COMPLETELY BOUNDED MAPS

and

OPERATOR ALGEBRAS

Solutions to Selected Exercises

1.1 Let $(T_{i,j})$ be in $M_n(B(\mathcal{H}))$. Verify that the linear transformation it determines on $\mathcal{H}^{(n)}$ is bounded and that, in fact, $\|(T_{i,j})\| \leq \left(\sum_{i,j=1}^{n} \|T_{i,j}\|^2\right)^{1/2}$.

Solution: Let $\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} \in \mathcal{H}^{(n)}$ be of unit length. We compute

$$\left\|(T_{i,j})\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}\right\|^2 = \left\|\begin{bmatrix} \sum_{j=1}^{n} T_{1,j}h_j \\ \vdots \\ \sum_{j=1}^{n} T_{n,j}h_j \end{bmatrix}\right\|^2 = \sum_{i=1}^{n}\left\|\sum_{j=1}^{n} T_{i,j}h_j\right\|^2 \tag{1.1}$$

$$\leq \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\|T_{i,j}h_j\|\right)^2 \quad \text{triangle inequality} \tag{1.2}$$

$$\leq \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\|T_{i,j}\|\,\|h_j\|\right)^2 \tag{1.3}$$

$$\leq \sum_{i=1}^{n}\left[\left(\sum_{j=1}^{n}\|T_{i,j}\|^2\right)\left(\sum_{j=1}^{n}\|h_j\|^2\right)\right] \tag{1.4}$$

$$= \sum_{i,j=1}^{n}\|T_{i,j}\|^2 \tag{1.5}$$

Taking square roots now yields the result.

1.2 Let $\pi : M_n(B(\mathcal{H})) \to B(\mathcal{H}^{(n)})$ be the identification given in the text.

i) Verify that π is a one-to-one, $*$-homomorphism.

ii) Let $E_j : \mathcal{H} \to \mathcal{H}^{(n)}$ be the map defined by $E_j(h)$ equal to the vector that has h for its j-th entry and is 0 elsewhere. Show that $E_j^* : \mathcal{H}^{(n)} \to \mathcal{H}$ is the map which sends a vector in $\mathcal{H}^{(n)}$ to its j-th component.

iii) Given $T \in B(\mathcal{H}^{(n)})$ set $T_{ij} = E_i^* T E_j$. Show that $\pi((T_{ij})) = T$ and that consequently π is onto.

Solution: Note that,

$$\sum_{k=1}^{n} E_k h_k = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}. \tag{1.6}$$

i) Let $(S_{i,j})$ and $(T_{i,j})$ be in $M_n(B(\mathcal{H}))$, let $h = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}, f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \in \mathcal{H}^{(n)}$ and $\alpha \in \mathbb{C}$. We have,

$$\pi(\alpha(S_{i,j}) + (T_{i,j}))h = \pi((\alpha S_{i,j} + T_{i,j}))h = \begin{bmatrix} \sum_{j=1}^n (\alpha S_{1,j}h_j + T_{1,j}h_j) \\ \vdots \\ \sum_{j=1}^n (\alpha S_{n,j}h_j + T_{n,j}h_j) \end{bmatrix} \tag{1.7}$$

$$= \alpha \begin{bmatrix} \sum_{j=1}^n S_{1,j}h_j \\ \vdots \\ \sum_{j=1}^n S_{n,j}h_j \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^n S_{1,j}h_j \\ \vdots \\ \sum_{j=1}^n S_{n,j}h_j \end{bmatrix} \tag{1.8}$$

$$= (\alpha\pi((S_{i,j})) + \pi((T_{i,j})))h. \tag{1.9}$$

Thus π is linear.
Let $(R_{ij}) = (S_{i,j})(T_{i,j})$.

$$\pi((R_{i,j}))h = \begin{bmatrix} \sum_{j=1}^n R_{1,j}h_j \\ \vdots \\ \sum_{j=1}^n R_{n,j}h_j \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j,k=1}^n S_{1,k}T_{k,j}h_j \\ \vdots \\ \sum_{j,k=1}^n S_{n,k}T_{k,j}h_j \end{bmatrix} \tag{1.10}$$

On the other hand,

$$[\pi((S_{i,j}))\pi((T_{i,j}))]h = \pi((S_{i,j})) \begin{bmatrix} \sum_{j=1}^n T_{1,j}h_j \\ \vdots \\ \sum_{j=1}^n T_{n,j}h_j \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=1}^n S_{1,k} \left(\sum_{j=1}^n T_{k,j}h_j\right) \\ \vdots \\ \sum_{k=1}^n S_{n,k} \left(\sum_{j=1}^n T_{k,j}h_j\right) \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{j,k=1}^n S_{1,k}T_{k,j}h_j \\ \vdots \\ \sum_{j,k=1}^n S_{n,k}T_{k,j}h_j \end{bmatrix} \tag{1.11}$$

Comparison of (1.10) and (1.11) shows that π is a homomorphism.

We now compute the adjoint of $\pi((T_{i,j}))$.

$$\left\langle \pi((T_{i,j})) \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} \sum_{j=1}^n T_{1,j}h_j \\ \vdots \\ \sum_{j=1}^n T_{n,j}h_j \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \right\rangle \tag{1.12}$$

$$= \sum_{i,j=1}^{n} \langle T_{i,j}h_j, f_i \rangle = \sum_{i,j=1}^{n} \langle h_j, T_{i,j}^* f_i \rangle \tag{1.13}$$

$$= \left\langle \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}, \begin{bmatrix} \sum_{i=1}^n T_{i,1}^* f_i \\ \vdots \\ \sum_{i=1}^n T_{i,n}^* f_i \end{bmatrix} \right\rangle \tag{1.14}$$

$$= \left\langle \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}, \pi((T_{j,i}^*)) \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \right\rangle. \tag{1.15}$$

This shows that $\pi((T_{i,j}))^* = \pi((T_{j,i}^*)) = \pi((T_{i,j})^*)$ and so π is a $*$-homomorphism.
Suppose $\pi((T_{i,j})) = 0$. Then for every $k \in \{1, \ldots, n\}$ and $h \in \mathcal{H}$ we have

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \pi((T_{ij}))E_k(h) = \begin{bmatrix} T_{1,k}h \\ \vdots \\ T_{n,k}h \end{bmatrix}. \tag{1.16}$$

Hence $T_{i,j}h = 0$ for all $h \in \mathcal{H}$ and for all $i, j \in \{1, \ldots, n\}$. It follows that $(T_{i,j}) = 0$ and π is one-to-one.

ii) Let $h \in \mathcal{H}$ and $\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} \in \mathcal{H}^{(n)}$. The definition of the adjoint gives us

$$\left\langle E_j^* \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}, h \right\rangle = \left\langle \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}, E_j h \right\rangle = \langle h_j, h \rangle \tag{1.17}$$

Therefore, E_j^* is the map that sends $\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$ to h_j.

iii) We have,

$$\left\langle \pi((T_{i,j})) \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} \sum_{j=1}^n T_{1,j}h_j \\ \vdots \\ \sum_{j=1}^n T_{n,j}h_j \end{bmatrix}, \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \right\rangle \tag{1.18}$$

$$= \sum_{i,j=1}^n \langle T_{i,j}h_j, f_i\rangle = \sum_{i,j=1}^n \langle E_i^* T E_j h_j, f_i\rangle \tag{1.19}$$

$$= \sum_{i,j=1}^n \langle TE_jh_j, E_if_i\rangle = \left\langle \sum_{j=1}^n TE_jh_j, \sum_{i=1}^n E_if_i \right\rangle \tag{1.20}$$

$$= \left\langle T(\sum_{j=1}^n E_jh_j), \sum_{i=1}^n E_if_i \right\rangle = \langle Th, f\rangle . \tag{1.21}$$

Thus $\pi((T_{i,j})) = T$ and π is onto.

1.3 Let $(T_{i,j})$ be in $M_n(B(\mathcal{H}))$. Prove that $(T_{i,j})$ is a contraction if and only if for every choice of $2n$ unit vectors $x_1, \dots, x_n, y_1, \dots, y_n$ in $\mathcal{H}$, the scalar matrix $(\langle T_{i,j}x_j, y_i\rangle)$ is a contraction.

Solution: We use the fact that if T is a bounded operator on a Hilbert space $\mathcal{H}$, then T is a contraction if and only if $|\langle Th, k\rangle| \le 1$ for all $h, k \in \mathcal{H}$ of unit length.

Suppose first that $(T_{i,j})$ is a contraction and that $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}, \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}$ are vectors of unit length in $\mathbb{C}^n$. A short calculation:

$$\left\| \begin{bmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{bmatrix} \right\|^2 = \sum_{k=1}^n \|\lambda_k x_k\|^2 = \sum_{k=1}^n |\lambda_k|^2 \, \|x_k\|^2 = 1, \tag{1.22}$$

shows that $\begin{bmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{bmatrix}$ and $\begin{bmatrix} \mu_1 y_1 \\ \vdots \\ \mu_n y_n \end{bmatrix}$ are of unit length in $\mathcal{H}^{(n)}$. It follows that,

$$\left| \left\langle (\langle T_{i,j}x_j, y_i\rangle) \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}, \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} \right\rangle \right| = \left| \sum_{i,j=1}^n \langle T_{i,j}x_j, y_i\rangle \, \lambda_j \overline{\mu_i} \right|$$

$$= \left| \sum_{i,j=1}^n \langle T_{i,j}(\lambda_j x_j), \mu_i y_i\rangle \right| \tag{1.23}$$

$$= \left| \left\langle (T_{i,j}) \begin{bmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{bmatrix}, \begin{bmatrix} \mu_1 y_1 \\ \vdots \\ \mu_n y_n \end{bmatrix} \right\rangle \right| \le 1 \tag{1.24}$$

and hence $(\langle T_{i,j}x_j, y_i\rangle)$ is a contraction in M_n.

