Quantum Chromatic Numbers

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Finite Input-Output Games

These are games where two "cooperating" players compete against a third, called the Referee.

For each "round" of the game, the cooperating players, Alice and Bob, each receives an input from the Referee from some finite set of inputs I.

They must each produce an output belonging to some finite set O.

The game $\mathcal{G}$ has "rules" given by a function

$$\lambda : I \times I \times O \times O \to \{0, 1\}$$

where $\lambda(x, y, a, b) = 1$ means that if Alice and Bob receive inputs x, y, respectively and produce respective outputs a, b, then they win. If $\lambda(x, y, a, b) = 0$, they lose.

They both know the rule function, but Alice and Bob must produce their outputs without knowing what input the other received and, without knowing what output the other produced. This is what is meant by "non-communicating".

Usually, many rounds are played and we try to compute the probability that Alice and Bob will win, assuming that the Referee uses each input an equal number of times.

Alternatively, we can ask if there is some "strategy" so that they always win.

The game is called **synchronous** if whenever they receive the same input, they must produce the same output, i.e.,

$\lambda(x, x, a, b) = 0$, $\forall a \neq b$.

The Graph Colouring Game

Given a graph $G = (V, E)$ with vertex set V and edges $E \subset V \times V$, a *c-colouring* of G is a function $f : V \to \{1, ..., c\}$ such that $(v, w) \in E \implies f(v) \neq f(w)$.

The least c for which a graph colouring exists is denoted $\chi(G)$.

In the *graph c-colouring game*, the inputs are $I = V$ and the outputs are $O = \{1, ..., c\}$. The "disallowed" are that

$\lambda(v, w, a, a) = 0, \forall (v, w) \in E, \forall a \in O,$

$\lambda(v, v, a, b) = 0, \forall v \in V, \forall a \neq b.$

Note that without the synchronous rule, a strategy that always won would be for A to answer "red" and B to answer "green".

There is no rule that says when A or B receives the same vertex at a later round, that they must use the same colour they used earlier. Hence even if Alice and Bob always give "allowed" answers, it is not clear if we could construct an actual coloring of the graph from their outputs.

Thus it is natural to wonder: for some $c < \chi(G)$, can Alice and Bob have a "random" strategy that wins the graph c-colouring game with probability 1?

The answer is "no" if they use "classical" random variables, but "yes" if they are allowed to use quantum experiments to generate their random answers!

Hadamard Graphs

Given N the Hadamard graph, Ω_N is the graph whose vertices are the N-tuples of ± 1's(so 2^N vertices) and two N-tuples x, y form an edge if and only if $x \cdot y = 0$.
So when N is odd, no vertices are connected. Hence, we'll always assume that N is even.
Frankl-Rodl proved that there is $r, 1 < r < 2$ such that for all even $N > N_0$ we have that $r^N \leq \chi(\Omega_N) \leq 2^N$.
But AHKS proved that $\chi_q(\Omega_N) \leq N$, for all N even. That is, using quantum experiments, one could win the graph coloring game for Ω_N with probability 1, using only N colours.

Strategies and Conditional Probabilities

A “random strategy” for a game generates conditional probabilities where $p(a, b|x, y)$ represents the probability that if A receives input x and B receives input y, then they produce outputs a and b, respectively.

A strategy is called **winning** or **perfect** provided:

$$\lambda(x, y, a, b) = 0 \implies p(a, b|x, y) = 0.$$

The tuple $p(a, b|x, y)$ is called a **local** or **classical correlation** if there is a probability space (Ω, μ) and for each pair of inputs x, y random variables,

$$f_x, g_y : \Omega \to O,$$

such that

$$p(a, b|x, y) = \mu(f_x = a, g_y = b).$$

A tuple $p(a,b|x,y)$ is called a **quantum correlation** if it arises as follows:
Alice and Bob have finite dimensional Hilbert spaces H_A, H_B and for each input x Alice has projections with $\sum_{a \in O} E_{x,a} = I_{H_A}$, i.e., a $|O|$-outcome quantum experiment and for each input y Bob has projections $\sum_{b \in O} F_{y,b} = I_{H_B}$ and they share a state $\psi \in H_A \otimes H_B$. In this case

$$p(a,b|x,y) := \langle E_{x,a} \otimes F_{y,b} \psi, \psi \rangle = 0$$

is the probability of getting outcomes a, b given that they conduct quantum experiments x, y.

