 \otimes U_B)$ on $\mathbb{C}^2 \otimes \mathcal{R} \otimes \mathbb{C}^2$.



Given a commuting operator framework, we say that $\psi \in \mathcal{R}$ *is a catalyst vector for perfect embezzlement of a Bell state in a commuting operator framework* provided that

$$(U_A \otimes id_B)(id_A \otimes U_B)(e_0 \otimes \psi \otimes e_0) = \frac{1}{\sqrt{2}}\big(e_0 \otimes \psi \otimes e_0 + e_1 \otimes \psi \otimes e_1\big).$$

Theorem (CLP)

Perfect embezzlement of a Bell state is possible in a commuting operator framework.

In the rest of this talk, I want to outline the proof and show why the fact that perfect embezzlement is possible in this commuting framework but not possible in a tensor product framework is closely related to the Tsirelson conjectures and to Connes' embedding conjecture.

Suppose that $\mathcal{H}_A = \mathbb{C}^n$ and identify $\mathbb{C}^n \otimes \mathcal{R} = \mathcal{R} \oplus \cdots \oplus \mathcal{R}$(n times). Using this identification, we write $U_A = (U_{i,j})$ where $U_{i,j} \in B(\mathcal{R})$, $0 \le i, j \le n-1$. Similarly, if $\mathcal{H}_B = \mathbb{C}^m$, then we may identify $U_B = (V_{k,l})$ where $V_{k,l} \in B(\mathcal{R})$, $0 \le k, l \le m-1$.

Lemma

$(U_A \otimes id_B)$ commutes with $(id_A \otimes U_B)$ if and only if $U_{i,j}V_{k,l} = V_{k,l}U_{i,j}$ and $U^*_{i,j}V_{k,l} = V_{k,l}U^*_{i,j}$ for all i, j, k, l.

This last condition is called **-commuting*.

Thus, we see that having commuting operator frameworks as above is exactly the same as having operator matrices $U_A = (U_{i,j})$ and $U_B = (V_{k,l})$ that yield unitaries and whose entries pairwise *-commute.

The C*-algebra $U_{nc}(n)$

L. Brown introduced a C*-algebra denoted $U_{nc}(n)$. It has n^2 generators $u_{i,j}$ and the "universal" property that whenever there are n^2 operators $U_{i,j}$ on a Hilbert space $\mathcal{R}$ such that $(U_{i,j})$ defines a unitary operator on $\mathbb{C}^n \otimes \mathcal{R}$ then there is a *-homomorphism $\pi : U_{nc}(n) \to B(\mathcal{R})$ with $\pi(u_{i,j}) = U_{i,j}$.
Thus, a representation of $U_{nc}(n) \otimes_{max} U_{nc}(m)$ corresponds to operators $U_{i,j}, V_{k,l}$ where the $U_{i,j}$'s *-commute with the $V_{k,l}$'s such that $(U_{i,j})$ and $(V_{k,l})$ are unitary operator matrices.

Theorem (CLP)

Perfect embezzlement of a Bell state is possible in a commuting operator framework if and only if there is a state s on $U_{nc}(2) \otimes_{max} U_{nc}(2)$ *satisfying* $s(u_{00} \otimes v_{00}) = s(u_{10} \otimes v_{10}) = 1/\sqrt{2}$ *and* $s(u_{00} \otimes v_{10}) = s(u_{10} \otimes v_{00}) = 0$.

To prove, take the GNS representation of any such state then $\psi = [1]$ is a catalyst vector for perfect embezzlement of a state in a commuting operator framework.

Corollary

The van Dam–Hayden approximate embezzlement results imply that there exists a state on $U_{nc}(2) \otimes_{min} U_{nc}(2)$ *satisfying the above equations. Hence, the conditions of the above result are met and perfect embezzlement of a state is possible in a commuting operator framework.*

The representation of $U_{nc}(2) \otimes_{min} U_{nc}(2)$ given by the Corollary can not decompose as a spatial tensor product of a representation of each factor or else we would contradict the fact that perfect embezzlement is impossible in a tensor product framework!
We now want to draw an analogy with quantum correlation matrices.

Suppose that Alice and Bob each have n quantum experiments and each experiment has m outcomes. We let $p(a,b|x,y)$ denote the conditional probability that Alice gets outcome a and Bob gets outcome b given that they perform experiments x and y respectively. There are several possible models for describing the set of all such tuples.

One model is that Alice and Bob have finite dimensional state spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. For each experiment x, Alice has projections $\{E_{x,a}, 1 \le a \le m\}$ such that $\sum_a E_{x,a} = I_A$. Similarly, for each y, Bob has projections $\{F_{y,b} : 1 \le b \le m\}$ such that $\sum_b F_{y,b} = I_B$. They share an entangled state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ and

$$p(a,b|x,y) = \langle \psi | E_{x,a} \otimes F_{y,b} | \psi \rangle.$$

We let $C_q(n,m) = \{p(a,b|x,y) :$ obtained as above $\} \subseteq \mathbb{R}^{n^2m^2}$.
We let $C_{qs}(n,m)$ denote the possibly larger set that we could obtain if we allowed the spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ to also be infinite dimensional.
We let $C_{qc}(n,m)$ denote the possibly larger set that we could obtain if instead of requiring the common state space to be a tensor product, we just required one common state space, and demanded that $E_{x,a}F_{y,b} = F_{y,b}E_{x,a}$ for all a,b,x,y, i.e., a commuting model.
Tsirelson was the first to examine these sets in the 1990's and study the relations between them. In fact, he wondered if they could all be equal. The equalities of various pairs are called the Tsirelson conjectures
Here are some of the things that we know/don't know about these sets.