Conversely, let $\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$ and $\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$ be unit vectors in $\mathcal{H}^{(n)}$. Choose unit vectors $x_1, \ldots, x_n, y_1, \ldots, y_n$ in $\mathcal{H}$ and scalars $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n$ such that $\lambda_j x_j = h_j$ and $\mu_i y_i = f_i$. We have,

$$1 = \left\| \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} \right\|^2 = \sum_{k=1}^{n} \|\lambda_k x_k\|^2 = \sum_{k=1}^{n} |\lambda_k|^2 \|x_k\|^2 = \sum_{k=1}^{n} |\lambda_k|^2 . \tag{1.25}$$

Which proves that the vectors $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}, \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}$ are of unit length in $\mathbb{C}^n$. A calculation similar to the one in (1.24) proves that $(T_{i,j})$ is a contraction.

1.4 Let$(T_{i,j})$ be in $M_n(\mathcal{H}^{(n)})$. Prove that $(T_{i,j})$ is positive if and only if for every choice of n vectors $x_1, \ldots, x_n$ in $\mathcal{H}$, the scalar matrix $(\langle T_{i,j}x_j, x_i\rangle)$ is positive.

Solution: If $\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \in \mathbb{C}^n$ and $x_1, \ldots, x_n$ are vectors in $\mathcal{H}$, then

$$\left\langle (\langle T_{i,j}x_j, x_i\rangle) \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}, \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \right\rangle = \left\langle (T_{i,j}) \begin{bmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{bmatrix}, \begin{bmatrix} \lambda_1 x_1 \\ \vdots \\ \lambda_n x_n \end{bmatrix} \right\rangle \tag{1.26}$$

If $(T_{i,j})$ is positive, (1.26) shows that $(\langle T_{i,j}x_j, x_i\rangle)$ is positive. If $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathcal{H}^{(n)}$ and $(\langle T_{i,j}x_j, x_i\rangle)$ is positive then set $\lambda_k = 1$ for all $k \in \{1, \ldots, n\}$ in (1.26) to prove $(T_{i,j})$ is positive.

1.5 Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras, and let $\pi : \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism with $\pi(\mathbf{1}) = \mathbf{1}$. Show that π is completely positive and completely bounded and that $\|\pi\| = \|\pi_n\| = \|\pi\|_{cb} = 1$.

Solution: If π is a $*$-homomorphism with $\pi(\mathbf{1}) = \mathbf{1}$, then π maps invertible elements in $\mathcal{A}$ to invertible elements in $\mathcal{B}$. Therefore $\sigma(\pi(a)) \subseteq \sigma(a)$ for any $a \in \mathcal{A}$. It follows that,

$$\|\pi(a)\|^2 = \|\pi(a)^*\pi(a)\| \tag{1.27}$$
$$= \|\pi(a^*a)\| = r(\pi(a^*a)) \tag{1.28}$$
$$\leq r(a^*a) = \|a^*a\| = \|a\|^2 . \tag{1.29}$$

where $r(x)$ denotes the spectral radius of x. Thus, $\|\pi\| \leq 1$. Since $\|\pi(\mathbf{1})\| = \|\mathbf{1}\| = 1$, we have $\|\pi\| = 1$.

If a is positive, then there exists an $x \in \mathcal{A}$ such that $a = x^*x$. Therefore,

$$\pi(a) = \pi(x^*x) = \pi(x)^*\pi(x) \geq 0. \tag{1.30}$$

This shows that π is positive.

We have proved that a $*$-homomorphism that maps the identity to the identity is a norm 1, positive map. The definitions of addition and multiplication in $M_n(\mathcal{A})$ and the definition of π_n imply that π_n is indeed a unital, $*$-homomorphism. Hence π is completely positive and

$$\|\pi\|_{cb} = \sup\{\|\pi_n\| : n \geq 1\} = 1 = \|\pi\|, \tag{1.31}$$

shows that π is completely bounded.

1.6 Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be C^*-algebras, and let $\varphi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{C}$ be (completely) positive maps. Show that $\psi \circ \varphi$ is (completely) positive.

Solution: We first prove that $\psi \circ \varphi$ is positive. Let $a \in \mathcal{A}$ be positive. Since φ is a positive map $\varphi(a)$ is positive in $\mathcal{B}$. The fact that ψ is a positive map shows that $(\psi \circ \varphi)(a) = \psi(\varphi(a)) \geq 0$, proving our claim.

It is straightforward to check that $(\psi \circ \varphi)_n = \psi_n \circ \varphi_n$. This fact and a similar argument to the one used in the preceeding paragraph shows us that $\psi \circ \varphi$ is completely positive when φ and ψ are completely positive.

1.7 Let $\{E_{i,j}\}_{i,j=1}^n$ be matrix units for M_n, let $A = (E_{j,i})_{i,j=1}^n$ and let $B = (E_{i,j})_{i,j=1}^n$ be in $M_n(M_n)$. Show that A is unitary and that $\dfrac{1}{n}B$ is a rank one projection.

Solution: We begin by noting the following facts: $E_{i,j}^* = E_{j,i}$, $\sum_{k=1}^n E_{k,k} = \mathbf{1}_n$, the $n \times n$ identity matrix, and

$$E_{i,j}E_{p,q} = \begin{cases} 0 & \text{if } j \neq p \\ E_{i,q} & \text{if } j = p \end{cases}. \tag{1.32}$$

We have $A^* = (E_{j,i})^* = (E_{i,j}^*) = (E_{j,i}) = A$, and so A is self-adjoint. Thus $AA^* = A^2 = A^*A$. When we compute the (i,j)-th entry of A^2 we get,

$$\sum_{k=1}^n E_{k,i}E_{j,k} = \begin{cases} 0 & \text{if } i \neq j \\ \sum_{k=1}^n E_{kk} = \mathbf{1}_n & \text{if } i = j \end{cases} \tag{1.33}$$

From which it follows that $A^2 = \mathbf{1}$ and A is unitary.

Since $B^* = (E_{i,j})^* = (E_{j,i}^*) = (E_{i,j}) = B$, $\frac{1}{n}B$ is self-adjoint. The (i,j)-th entry of B^2 is

$$\sum_{k=1}^n E_{i,k}E_{k,j} = \sum_{k=1}^n E_{i,j} = nE_{i,j}. \tag{1.34}$$

Therefore the (i,j)-th entry of $\frac{1}{n^2}B^2$ is $\frac{1}{n}E_{i,j}$ and so $\left(\frac{1}{n}B\right)^2 = \frac{1}{n}B$. Thus $\frac{1}{n}B$ is a projection. It remains to be shown that B has rank one. The identification of $M_n(M_n)$ with M_{n^2} allows us to treat B as an operator on $\mathbb{C}^{n^2} = \underbrace{\mathbb{C}^n \oplus \ldots \oplus \mathbb{C}^n}_{n \text{ copies}}$. We compute the trace of B and note for a projection P that $\operatorname{rank} P = \operatorname{trace} P$. Note that the diagonal entries of B are 0 except the $((k-1)n+k, (k-1)n+k)$ entries for $k = 1, \ldots, n$ and so $\operatorname{trace} B = n$ which shows that $\operatorname{trace} \frac{1}{n}B = 1$. We now compute the range of B. Let $h_k = \begin{bmatrix} \lambda_{k,1} \\ \vdots \\ \lambda_{k,n} \end{bmatrix} \in \mathbb{C}^n$.

Now,

$$B(h_1 \oplus \ldots \oplus h_n) = \left(\sum_{j=1}^n E_{1,j}h_j\right) \oplus \ldots \oplus \left(\sum_{j=1}^n E_{n,j}h_j\right) = (\lambda_{1,1} + \ldots + \lambda_{n,n})(e_1 \oplus \ldots \oplus e_n). \tag{1.35}$$

Which shows that B, and hence $\frac{1}{n}B$, is the projection onto the span of the n^2-tuple whose entries are equal to 1 in the 1st, $(n+1)$-th, ..., $((n-1)n+1)$-th place and is 0 otherwise.

1.8 Let $\{E_{i,j}\}_{i,j=1}^n$ be a system of matrix units for $B(\mathcal{H})$, let $A = (E_{j,i})_{i,j=1}^n$ and let $B = (E_{i,j})_{i,j=1}^n$ be in $M_n(B(\mathcal{H}))$. Show that A is a partial isometry and that $\dfrac{1}{n}B$ is a rank one projection. Show that $\varphi_n(A) = B$ and $\|\varphi_n(A)\| = n$.

Solution: φ is the transpose map. We begin by noting the following facts: $E_{i,j}^* = E_{j,i}$,

$$E_{i,j}E_{l,m} = \begin{cases} 0 & \text{if } j \neq l \\ E_{i,m} & \text{if } j = l \end{cases}. \tag{1.36}$$

and

$$\sum_{k=1}^{n} E_{k,k} = \begin{bmatrix} \mathbf{1}_n & 0 \\ 0 & 0 \end{bmatrix} = P_n, \tag{1.37}$$

where $\mathbf{1}_n$ denotes the $n \times n$ identity matrix.