We call $p(a,b|x,y)$ **quantum commuting correlation** if there is a single Hilbert space H, and for each input x Alice has projections $\sum_{a\in O} E_{x,a} = I_H$, for each input y Bob has projections $\sum_{b\in O} F_{y,b} = I_H$, $E_{x,a}F_{y,b} = F_{y,b}E_{x,a}$, $\forall x,y,a,b$ they share a state $\psi \in H$ and

$$p(a,b|x,y) = \langle E_{x,a}F_{y,b}\psi, \psi\rangle.$$

When $|I| = n$ and $|O| = m$, we let $LOC(n, m) = C_{loc}(n, m)$ denote the set of all n^2m^2-tuples $p(a, b|x, y)$ of local outcomes.
We let $C_q(n, m)$ denote the set of all $p(a, b|x, y)$ that are quantum and $C_{qc}(n, m)$ denote the set of all those obtained from a quantum commuting.
We call $p(a, b|x, y)$ **synchronous** if $p(a, b|x, x) = 0, \ \forall x, \ \forall a \neq b$ and let C^s denote the synchronous elements of each of these sets.

The Conjectures of Connes and Tsirelson

Here are some of the things known and conjectured about these correlations:

- $C_{loc}(n,m) \subseteq C_q(n,m) \subseteq C_{qc}(n,m)$.
- $C_{loc}(n,m)$ and $C_{qc}(n,m)$ are closed.
- Strong Tsirelson conjecture: $C_q(n,m) = C_{qc}(n,m), \forall n,m$.
- JNPPSW, Ozawa: Connes' embedding conjecture is true iff $C_q(n,m)^- = C_{qc}(n,m), \forall n,m$.
- Dykema-P: Connes' embedding conjecture is true iff $C_q^s(n,m)^- = C_{qc}^s(n,m), \forall n,m$.
- "Bounded Entanglement Conjecture": $C_q(n,m)$ is closed $\forall n,m$.

P-Todorov: Can we distinguish these sets of correlations by studying existence/non-existence of perfect strategies for games?

PSSTW: Showed $C_{qc}(15,7) \subsetneq C_{vect}(15,7)$ by showing that there is a graph on 15 vertices with no perfect qc-strategy for 7 colouring, but a perfect vect-strategy for 7 colouring.

W. Slofstra: There is a BCS game with a perfect qc-strategy but no perfect q-strategy. Hence the Strong Tsirelson conjecture is false.

The Quantum Chromatic Numbers

Set $C_{qa}(n,m) = C_q(n,m)^-$.Given a graph G and for $t \in \{loc, q, qa, qc\}$ we define the **quantum chromatic number** by

$$\chi_t(G) = min\{c : \exists p(a,b|x,y) \in C_t(n,c),$$
$$(x,y) \in E(G) \implies p(a,a|x,y) = 0,$$
$$p(a,b|x,x) = 0, \forall a \neq b\}.$$

So $\chi_{loc}(G) \geq \chi_q(G) \geq \chi_{qa}(G) \geq \chi_{qc}(G)$, and if the corresponding sets are equal then these integers will need to be equal.
Key Problem: Can we find graphs that separate, χ_q, χ_{qa} and χ_{qc} ?
How to compute these colouring numbers?

Some Known Results

- $\chi_{loc}(G) = \chi(G)$.
- $\chi(G) \leq 3$ is NP-complete,
- Z. Ji: $\chi_q(G) \leq 3$ is NP-hard, no known algorithm to compute,
- PSSTW: For each fixed $|G| = n$ and m, $\exists$ a semidefinite program(SDP) that decides if $\chi_{qc}(G) \leq m$
- P-Todorov: If $V(G) \subseteq \mathbb{T}^N$ with $(v, w) \in E(G) \iff v \perp w$, then $\chi_q(G) \leq N$.

