A SURVEY OF FUNCTIONAL ANALYSIS METHODS FOR QIT

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This course will survey the key concepts and results from functional analysis that need to be in the tool kit of researchers in quantum information theory. Concepts will be developed but proofs will only be presented as needed.

Graduate students in Pure Mathematics with interests in functional analysis are encouraged to take Functional analysis PMATH 753 and Banach algebras PMATH 810.

Topics to be covered include:

- Review of metric spaces
  - Definition of metric and examples
  - Convergence of sequences
  - $\epsilon - \delta$ definition of Continuity
  - Open, closed and compact sets
  - Sequential characterizations
  - Connected and pathwise connected sets
  - Equivalence and Uniform equivalence of metrics
  - Cauchy sequences and completeness
  - $C(\mathbb{R})$ as a metric space
  - Baire's theorem
- General topological spaces and nets
  - Open and closed sets, continuity
  - Nets and directed sets
  - Characterizations of closed sets, compact sets, and continuity in terms of nets
- Normed spaces
  - ℓ^p spaces, Holder and Minkowski inequalities, Riesz-Fischer
  - The space $C(X)$
  - Quotients and conditions for completeness, the 2/3's theorem
  - Finite dimensional normed spaces, equivalence of norms
  - Convexity, Absolute convexity, the bipolar theorem
  - Consequences of Baire's theorem:
    - Principle of Uniform boundedness, Resonance principle
    - Open mapping, closed graph and bounded inverse theorems
  - Hahn-Banach theorem
  - Krein-Milman theorem
  - Dual spaces and adjoints

The double dual
Weak topologies, weak convergences

- Hilbert spaces
  - Cauchy-Schwarz inequality
  - Polarization identity. Parallelogram Law
  - Jordan-vonNeumann theorem
  - Orthonormal bases and Parseval identities
  - Direct sums
  - Bilinear maps and Tensor products of Banach and Hilbert spaces
  - Infinite tensor products and quantum spin chains
- Operators on Hilbert spaces
  - Special families of operators: adjoints, projections, Hermitian, unitaries, partial isometries, polar decomposition
  - Density matrices and trace class operators
  - B(H) as dual of trace class
- Spectral Theory
  - Spectrum and resolvent
  - Spectrum versus point spectrum; approximate points spectrum
  - Neumann series
  - $\sigma(T)$ is non-empty and compact
  - Unbounded operators and self-adjointness problems
- The Riesz functional calculus
  - The functional calculus for normal operators
- Compact operators
  - Singular values and Schmidt's theorem
  - Schmidt decomposition of tensors
  - The Schatten classes
- Unbounded operators
  - Hellinger-Toeplitz theorem
  - Closable and non-closable operators
  - Self-adjointness problems
  - Stone's theorem
- Von Neumann algebras
  - Measurement operators and observables
  - The Double Commutant theorem
  - Theory of types
  - Connes' embedding conjecture and the Tsirelson problems
- C*-algebras
  - Key examples
  - GNS theorem
- Completely Positive and Completely Bounded Maps
  - Matrix norm and matrix order
  - Stinespring's theorem and Choi-Kraus
  - Arveson's Hahn-Banach and Radon-Nikodym theorems
  - Wittstock's extension and decomposition theorems