A calculation similar to that in excercise 1.7 shows that A is self-adjoint and that

$$AA^* = A^2 = \begin{bmatrix} P_n & 0 & \cdots & 0 \\ 0 & P_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_n \end{bmatrix} \tag{1.38}$$

which is the projection of $\mathcal{H}^{(n)}$ onto the subspace spanned by vectors of the form $\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$ where all except possibly the first n entries of each h_k are 0. Therefore A is a partial isometry.

A similar argument to the one for Exercise 1.7 shows that $\frac{1}{n}B$ is a non-zero projection. We have,

$$\varphi_n(A) = \varphi_n((E_{j,i})) = (E_{j,i}^t) = (E_{i,j}) = B, \tag{1.39}$$

and

$$\|\varphi_n(A)\| = \|B\| = n\left\|\frac{1}{n}B\right\| = n. \tag{1.40}$$

1.9 Let A be in M_n and let A^t denote the transpose of A. Prove that A is positive if and only if A^t is positive and that $\|A\| = \|A^t\|$. Prove that these same results hold for operators on separable, infinite dimensional Hilbert space, when we fix an orthonormal basis, regard operators as infinite matrices and use this to define the transpose.

Solution: We prove the second part of the exercise which concerns Hilbert space, the first part being a special case of the second. To see that A^t is positive we write $A = B^*B$ for some $B \in B(\mathcal{H})$. Thus,

$$A^t = (B^*B)^t = B^t(B^*)^t = B^t(B^t)^* \geq 0. \tag{1.41}$$

By interchanging the roles of A and A^t we see that A is positive whenever A^t is positive.

Since $\mathcal{H}$ is a separable, infinite-dimensional Hilbert space we can assume that $\mathcal{H} = \ell^2$ with the orthonormal basis $\{e_n\}_{n\geq 1}$, where e_n is the sequence whose n-th entry is 1 and whose other entries are 0. Let $(\alpha_{i,j})$ be the matrix of A with respect to this orthonormal basis and let $(x_1, x_2, \ldots)$ be an element of $\mathcal{H}$. Denote by $\overline{A}$ and $\overline{x}$ the matrix $(\overline{a}_{i,j})$ and the sequence $(\overline{x}_1, \overline{x}_2, \ldots)$. Note that $\|x\| = \|\overline{x}\|$. It follows from the definition of the inner-product that

$$\langle x, y\rangle = \overline{\langle \overline{x}, \overline{y}\rangle}. \tag{1.42}$$

It is also straightforward to show that $\overline{Ax} = \overline{A}\overline{x}$.

Let $\|x\| \leq 1$ and consider,

$$\left\|\overline{A}x\right\|^2 = \left\langle \overline{A}x, \overline{A}x\right\rangle = \overline{\langle A\overline{x}, A\overline{x}\rangle} = \|A\overline{x}\|^2 \leq \|A\|^2. \tag{1.43}$$

Therefore $\left\|\overline{A}\right\| \leq \|A\|$. By interchanging the roles of A and $\overline{A}$ we see that $\|A\| = \|\overline{\overline{A}}\| \leq \left\|\overline{A}\right\|$. Therefore $\|A\| = \left\|\overline{A}\right\|$. Since $\overline{A^t} = A^*$ we get,

$$\left\|A^t\right\| = \|\overline{A^t}\| = \|A^*\| = \|A\|. \tag{1.44}$$

The idea in the last two paragraphs gives an alternate proof of the positivity result. A short calculation,

$$\langle Ax, x\rangle = \langle x, A^*x\rangle = \left\langle x, \overline{A^t}x\right\rangle = \left\langle x, \overline{A^t\overline{x}}\right\rangle = \overline{\langle \overline{x}, A^t\overline{x}\rangle}, \tag{1.45}$$

shows that A is positive if and only if A^t is positive.

1.10 Prove that the map $\pi : M_n(\mathcal{A}) \to \mathcal{A} \otimes M_n$ defined by $\pi((a_{i,j})) = \sum_{i,j=1}^n a_{i,j} \otimes E_{i,j}$ is an algebra isomorphism.

Solution: Let α be a scalar. The addition and scalar multiplication of tensors satisfies $(a \otimes A) + (a \otimes B) = a \otimes (A + B)$ and $\alpha(a \otimes A) = (\alpha a) \otimes A = a \otimes (\alpha A)$ and therefore π is linear. Let $(c_{i,j}) = (a_{i,j})(b_{i,j})$. We have,

$$\pi((c_{i,j})) = \sum_{i,j=1}^{n} c_{i,j} \otimes E_{i,j} = \sum_{i,j,k=1}^{n} a_{i,k}b_{k,j} \otimes E_{i,j}. \tag{1.46}$$

Now,

$$\begin{aligned}
\pi((a_{i,j}))\pi((b_{i,j})) &= \left(\sum_{i,j=1}^{n} a_{i,j} \otimes E_{i,j}\right)\left(\sum_{k,l=1}^{n} a_{k,l} \otimes E_{k,l}\right) && (1.47)\\
&= \sum_{i,j,k,l=1}^{n} a_{i,j}b_{k,l} \otimes E_{i,j}E_{k,l} = \sum_{i,l=1}^{n}\sum_{j,k=1}^{n} a_{i,j}b_{k,l} \otimes E_{i,j}E_{k,l} && (1.48)\\
&= \sum_{i,l=1}\sum_{j=1} a_{i,j}b_{j,l} \otimes E_{i,l} \qquad \text{by using (1.33)}, && (1.49)
\end{aligned}$$

which (after relabeling the indices) is equal to (1.46). Thus π is a homomorphism. The involution in $\mathcal{A} \otimes M_n$ is defined by $(a \otimes A)^* = a^* \otimes A^*$ which proves that π is $*$-homomophism. We have,

$$\pi(1) = \pi\left(\begin{bmatrix} 1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix}\right) = \sum_{i=1} 1 \otimes E_{i,i} = 1 \otimes \sum_{i=1} E_{i,i} = 1 \otimes 1, \tag{1.50}$$

which shows that π is unital.

It remains to show π is bijective. Suppose that $\pi((a_{i,j})) = 0$, we want to show that $a_{k,l} = 0$ for all $k, l \in \{1, \ldots, n\}$. We compute,

$$(\mathbf{1} \otimes E_{k,k})\pi((a_{i,j}))(\mathbf{1} \otimes E_{l,l}) = (\mathbf{1} \otimes E_{k,k}) \left(\sum_{i,j=1}^{n} a_{i,j} \otimes E_{i,j} \right) (\mathbf{1} \otimes E_{l,l}) \tag{1.51}$$

$$= (\mathbf{1} \otimes E_{k,k}) \left(\sum_{i=1}^{n} a_{i,l} \otimes E_{i,l} \right) \tag{1.52}$$

$$= \sum_{i=1}^{n} a_{i,l} \otimes E_{k,k}E_{i,l} = a_{k,l} \otimes E_{k,l} \tag{1.53}$$

If we now use the fact that $\|a_{k,l} \otimes E_{k,l}\| = \|a_{k,l}\| \, \|E_{k,l}\|$ and $\|E_{k,l}\| = 1$ we see that $\|a_{k,l}\| = 0$ and so $a_{k,l} = 0$. Now suppose that $a \otimes A$ is an elementary tensor in $\mathcal{A} \otimes M_n$. If $A = (\alpha_{i,j})$, then we can write

$$a \otimes A = \sum_{i,j=1}^{n} a \otimes (\alpha_{i,j} E_{i,j}) = \sum_{i,j=1}^{n} (\alpha_{i,j} a) \otimes E_{i,j} = \pi((\alpha_{i,j} a)). \tag{1.54}$$

Thus, since π is linear, the range of π contains the span of the elementary tensors. The fact that π is a one-to-one, $*$-homomorpism implies that it is an isometry and has closed range. This together with the fact that the set of elementary tensors is dense in $\mathcal{A} \otimes M_n$ proves range $\pi = \mathcal{A} \otimes M_n$ and so π is onto.

2.1 Let $\mathcal{S}$ be an operator system, $\mathcal{B}$ be a C^*-algebra and $\varphi : \mathcal{S} \to \mathcal{B}$ a postive map. Prove that φ is *self-sdoint*, i.e., that $\varphi(x^*) = \varphi(x)^*$.

Solution: We first prove the result when $x = x^*$. We can write $x = p_1 - p_2$ with p_1 and p_2 positive. Since φ is positive $\varphi(p_1)$ and $\varphi(p_2)$ are positive and so $\varphi(x) = \varphi(p_1) - \varphi(p_2)$, which is the difference of two postive elements, is self-adjoint.

Now let $x \in \mathcal{S}$ and write $x = x_1 + ix_2$ with x_1 and x_2 self-adjoint. Then we have,

$$\varphi(x^*) = \varphi(x_1 - ix_2) = \varphi(x_1) - i\varphi(x_2) = \varphi(x_1)^* - i\varphi(x_2)^* = (\varphi(x_1) + i\varphi(x_2))^* = \varphi(x)^*, \tag{2.1}$$

which completes the proof.

2.2 Let $\mathcal{S}$ be an operator system, $\mathcal{B}$ be a C^*-algebra and $\varphi : \mathcal{S} \to \mathcal{B}$ be a positive map. Prove that φ extends to a positive map on the norm closure of $\mathcal{S}$.