Note $|V(G)|$ can be arbitrarily large!

Fractional Quantum Chromatic Number

In classical graph theory, the "fractional chromatic number" is used to give a lower bound on the chromatic number. In PSSTW we found a game theoretic description of this value, which allowed us to generalize the inequality to other chromatic numbers.
Given $G = (V, E)$ with $|V| = n$, consider a game where $I = V$, $O = \{R, G\}$ and rules:

$\lambda(v, v, a, b) = 0$ when $a \neq b$ and $\lambda(v, w, R, R) = 0$ when $(v, w) \in E$.

So the referee only cares if you colour two adjacent vertices both Red.
So we could win by always using Green, but we would like to maximize the probability that we use Red.
Given p, let $p_A(R|v) = \sum_b p(R, b|v, w)$ and $p_B(R|w) = \sum_a p(a, R|v, w)$.

For $t \in \{loc, q, qa, qc\}$ let

$$\xi_t(G)^{-1} = sup\{min_{v,w}\{p_A(R|v), p_B(R|w)\} : p \in C_t(n,2),\ \text{p perfect}\}.$$

Theorem (PSSTW)

- *$\xi_{loc}(G)$ is the classical fractional chromatic number,*
- *$\xi_q(G)$ is Roberson's quantum fractional chromatic number,*
- *$\xi_t(G) \leq \chi_t(G)$ for $t = loc, q, qa, qc$,*
- *$\xi_t(G \boxtimes H) = \xi_t(G)\xi_t(H)$.*

Given a graph G and c, we let $\mathcal{A}(G, K_c)$ denote the "universal" unital *-algebra generated by projections $\{e_{v,i} : v \in V(G), 1 \le i \le c\}$ satisfying:

- $e_{v,i}e_{v,j} = 0, \ \forall i \neq j,$
- $\sum_i e_{v,i} = I, \ \forall v,$
- $(v, w) \in E(G) \implies e_{v,i}e_{w,i} = 0, \ \forall i.$

This *-algebra might be trivial, i.e., 1 might be in the ideal generated by the relations. In which case we say $\mathcal{A}(G, K_c)$ is *trivial*. We do not know NASC on G and c for $\mathcal{A}(G, K_c)$ to be non-trivial or the complexity level of this problem.
We also do not know NASC on G and c for there to exist a family of projections on a Hilbert space satisfying these conditions or the complexity level of this problem.

Theorem (Ortiz-P)

Let G be a graph.

- $\chi(G) \leq m \iff \mathcal{A}(G, K_m)$ *has a 1-dimensional representation,*
- $\chi_q(G) \leq m \iff \mathcal{A}(G, K_m)$ *has a finite dimensional representation,*
- $\chi_{qc}(G) \leq m \iff \mathcal{A}(G, K_m)$ *has a tracial state.*

Similar theory for any synchronous game $\mathcal{G}$, i.e., $\exists$ an algebra of the game $\mathcal{A}(\mathcal{G})$.

Corollary (Ortiz-P)

Let G be a graph on n vertices.

- *There exist algorithms for deciding if $\mathcal{A}(G, K_m)$ has a 1-dimensional representation, but the problem of deciding if $\mathcal{A}(G, K_3)$ has a 1-dimensional representation is NP-complete.*
- *The problem of deciding if $\mathcal{A}(G, K_3)$ has a finite dimensional representation is NP-hard and currently there is no known algorithm.*
- *There exists an SDP that decides if $\mathcal{A}(G, K_3)$ has a trace.*

To reiterate, don't know the complexity level of determining if $\mathcal{A}(G, K_3)$ is non-trivial, or if any representations on Hilbert space exist. Don't understand qa cases.

References

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Thanks!