- $C_q(n,m) \subseteq C_{qs}(n,m) \subset C_{qc}(n,m)$ and this larger set is closed.
- $C_q(n,m)^- = C_{qs}(n,m)^-$ and this can be identified with the states on a minimal tensor product.
- (JNPPSW + Ozawa, 2013)$C_q(n,m)^- = C_{qc}(n,m), \forall n,m$ iff Connes' Embedding conjecture has an affirmative answer.
- (Slofstra, April 2016) there exists an n,m(very large) such that $C_{qs}(n,m) \neq C_{qc}(n,m)$.
- (Slofstra, March, 2017) proved that the sets $C_q(n,m)$ and $C_{qs}(n,m)$ are not closed, for n,m sufficiently large.

Sam Harris's Results

Theorem (Harris)

The following are equivalent.

1. *Connes' Embedding conjecture is true.*
2. $U_{nc}(n) \otimes_{min} U_{nc}(m) = U_{nc}(n) \otimes_{max} U_{nc}(m), \forall n, m.$
3. $U_{nc}(2) \otimes_{min} U_{nc}(2) = U_{nc}(2) \otimes_{max} U_{nc}(2).$
4. *Certain unitary correlation sets satisfy* $UC_q(n, m)^- = UC_{qc}(n, m), \forall n, m.$

The equivalence of the first three, is the analogue of Kirchberg's theorem relating Connes to tensor products of free group C*-algebras. The equivalence of the first and last is the analogue of the result of [Junge $\cdots$ Ozawa].

Reduced Unitary Correlation Sets

We set

$$B_q(n,m) = \{\langle U_{ij} \otimes V_{kl}\psi, \psi\rangle : (U_{i,j}), (V_{k,l}) \text{ are unitary}, \\ U_{i,j} \in M_p, V_{k,l} \in M_q, \exists p, q, ||\psi|| = 1\} \\ \subset M_n \otimes M_m.$$

For the set $B_{qs}(n,m)$ we drop the requirement that each $U_{i,j}$ and $V_{k,l}$ act on finite dimensional spaces.
For the set $B_{qc}(n,m)$ we replace the tensor product of two spaces by a single space and instead demand that the $U_{i,j}$'s *-commute with the $V_{k,l}$'s.
Here are some of the things that we know/don't know about these sets.

Theorem (Harris-P)

- $B_q(n,m) \subseteq B_{qs}(n,m) \subseteq B_{qc}(n,m)$.
- *For each* n, m, $B_q(n,m)$ *and* $B_{qs}(n,m)$ *are not closed–consequence of embezzlement theory*
- $B_{qc}(n,m) = \{ \big(s(u_{ij} \otimes v_{kl}) \big) | s : U_{nc}(n) \otimes_{max} U_{nc}(m) \to \mathbb{C} \text{ is a state } \}$.
- $B_q(n,m)^- = B_{qs}(n,m)^- = \{ \big(s(u_{ij} \otimes v_{kl}) \big) | s : U_{nc}(n) \otimes_{min} U_{nc}(m) \to \mathbb{C} \text{ is a state } \}$.
- $B_{qs}(n,m) \neq B_{qc}(n,m), \forall n, m \geq 2$.
- $B_q(n,m)^- = B_{qc}(n,m), \ \forall n, m \iff$ *Connes Embedding is true.*

- The sets $B_q(n,m)^-$ and $B_{qc}(n,m)$ are the closed unit balls of norms on $M_n \otimes M_m$ that are reasonable cross norms in the sense of Grothendieck.
- CEC is true iff these tensor norms are equal for all n, m iff they are equal for all $n = m$.
- Both of these tensor norms are larger than the operator norm on $M_n \otimes M_m$.
- If $\psi = \sum_{i,k} x_{i,k} e_i \otimes e_k \in \mathbb{C}^n \otimes \mathbb{C}^n$ is a unit vector with Schmidt rank n, then the matrix $(a_{i,j,k,l})$ given by $a_{i,0,j,0} = x_{i,j}$, and $a_{i,j,k,l} = 0$ for all other indices is an extreme point of both $B_q(n,n)^-$ and $B_{qc}(n,m)$.
- If $(a_{i,j,k,l}) \in B_{qc}(n,n)$ with $a_{i,0,k,0} = x_{i,k}$ then necessarily $a_{i,j,k,l} = 0, \forall (j,l) \neq (0,0)$.

What are the extreme points of $B_{qc}(n,n)$?

Thanks!