Solution: The positivity of φ implies that φ is bounded and in particular is uniformly continuous. It follows that φ extends in a unique way to a continuous map $\tilde{\varphi}$ on the norm closure of $\mathcal{S}$. It remains, therefore, to show that $\tilde{\varphi}$ is positive.

Let p be a positive element in $\overline{\mathcal{S}}$. Then, there is a sequence $\{x_n\}$ in $\mathcal{S}$ such that $\|p - x_n\| \to 0$ as $n \to \infty$. The fact that p is self-adjoint tells us that $\|p - x_n^*\| = \|p - x_n\|$ and so,

$$\left\| p - \frac{x_n + x_n^*}{2} \right\| \to 0 \text{ as } n \to \infty. \tag{2.2}$$

Set $h_n = (x_n + x_n^*)/2$ and note that h_n is self-adjoint. Let $\varepsilon > 0$. We claim that $h_n + \varepsilon\mathbf{1}$ is positive for sufficiently large values of n. We may assume that h_n is a sequence of operators on a Hilbert space $\mathcal{H}$. Let $x \in \mathcal{H}$ and choose N so large that $n \geq N$ implies $\|h_n - p\| < \varepsilon/2$. Then,

$$\langle (h_n + \varepsilon\mathbf{1})x, x\rangle = \langle (h_n + \varepsilon\mathbf{1} - (p + \varepsilon\mathbf{1}))x, x\rangle + \langle (p + \varepsilon\mathbf{1})x, x\rangle\,. \tag{2.3}$$

Since p is positive

$$\langle (p + \varepsilon\mathbf{1})x, x\rangle \geq \varepsilon\,\|x\|^2\,, \tag{2.4}$$

and the Cauchy-Schwarz inequality gives,

$$|\langle (h_n + \varepsilon\mathbf{1} - (p + \varepsilon\mathbf{1}))x, x\rangle| \leq \|h_n + \varepsilon\mathbf{1} - (p + \varepsilon\mathbf{1})\|\,\|x\|^2 = \|h_n - p\|\,\|x\|^2 < \frac{\varepsilon}{2}\,\|x\|^2\,. \tag{2.5}$$

Using (2.3), (2.4) and (2.5) we get,

$$\langle (h_n + \varepsilon\mathbf{1})x, x\rangle \geq (\varepsilon - \frac{\varepsilon}{2})\,\|x\|^2 = \frac{\varepsilon}{2}\,\|x\|^2\,, \tag{2.6}$$

which establishes our claim.

Since $h_n + \varepsilon\mathbf{1} \to p + \varepsilon\mathbf{1}$ we have,

$$\tilde{\varphi}(p + \varepsilon\mathbf{1}) = \lim_{n\to\infty} \varphi(h_n + \varepsilon\mathbf{1}) \geq 0. \tag{2.7}$$

Thus,

$$\tilde{\varphi}(p) + \varepsilon\tilde{\varphi}(1) \geq 0, \tag{2.8}$$

for every positive ε. When we let $\varepsilon \to 0$ we see that $\tilde{\varphi}(p) \geq 0$, which completes the proof.

2.3 Let $\mathcal{S}$ be an operator system and let $\varphi : \mathcal{S} \to \mathbb{C}$ be positive. Prove that $\|\varphi\| \leq \varphi(1)$.

Solution: Let $a \in \mathcal{S}$ and choose a unimodular $\lambda \in \mathbb{C}$ such that $|\varphi(a)| = \lambda\varphi(a) = \varphi(\lambda a)$. We have $|\varphi(a)| = \overline{\varphi(\lambda a)} = \varphi((\lambda a)^*)$, where the last equality follows from the fact that φ is positive. For any element a of a C^*-algebra $\mathrm{Re}\, a \leq \|a\|\, 1$. Thus,

$$\begin{aligned} |\varphi(a)| &= \frac{1}{2}(\varphi(\lambda a) + \varphi((\lambda a)^*)) = \varphi\left(\frac{\lambda a + (\lambda a)^*}{2}\right) && (2.9)\\ &= \varphi(\mathrm{Re}\,(\lambda a)) \leq \varphi(\|\lambda a\|\, 1) = \|a\|\, \varphi(1). && (2.10)\end{aligned}$$

Therefore, $\|\varphi\| \leq \varphi(1)$.

2.4 Let $\mathcal{S}$ be an operator system and let $\varphi : \mathcal{S} \to C(X)$ where $C(X)$ denotes the continuous functions on a compact Hausdorff space X. Prove that if φ is positive, then $\|\varphi\| \leq \|\varphi(1)\|$.

Solution: For each x in X let $\pi_x : C(X) \to \mathbb{C}$ denote the function that maps f to $f(x)$. π_x is positive and so the map $\pi_x \circ \varphi : \mathcal{S} \to \mathbb{C}$ is positive. From Exercise 2.3 we get,

$$|\varphi(a)(x)| = |(\pi_x \circ \varphi)(a)| \leq ((\pi_x \circ \varphi)(1))\, \|a\| = \varphi(1)(x)\, \|a\| \leq \|\varphi(1)\|\, \|a\|\,. \quad (2.11)$$

Thus, $\|\varphi(a)\| \leq \|\varphi(1)\|\, \|a\|$ and it follows that $\|\varphi\| \leq \|\varphi(1)\|$.

2.5 (Schwarz Inequality) Let $\mathcal{A}$ be a C^*-algebra and let $\varphi : \mathcal{A} \to \mathbb{C}$ be a positive linear functional. Prove that $|\varphi(x^*y)|^2 \leq \varphi(x^*x)\varphi(y^*y)$.

Solution: Let $t \in \mathbb{R}$ and choose a unimodular complex number λ such that $|\varphi(x^*y)| = \lambda\varphi(x^*y)$. Using the fact that φ is positive we get,

$$\begin{aligned} 0 &\leq \varphi((x+ty)^*(x+ty)) && (2.12)\\ &= \varphi(x^*x + t(x^*y + y^*x) + t^2y^*y) && (2.13)\\ &= \varphi(x^*x) + t(\varphi(x^*y) + \varphi(y^*x)) + t^2\varphi(y^*y) && (2.14)\\ &= \varphi(x^*x) + t(\varphi(x^*y) + \overline{\varphi(x^*y)}) + t^2\varphi(y^*y) && (2.15)\\ &= \varphi(x^*x) + 2\mathrm{Re}\,\varphi(x^*y)t + \varphi(y^*y)t^2 && (2.16)\end{aligned}$$

where (2.15) follows from the fact that φ is positive and therefore self-adjoint. The expression in (2.16) is a quadratic in t with real coefficients. Since this quadratic is always non-negative it follows that its discriminant is never positive. Thus,

$$(2\mathrm{Re}\,\varphi(x^*y))^2 \leq 4\varphi(x^*x)\varphi(y^*y). \quad (2.17)$$

By replacing x by λx and dividing out the factor of 4 we get,

$$|\varphi(x^*y)|^2 \leq \varphi(x^*x)\varphi(y^*y). \quad (2.18)$$

2.6 Let T be an operator on a Hilbert space $\mathcal{H}$, the *numerical radius* of T is defined by

$$w(T) = \sup\{|\langle Tx, x\rangle| :\ x \in \mathcal{H}, \|x\| = 1\}. \quad (2.19)$$

Prove that if $\varphi : \mathcal{S} \to B(\mathcal{H})$ is positive and $\varphi(1) = 1$, then $w(\varphi(a)) \leq \|a\|$.

Solution: Let $a \in \mathcal{S}$ and let x be a unit vector in $\mathcal{H}$. Note that the linear functional ρ_x that maps an operator $T \in B(\mathcal{H})$ to $\langle Tx, x\rangle$ is positive. Therefore $\rho_x \circ \varphi : \mathcal{S} \to \mathbb{C}$ is positive. Hence by exercise 2.3,

$$|\langle\varphi(a)x, x\rangle| = |\rho_x \circ \varphi(a)| \leq (\rho_x \circ \varphi(1))\, \|a\| = \rho_x(1)\, \|a\| = \|x\|^2\, \|a\| = \|a\|\,. \quad (2.20)$$

From which we see that $w(\varphi(a)) \leq \|a\|$.

2.7 Let T be an operator on a Hilbert space. Prove that $w(T) \leq 1$ if and only if $2 + (\lambda T) + (\lambda T)^* \geq 0$ for all complex numbers λ with $|\lambda| = 1$.

Solution: First note that $2 + (\lambda T) + (\lambda T)^*$ is positive if and only if $1 + \operatorname{Re} \lambda \langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$. If $w(T) \leq 1$, then,

$$-\operatorname{Re} \lambda \langle Tx, x \rangle \leq |\lambda \langle Tx, x \rangle| = |\langle Tx, x \rangle| \leq w(T) \leq 1, \tag{2.21}$$

which proves that $2 + (\lambda T) + (\lambda T)^* \geq 0$.

For the converse choose a λ, (depending on x), of modulus 1 such that $\lambda \langle Tx, x \rangle = -|\langle Tx, x \rangle|$. We have,

$$|\langle Tx, x \rangle| = -\lambda \langle Tx, x \rangle = -\operatorname{Re} \lambda \langle Tx, x \rangle \leq 1, \tag{2.22}$$

and so $w(T) \leq 1$.

2.8 Prove that $w(T)$ defines a norm on $B(\mathcal{H})$, with $w(T) \leq \|T\| \leq 2w(T)$. Show that both inequalities are sharp.

Solution: $w(T)$ is defined as the supremum of a set of non-negative numbers and is therefore non-negative. Also $w(T) = 0$ if and only if $\langle Tx, x \rangle = 0$ for all x such that $\|x\| = 1$ if and only if $T = 0$. If $\alpha \in \mathbb{C}$, then

$$\begin{aligned} w(\alpha T) &= \sup\{|\langle \alpha Tx, x \rangle| : \|x\| = 1\} = \sup\{|\alpha| \, |\langle Tx, x \rangle| : \|x\| = 1\} && (2.23) \\ &= |\alpha| \sup\{|\langle Tx, x \rangle| : \|x\| = 1\} = |\alpha| \, w(T) && (2.24) \end{aligned}$$

If R, T are two operators and $\|x\| = 1$, then

$$|\langle (R+T)x, x \rangle| = |\langle Rx, x \rangle + \langle Tx, x \rangle| \leq |\langle Rx, x \rangle| + |\langle Tx, x \rangle| \leq w(R) + w(T) \tag{2.25}$$

implies that $w(R+T) \leq w(R) + w(T)$. This proves that $w(T)$ defines a norm on $B(\mathcal{H})$.

If $\|x\| = 1$, then an application of the Cauchy-Schwarz shows that,

$$|\langle Tx, x \rangle| \leq \|Tx\| \, \|x\| \leq \|T\|, \tag{2.26}$$

and so $w(T) \leq \|T\|$. For the other inequality begin by writing $T = A + iB$ where A and B are self-adjoint. Note that $w(T) = w(T^*)$. We use the fact that for a self-adjoint operator A, $\|A\| = \sup\{|\langle Ax, x \rangle| : \|x\| = 1\} = w(A)$. We have,

$$\begin{aligned} \|T\| &= \|A + iB\| \leq \|A\| + \|B\| && (2.27) \\ &= w(A) + w(B) = w\left(\frac{T+T^*}{2}\right) + w\left(\frac{T-T^*}{2i}\right) && (2.28) \\ &\leq \frac{w(T) + w(T^*) + w(T) + w(T^*)}{2} = 2w(T) && (2.29) \end{aligned}$$

The identity map $\mathbf{1}$ on any Hilbert space satisfies $w(\mathbf{1}) = 1 = \|\mathbf{1}\|$. To see that the best upper bound is 2 we consider the operator defined on $\mathbb{C}^2$ by $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}$. We have,

$$\left\| T \begin{bmatrix} x \\ y \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} y \\ 0 \end{bmatrix} \right\|^2 = |y|^2 \leq |x|^2 + |y|^2 = \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|^2, \tag{2.30}$$

and so $\|T\| \le 1$. By considering $T\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ we see that $\|T\| = 1$. Now suppose that $\begin{bmatrix} x \\ y \end{bmatrix}$ has unit length and consider,

$$\left|\left\langle T\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right\rangle\right| = |yx| \le \frac{|x|^2 + |y|^2}{2} = \frac{1}{2}. \tag{2.31}$$

Hence,

$$\|T\| \le 2w(T) \le 1 = \|T\|, \tag{2.32}$$

which shows that the other inequality is sharp.

2.9 Let $\mathcal{S}$ be an operator system, $\mathcal{B}$ a C^*-algebra and $\varphi : \mathcal{S} \to \mathcal{B}$ a linear map such that $\varphi(1)$ is positive, $\|\varphi(1)\| = \|\varphi\|$. Give an example to show that φ need not be positive. In similar vein, show that if $\mathcal{M}$ is as in proposition 2.12, $\varphi(1)$ is positive, with $\|\varphi(1)\| = \|\varphi\|$, then $\tilde{\varphi}$ need not be well-defined.

Solution: Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in M_2 \tag{2.33}$$

and define $\varphi : M_2 \to M_2$ by

$$\varphi(A) = JA = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix}. \tag{2.34}$$

Since φ is left multiplication by J, we have $\|\varphi\| = \|\varphi(1)\| = 2$. Further $\varphi(1) = J \ge 0$. To see that φ is not positive consider

$$\varphi\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \tag{2.35}$$

which is not self-adjoint.

For the second part let $\mathcal{M}$ be the set of 2×2 upper triangular matrices and restrict the map φ in the previous paragraph to $\mathcal{M}$. Note that if $a \in \mathcal{M}$ is self-adjoint but $\varphi(a) \neq \varphi(a)^*$, then $\tilde{\varphi}$ is not well-defined. Consider the self-adjoint matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.36}$$

and note that

$$\varphi\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}, \tag{2.37}$$

which is not self-adjoint.

2.10 (Krein) Let $\mathcal{S}$ be an operator system contained in the C^*-algebra $\mathcal{A}$, and let $\varphi : \mathcal{S} \to \mathbb{C}$ be positive. Prove that φ can be extended to a positive map on $\mathcal{A}$.

Solution: The fact that φ is positive tells us that $\varphi(1)$ is a non-negative real number. If $\varphi(1) = 0$ then $\|\varphi\| \le \varphi(1) = 0$ and this map extends to the zero functional. Assume that $\varphi(1) \neq 0$ and let $\psi(a) = \varphi(1)^{-1}\varphi(a)$ and note that ψ is a positive map on $\mathcal{S}$ with $\|\psi\| = \psi(1) = 1$. By the Hahn-Banach theorem ψ extends to a map $\tilde{\psi}$ on $\mathcal{A}$ with $\|\tilde{\psi}\| = 1$. $\tilde{\psi}$ is a unital contraction and is therefore a positive map. The map $\tilde{\varphi}$ defined by $\tilde{\varphi}(a) = \varphi(1)\tilde{\psi}(a)$ is a positive map which extends φ.

2.15 In this exercise we give an alternate proof of von Neumann's inequality. We assume that the reader has some familiarity with integration of operator valued functions. Let $T \in B(\mathcal{H})$ with $\|T\| < 1$ and, let p and q denote arbitrary polynomials.

(a) Let $P(t,T) = (1 - e^{-it}T)^{-1} + (1 - e^{it}T^*)^{-1} - 1$, and show that $P(t,T) \geq 0$ for all t.

(b) Show that $p(T) + q(T)^* = \frac{1}{2\pi}\int_0^{2\pi}(p(e^{it}) + \overline{q(e^{it})})P(t,T)\,dt$.

(c) Deduce von Neumann's inequality.

Solution:

(a) We know that if $\|T\| < 1$, then $1 - e^{-it}T$ is invertible. Let $R = e^{-it}T$ and note that the adjoint of $S = (1-R)^{-1}$ is $S^* = (1-R^*)^{-1}$. Let $h \in \mathcal{H}$ and set $k = Sh$. Then,

$$\begin{aligned}
\langle (S + S^* - 1)h, h\rangle &= \langle Sh, h\rangle + \langle S^*h, h\rangle - \langle h, h\rangle && (2.38)\\
&= \langle k, S^{-1}k\rangle + \langle S^{-1}k, k\rangle - \langle S^{-1}k, S^{-1}k\rangle && (2.39)\\
&= \langle k, (1-R)k\rangle + \langle (1-R)k, k\rangle - \langle (1-R)k, (1-R)k\rangle && (2.40)\\
&= \|k\|^2 - \|Rk\|^2 \geq 0, && (2.41)
\end{aligned}$$

since R is a contraction.

(b) We know that if $\|T\| < 1$, then the operator $1 - e^{-it}T$ is invertible with inverse given by,

$$(1 - e^{-it}T)^{-1} = \sum_{n=0}^{\infty} e^{-int}T^n. \qquad (2.42)$$

Let $p(z) = \alpha_0 + \ldots + \alpha_k z^k$. We have,

$$\int_0^{2\pi} p(e^{it})P(t,T)\,dt = \int_0^{2\pi} p(e^{it})(1 - e^{-it}T)^{-1}\,dt + \int_0^{2\pi} p(e^{it})(1 - e^{it}T^*)^{-1}\,dt - \int_0^{2\pi} p(e^{it})1\,dt \qquad (2.43)$$

Consider the first term on the right side of (2.43),

$$\begin{aligned}
\int_0^{2\pi} p(e^{it})(1 - e^{-it}T)^{-1}\,dt &= \int_0^{2\pi} p(e^{it})\left(\sum_{n=0}^{\infty} e^{-int}T^n\right)dt && (2.44)\\
&= \sum_{n=0}^{\infty}\int_0^{2\pi} p(e^{it})e^{-int}T^n\,dt && (2.45)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{k}\int_0^{2\pi} \alpha_m e^{imt}e^{-int}T^n\,dt && (2.46)\\
&= 2\pi(\alpha_0 1 + \alpha_1 T + \ldots \alpha_k T^k) = 2\pi p(T) && (2.47)
\end{aligned}$$

A similar calculation shows that,

$$\begin{aligned}
\int_0^{2\pi} p(e^{it})(1 - e^{it}T^*)^{-1}\,dt &= \int_0^{2\pi} p(e^{it})\left(\sum_{n=0}^{\infty} e^{int}T^{*n}\right)dt = \sum_{n=0}^{\infty}\int_0^{2\pi} p(e^{it})e^{int}T^{*n}\,dt && (2.48)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{k}\int_0^{2\pi} \alpha_m e^{imt}e^{int}T^{*n}\,dt = 2\pi\alpha_0 1 && (2.49)
\end{aligned}$$

and

$$\int_0^{2\pi} p(e^{it})1\,dt \;=\; \int_0^{2\pi} \sum_{m=0}^{k} \alpha_k e^{it}1\,dt = 2\pi\alpha_0 1. \tag{2.50}$$

Therefore,

$$\frac{1}{2\pi}\int_0^{2\pi} p(e^{it})P(t,T)\,dt = p(T). \tag{2.51}$$

The same argument can be used to show that

$$q(T)^* = \frac{1}{2\pi}\int_0^{2\pi} \overline{q(e^{it})}P(t,T)\,dt. \tag{2.52}$$

(c) Suppose that $p + \overline{q}$ is positive, then the integral

$$\frac{1}{2\pi}\int_0^{2\pi} \left(p(e^{it}) + \overline{q(e^{it})}\right) P(t,T)\,dt, \tag{2.53}$$

is positive, being a limit of Riemann sums, each of which is a positive operator. This last statement follows from the fact that $P(t,T)$ and $p+\overline{q}$ are positive. This proves Theorem 2.6 and von Neumann's inequality follows for operators T with $\|T\| < 1$. We now prove the case $\|T\| = 1$. Let $0 < r < 1$ and note that $\|rT\| < 1$. Thus for any polynomial p, $\|p(rT)\| \leq \|p\|_\infty$. Therefore,

$$\|p(T)\| = \lim_{r\to 1^-} \|p(rT)\| \leq \|p\|_\infty\,. \tag{2.54}$$

2.16 (Wermer) In this exercise we give an alternate proof of von Neumann's inequality which is only valid for matrices. We assume that the reader is familiar with the singular value decomposition of a matrix. Let $T \in M_n$ with $\|T\| \leq 1$ and write $T = USV$ with U, V unitary and $S = \operatorname{diag}(s_1, \dots, s_n)$ a positive diagonal matrix, $0 \leq s_i \leq 1$, $i = 1, \dots, n$. Define an analytic matrix-valued function $T(z_1, \dots, z_n) = UZV$ where $Z = \operatorname{diag}(z_1, \dots, z_n)$, $|z_i| \leq 1$, $i = 1, \dots, n$. Fix a polynomial p.

i) Let $x, y \in \mathbb{C}^n$ and let $f(z_1, \dots, z_n) = \langle p(T(z_1, \dots, z_n))x, y\rangle$. Deduce that f achieves its maximum modulus at a point where $|z_1| = \ldots = |z_n| = 1$. Note that at such a point $T(z_1, \dots, z_n)$ is unitary.

ii) Deduce that $\sup_{|z_i|\leq 1} \|p(T(z_1, \dots, z_n))\|$ is attained at a point where $T(z_1, \dots, z_n)$ is unitary.

iii) Deduce that $\|p(T)\| \leq \sup \|p(W)\|$ over $W \in M_n$, unitary.

iv) Show that for W unitary $\|p(W)\| \leq \|p\|_\infty$.

v) Deduce von Neumann's inequality for $T \in M_n$.

Solution:

i) Note that f is an analytic function. We have

$$\sup_{|z_i|\leq 1} |f(z_1, \dots, z_n)| \;=\; \sup_{|z_i|\leq 1, i\neq 1}\ \sup_{|z_1|\leq 1} |f(z_1, \dots, z_n)| \tag{2.55}$$

$$\;=\; \sup_{|z_i|\leq 1, i\neq 1} |f(w_1, z_2, \dots, z_n)|\,, \tag{2.56}$$

where w_1 is a complex number of modulus 1, by the maximum modulus principle. Repeating this argument we see that f achieves its maximum modulus at a point where $|z_i| = 1$ for all $i = 1, \dots, n$. In this case the matrix Z is diagonal matrix with diagonal entries of absolute value 1, thus Z, and consequently $T(z_1, \dots, z_n) = UZV$ is unitary.

ii) We have,

$$\sup_{|z_i|\leq 1} \|p(T(z_1,\ldots,z_n))\| = \sup_{|z_i|\leq 1} \sup_{x,y\in\mathbf{C}^n} |\langle p(T(z_1,\ldots,z_n))x,y\rangle| \tag{2.57}$$

$$= \sup_{x,y\in\mathbf{C}^n} \sup_{|z_i|\leq 1} |f(z_1,\ldots,z_n)|, \tag{2.58}$$

from which it follows by i) that the supremum occurs at a point where $T(z_1,\ldots,z_n)$ is unitary.

iii) Since the singular values of T satisfy $0 \leq s_i \leq 1$, we have,

$$\|p(T)\| = \|p(T(s_1,\ldots,s_n))\| \leq \sup_{|z_i|\leq 1} \|p(T(z_1,\ldots,z_n))\| \leq \sup \|p(W)\|. \tag{2.59}$$

where the last inequality follows from part ii).

iv) If W is unitary, then $p(W)$ is normal and so $\|p(W)\| = r(p(W))$ where r denotes the spectral radius. Let $\lambda \in \sigma(W)$. Since the eigenvalues of W have modulus 1, $|p(\lambda)| \leq \sup_{|z|=1} |p(z)| = \|p\|_\infty$, and so

$$\|p(W)\| = \sup\{|p(\lambda)| : \lambda \in W\} \leq \|p\|_\infty. \tag{2.60}$$

v) By combining the results from parts iii) and iv) we get

$$\|p(T)\| \leq \sup \|p(W)\| \leq \|p\|_\infty. \tag{2.61}$$

2.20 (Korovkin) Let $f \in C([0,1])$ and let $g_x(t) = (t-x)^2$.

i) Given $\varepsilon > 0$, show that there exists a constant $c > 0$ depending only on ε and f such that

$$|f(t) - f(x)| \leq \varepsilon + c g_x(t) \text{ for all } 0 \leq x, t \leq 1. \tag{2.62}$$

ii) Let $\varphi : C([0,1]) \to C([0,1])$ be a positive map with $\varphi(1) = 1$. Show that

$$-\varepsilon - c\varphi(g_x)(x) \leq \varphi(f)(x) - f(x) \leq \varepsilon + c\varphi(g_x)(x), \tag{2.63}$$

and deduce that $\|\varphi(f) - f\| \leq \varepsilon + c \sup_x |\varphi(g_x)(x)|$.

iii) Let $\varphi_n : C([0,1]) \to C([0,1])$ be a sequence of positive maps. Prove that if $\|\varphi_n(f_i) - f_i\| \to 0$ as $n \to \infty$ for $f_i(t) = t^i$, $i = 0,1,2$, then $\|\varphi_n(f) - f\| \to 0$ as $n \to \infty$ for all $f \in C([0,1])$.

Solution:

i) Since f is uniformly continuous we may choose $\delta > 0$ such that $|f(t) - f(x)| < \varepsilon$ for $|t - x| < \delta$ with $t \in [0,1]$. Let $M = \sup\{|f(t)| : t \in [0,1]\}$. For $|t-x| < \delta$ we have $|f(t) - f(x)| \leq \varepsilon$ and for $|t - x| \geq \delta$ we have,

$$|f(t) - f(x)| \leq 2M \leq \frac{2M}{\delta^2} g_x(t) = c g_x(t). \tag{2.64}$$

Combining these inequalities we get,

$$|f(t) - f(x)| \leq \varepsilon + c g_x(t). \tag{2.65}$$

ii) From (2.65),

$$-\varepsilon\mathbf{1} - cg_x \leq f - f(x) \leq \varepsilon\mathbf{1} + cg_x. \tag{2.66}$$

Applying φ to both sides of (2.66) and noting that φ is unital yields

$$-\varepsilon - c\varphi(g_x)(x) \leq \varphi(f)(x) - f(x) \leq \varepsilon + c\varphi(g_x)(x). \tag{2.67}$$

Hence,

$$|\varphi(f)(x) - f(x)| \leq \varepsilon + c\,|\varphi(g_x)(x)|\,, \tag{2.68}$$

and taking the supremum over $x \in [0,1]$ yields,

$$\|\varphi(f) - f\| \leq \varepsilon + c\sup_x |\varphi(g_x)(x)|\,. \tag{2.69}$$

iii) Note that,

$$\varphi_n(g_x) - g_x = \varphi_n(f_2) - f_2 - 2x(\varphi_n(f_1) - f_1) + x^2(\varphi_n(f_0) - f_0). \tag{2.70}$$

Therefore,

$$\|\varphi_n(g_x) - g_x\| \leq 6 \max_{i=0,1,2} \|\varphi_n(f_i) - f_i\| \leq \varepsilon \tag{2.71}$$

for sufficiently large values of n. Note that $g_x(x) = 0$ and so

$$|\varphi_n(g_x)(x)| = |\varphi_n(g_x)(x) - g_x(x)| \leq \varepsilon. \tag{2.72}$$

Thus

$$\|\varphi_n(f) - f\| \leq \varepsilon + c\sup_x |\varphi_n(g_x)(x)| \leq (1+c)\varepsilon, \tag{2.73}$$

which shows $\varphi_n(f) \to f$ uniformly.

2.21 The *Bernstein maps* $\varphi_n : C([0,1]) \to C([0,1])$ are defined by

$$\varphi_n(f)(t) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) t^k(1-t)^{n-k}. \tag{2.74}$$

i) Verify that the Bernstein maps are positive maps with $\varphi_n(\mathbf{1}) = \mathbf{1}$, $\varphi_n(t) = t$, $\varphi_n(t^2) = t^2 + \dfrac{t-t^2}{n}$.

ii) Deduce that $\|\varphi_n(f) - f\| \to 0$ for all $f \in C([0,1])$.

iii) Deduce the Weierstrass theorem, i.e., prove that the polynomials are dense in $C([0,1])$.

Solution:

i) Note that the quantity $t^k(1-t)^{n-k}$ is non-negative on $[0,1]$. Thus if $f \geq 0$, then,

$$\binom{n}{k} f\left(\frac{k}{n}\right) t^k(1-t)^{n-k}, \tag{2.75}$$

is non-negative. This implies that $\varphi_n(f) \geq 0$.
Denote by f_i the functions $f_i(t) = t^i$. We have by the binomial theorem that,

$$\varphi_n(1)(t) = \sum_{k=0}^{n} \binom{n}{k} t^k(1-t)^{n-k} = (t + (1-t))^n = 1. \tag{2.76}$$

Now,

$$\begin{aligned}
\varphi_n(f_1)(t) &= \sum_{k=0}^{n} \binom{n}{k} f_1\left(\frac{k}{n}\right) t^k(1-t)^{n-k} && (2.77)\\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} t^k(1-t)^{n-k} && (2.78)\\
&= \sum_{k=1}^{n} \binom{n}{k} \frac{k}{n} t^k(1-t)^{n-k} && (2.79)\\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} t^k(1-t)^{n-k} && (2.80)\\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} t^{k+1}(1-t)^{n-1-k} && (2.81)\\
&= t\sum_{k=0}^{n-1} t^k(1-t)^{n-1-k} && (2.82)\\
&= t(t+(1-t))^{n-1} = t. && (2.83)
\end{aligned}$$

A similar calculation,

$$\begin{aligned}
\varphi_n(f_2)(t) &= \sum_{k=0}^{n} \binom{n}{k} \frac{k^2}{n^2} t^k(1-t)^{n-k} && (2.84)\\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{k}{n} t^k(1-t)^{n-k} && (2.85)\\
&= t\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{k}{n} t^{k-1}(1-t)^{n-k} && (2.86)\\
&= t\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{k+1}{n} t^k(1-t)^{n-1-k} && (2.87)\\
&= t\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{k}{n} t^k(1-t)^{n-1-k} + t\sum_{k=0}^{n-1} \frac{1}{n} t^k(1-t)^{n-1-k} && (2.88)\\
&= \frac{n-1}{n} t\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{k}{n-1} t^k(1-t)^{n-1-k} + \frac{t}{n}\sum_{k=0}^{n-1} t^k(1-t)^{n-1-k} && (2.89)\\
&= \frac{n-1}{n} t^2 + \frac{1}{n} t = t^2 + \frac{t-t^2}{n}. && (2.90)
\end{aligned}$$

ii) We will make use of the results from exercise 20. Note that $\|\varphi_n(f_0) - f_0\| = \|\varphi_n(f_1) - f_1\| = 0$ and $\|\varphi_n(f_2) - f_2\| = \left\|\frac{f_1 - f_2}{n}\right\| \leq \frac{2}{n} \to 0$ as $n \to \infty$. By the result in Exercise 20.iii this implies that $\|\varphi_n(f) - f\| \to 0$ for all $f \in C([0,1])$.

iii) From the part ii) of this exercise we know that the collection $\{\varphi_n(f) : f \in C([0,1])\}$ is dense in $C([0,1])$. This collection is contained in the set of polynomials and hence the polynomials are dense in $C([0,1])$.

2.22 A sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers is called a *Hausdorff moment sequence* if there exists a positive (finite) Borel measure μ on $[0,1]$ such that $a_n = \int_0^1 t^n \, d\mu(t)$ for all n. Set $b_{n,m} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_{k+m}$ for all $n, m \geq 0$.

i) Assuming the existence of such a measure μ, show that $b_{n,m} = \int_0^1 t^m (1-t)^n \, d\mu(t)$ and deduce that necessarily $b_{n,m} \geq 0$ for all $n, m \geq 0$.

ii) Let $\mathcal{P} \subseteq C([0,1])$ denote the set of polynomials and define $\varphi : \mathcal{P} \to \mathbb{C}$ by setting $\varphi(t^n) = a_n$. Show that if $b_{n,m} \geq 0$ for all $n, m \geq 0$ then φ is a positive map.

iii) Prove that $\{a_n\}_{n=0}^{\infty}$ is a Hausdorff moment sequence if and only if $b_{n,m} \geq 0$ for all $m, n \geq 0$.

Solution:

i) We have,

$$
\begin{aligned}
b_{n,m} &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_{k+m} \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \int_0^1 t^{k+m} \, d\mu(t) && (2.91) \\
&= \int_0^1 t^m \sum_{k=0}^{n} \binom{n}{k} (-1)^k t^k \, d\mu(t) && (2.92) \\
&= \int_0^1 t^m (1-t)^n \, d\mu(t) \geq 0. && (2.93)
\end{aligned}
$$

Since μ is positive and the function $f(t) = t^m (1-t)^n$ is non-negative on $[0,1]$.

ii) Let $\mathcal{P}_j$ denote the set of polynomials of degree at most j and let $f_j \in \mathcal{P}_j$ be defined by $f_j(t) = t^j$. We begin by proving that each of the Bernstein maps $\varphi_n : \mathcal{P}_j \to \mathcal{P}_j$. It is clear that $\varphi_n(\mathbf{1}) = \mathbf{1}$ and so the claim is true for $j = 0$. We proceed by induction. Assuming the result for all polynomials of degree $j-1$ or less we prove it for polynomials of degree j. Since φ_n is linear for all n we need only check that

$\varphi_n(f_j) \in \mathcal{P}_j$. We have,

$$\varphi_n(f_j)(t) = \sum_{k=0}^{n} \binom{n}{k} f_j\left(\frac{k}{n}\right) t^k(1-t)^{n-k} \tag{2.94}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \frac{k^j}{n^j} t^k(1-t)^{n-k} \tag{2.95}$$

$$= \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{k^{j-1}}{n^{j-1}} t^k(1-t)^{n-k} \tag{2.96}$$

$$= t\sum_{k=1}^{n} \binom{n-1}{k-1} \frac{k^{j-1}}{n^{j-1}} t^{k-1}(1-t)^{n-k} \tag{2.97}$$

$$= t\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(k+1)^{j-1}}{n^{j-1}} t^k(1-t)^{n-1-k} \tag{2.98}$$

$$= t\varphi_{n-1}(g_{n,j})(t) \tag{2.99}$$

where $g_{n,j}(t) = \left(t + \frac{1}{n}\right)^{j-1} \in \mathcal{P}_{j-1}$. By the induction hypothesis $\varphi_{n-1}(g) \in \mathcal{P}_{j-1}$ and so $\varphi_n(f_j) = f_1\varphi_{n-1}(g_{n,j}) \in \mathcal{P}_j$.

We also have,

$$0 \leq b_{n,m} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_{k+m} \tag{2.100}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \varphi(t^{k+m}) \tag{2.101}$$

$$= \varphi\left(t^m \sum_{k=0}^{n} \binom{n}{k} (-1)^k t^k\right) \tag{2.102}$$

$$= \varphi\left(t^m(1-t)^n\right) \tag{2.103}$$

This shows that if $f \geq 0$ then,

$$(\varphi \circ \varphi_n(f))(t) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) \varphi(t^k(1-t)^{n-k}) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) b_{k,n-k} \geq 0. \tag{2.104}$$

Now assume that $f \in \mathcal{P}$ and suppose that f has degree j. We have seen that $\varphi_n(f) \in \mathcal{P}_j$ for all n. The map $\varphi|_{\mathcal{P}_j}$ being a linear map on a finite dimensional space must be continuous. Therefore,

$$\varphi(f) = \varphi(\lim_{n\to\infty} \varphi_n(f)) = \lim_{n\to\infty} \varphi(\varphi_n(f)) \geq 0. \tag{2.105}$$

iii) We have seen in part i) that if $\{a_n\}_{n=0}^{\infty}$ is a Hausdorff moment sequence, then $b_{n,m}$ is non-negative. For the converse assume that $b_{n,m}$ is positive for all $n, m \geq 0$ and note this implies that the linear functional $\varphi : \mathcal{P} \to \mathbb{C}$ is positive. By Exercise 2.2, φ has a positive extension to the closure of $\mathcal{P}$, which is $C([0,1])$. The Riesz representation theorem implies that there is a positive (finite) Borel measure such that

$$a_n = \varphi(t^n) = \int_0^1 t^n \, d\mu(t). \tag{2.106}$$

3.1 Prove that $\|\varphi_n\| \le \|\varphi_k\|$ for $n \le k$ and that if φ_k is positive, then φ_n is positive.

Solution: Note that the map $\Phi : M_n(\mathcal{A}) \to M_k(\mathcal{A})$ defined by,

$$\Phi(A) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.1}$$

is a one-to-one, $*$-homomorphism. This allows us to identify $M_n(\mathcal{A})$ as a C^*-subalgebra of $M_k(\mathcal{A})$. Under this identification φ_n is the compression, to $M_n(\mathcal{A})$, of φ_k. It follows that φ_n is positive whenever φ_k is positive and that $\|\varphi_n\| \le \|\varphi_k\|$.

3.2 Let P, Q, A be operators on some Hilbert space $\mathcal{H}$ with P, Q positive.

i) Show that $\begin{bmatrix} P & A \\ A^* & Q \end{bmatrix} \ge 0$ if and only if $|\langle Ax, y\rangle|^2 \le \langle Py, y\rangle \langle Qy, y\rangle$ for all x, y in $\mathcal{H}$.

ii) Prove that $\begin{bmatrix} 1 & a \\ a^* & b \end{bmatrix}$ is positive in $M_2(\mathcal{A})$ if and only if $a^*a \le b$.

iii) Show that if $\begin{bmatrix} P & A \\ A^* & Q \end{bmatrix} \ge 0$ then for any x in $\mathcal{H}$, we have that

$$0 \le \langle (P + A^* + A + Q)x, x\rangle \le (\sqrt{\langle Px, x\rangle} + \sqrt{\langle Qx, x\rangle})^2, \tag{3.2}$$

and hence $\|P + A + A^* + Q\| \le (\|P\|^2 + \|Q\|^2)^{1/2}$.

iv) Show that if $\begin{bmatrix} P & A \\ A^* & P \end{bmatrix} \ge 0$, then $A^*A \le \|P\|\, P$ and in particular $\|A\| \le \|P\|$.

Solution:

i) Let $x, y \in \mathcal{H}$. We recall the fact (Exercise 1.3) that the operator matrix $((T_{i,j}))$ is positive if and only if for any choice of n vectors $x_1, \dots, x_n$ the scalar matrix $(\langle T_{i,j}x_j, x_i\rangle)$ is positive. Using this we see that

$$\begin{bmatrix} P & A \\ A^* & Q \end{bmatrix} \tag{3.3}$$

is positive if and only if the matrix

$$\begin{bmatrix} \langle Py, y\rangle & \langle Ax, y\rangle \\ \langle A^*y, x\rangle & \langle Qx, x\rangle \end{bmatrix}. \tag{3.4}$$

A 2×2 scalar matrix is positive if and only if it has non-negative trace and determinant. Since P, Q are positive the matrix in (3.4) has positive trace. Thus, positivity of (3.4) is equivalent to the matrix having non-negative determinant. i.e.,

$$\langle Py, y\rangle \langle Qx, x\rangle \ge \langle Ax, y\rangle \langle A^*y, x\rangle = \langle Ax, y\rangle \langle y, Ax\rangle = |\langle Ax, y\rangle|^2. \tag{3.5}$$

ii) We can assume that $\mathcal{A}$ is a C^*-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. We know that

$$\begin{bmatrix} 1 & A \\ A^* & B \end{bmatrix} \ge 0 \tag{3.6}$$

if and only if $|\langle Ax,y\rangle|^2 \leq \langle y,y\rangle \langle Bx,x\rangle$. Setting $y = Ax$ yields

$$|\langle Ax,Ax\rangle|^2 \leq \langle Ax,Ax\rangle \langle Bx,x\rangle\,, \tag{3.7}$$

and so $\langle A^*Ax,x\rangle = \langle Ax,Ax\rangle \leq \langle Bx,x\rangle$, which proves that $A^*A \leq B$.
Conversely if we know that $\langle A^*Ax,x\rangle \leq \langle Bx,x\rangle$ then by multiplying both sides by $\langle y,y\rangle$ and using the Cauchy-Schwarz inequality we get,

$$|\langle Ax,y\rangle|^2 \leq \langle Ax,Ax\rangle \langle y,y\rangle \leq \langle Bx,x\rangle \langle y,y\rangle\,. \tag{3.8}$$

iii) Notice that,

$$\langle (P+A^*+A+Q)x,x\rangle = \left\langle \begin{bmatrix} P & A \\ A^* & Q \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} x \\ x \end{bmatrix} \right\rangle \geq 0. \tag{3.9}$$

For the other ineqaulity we use the fact that $|\langle Ax,x\rangle|^2 \leq \langle Px,x\rangle \langle Qx,x\rangle$. We have,

$$\begin{aligned}
\langle (P+A^*+A+Q)x,x\rangle &= \langle Px,x\rangle + \langle A^*x,x\rangle + \langle Ax,x\rangle + \langle Qx,x\rangle && (3.10)\\
&\leq \langle Px,x\rangle + 2\sqrt{\langle Px,x\rangle}\sqrt{\langle Qx,x\rangle} + \langle Qx,x\rangle && (3.11)\\
&= (\sqrt{\langle Px,x\rangle} + \sqrt{\langle Qx,x\rangle})^2 && (3.12)\\
&\leq (\|P\|^{1/2}\|x\| + \|Q\|^{1/2}\|x\|)^2 \leq (\|P\|^{1/2} + \|Q\|^{1/2})^2 \|x\|^2\,. && (3.13)
\end{aligned}$$

which proves the other inequality. Since $P + A + A^* + Q$ is self-adjoint the last inequality implies that $\|P + A + A^* + Q\| \leq (\|P\|^2 + \|Q\|^2)^{1/2}$.

iv) The positivity of

$$\begin{bmatrix} P & A \\ A^* & P \end{bmatrix} \tag{3.14}$$

gives,

$$|\langle Ax,Ax\rangle|^2 \leq \langle PAx,Ax\rangle \langle Px,x\rangle \leq \|P\| \langle Ax,Ax\rangle \langle Px,x\rangle\,. \tag{3.15}$$

It follows that $\langle Ax,Ax\rangle \leq \|P\| \langle Px,x\rangle$ which implies $A^*A \leq \|P\|\, P$ and $\|A\| \leq \|P\|$.

3.3 Prove a non-unital version of proposition 3.2.

Solution: Let $\mathcal{S}$ be an operator system, $\mathcal{B}$ a C^*-algebra and $\varphi : \mathcal{S} \to \mathcal{B}$ a 2-positive map. Let $a \in \mathcal{S}$ with $\|a\| \leq 1$. Since φ is 2-positive we have,

$$\varphi_2 \begin{bmatrix} \mathbf{1} & a \\ a^* & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \varphi(\mathbf{1}) & \varphi(a) \\ \varphi(a)^* & \varphi(\mathbf{1}) \end{bmatrix} \geq 0. \tag{3.16}$$

By 3.2.iii we have $\|\varphi(a)\| \leq \|\varphi(\mathbf{1})\|$.

3.4 (Modified Schwarz Inequality for 2-positive maps) Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, $\varphi : \mathcal{A} \to \mathcal{B}$ 2-positive. Prove that $\varphi(a)^*\varphi(a) \leq \|\varphi(\mathbf{1})\|\, \varphi(a^*a)$ and that $\|\varphi(a^*b)\|^2 \leq \|\varphi(a^*a)\|\, \|\varphi(b^*b)\|$.

Solution: If

$$\begin{bmatrix} P & A \\ A^* & Q \end{bmatrix} \tag{3.17}$$

is a positive operator matrix then,

$$\langle Ax,Ax\rangle^2 \leq \langle PAx,Ax\rangle \langle Qx,x\rangle \leq \|P\| \langle Ax,Ax\rangle \langle Qx,x\rangle\,, \tag{3.18}$$

from which we get that $A^*A \le \|P\|\, Q$ and $\|A\|^2 \le \|P\|\,\|Q\|$.

Notice that the matrix,

$$\begin{bmatrix} 1 & a \\ a^* & a^*a \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix} \ge 0, \tag{3.19}$$

and since φ is 2-positive

$$\varphi\left(\begin{bmatrix} 1 & a \\ a^* & a^*a \end{bmatrix}\right) = \begin{bmatrix} \varphi(1) & \varphi(a) \\ \varphi(a)^* & \varphi(a^*a) \end{bmatrix} \ge 0. \tag{3.20}$$

From our observation in the first paragraph $\varphi(a)^*\varphi(a) \le \|\varphi(1)\|\, \varphi(a^*a)$.

Next consider the matrix

$$\begin{bmatrix} a^*a & a^*b \\ b^*a & b^*b \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \ge 0. \tag{3.21}$$

The 2-positivity of φ implies that the matrix,

$$\begin{bmatrix} \varphi(a^*a) & \varphi(a^*b) \\ \varphi(b^*a) & \varphi(b^*b) \end{bmatrix} = \begin{bmatrix} \varphi(a^*a) & \varphi(a^*b) \\ \varphi(a^*b)^* & \varphi(b^*b) \end{bmatrix} \ge 0, \tag{3.22}$$

and so $\varphi(a^*b)^*\varphi(a^*b) \le \|\varphi(a^*a)\|\, \varphi(b^*b)$ and $\|\varphi(a^*b)\|^2 \le \|\varphi(a^*a)\|\, \|\varphi(b^*b)\|$.

3.5 Let $\mathcal{A}$ be a C^*-algebra with unit. Show that the maps $\mathrm{Tr}\,, \sigma : M_n(\mathcal{A}) \to \mathcal{A}$ defined by $\mathrm{Tr}\,((a_{i,j})) = \sum_{i=1}^n a_{i,i}$, and $\sigma((a_{i,j})) = \sum_{i,j=1}^n a_{i,j}$ are completely positive maps. Deduce that if $\|(a_{i,j})\| \le 1$, then $\left\|\sum_{i,j=1}^n a_{i,j}\right\| \le n$.

Solution: We identify $\mathcal{A}$ with a C^*-subalgebra of $B(\mathcal{H})$. Let E_j be the operator in $B(\mathcal{H}, \mathcal{H}^{(n)})$ defined by $E_j(h) = (0, \ldots, h, \ldots, 0)$ where the h appears in the j-th entry. Let $A = (A_{i,j})_{i,j=1}^n$ be an element of $M_n(B(\mathcal{H}))$. A calculation done in Exercise 1.2.iii shows that

$$E_i^* A E_j = A_{i,j} \tag{3.23}$$

and so

$$\mathrm{Tr}\,(A) = \sum_{i=1}^n A_{i,i} = \sum_{i=1}^n E_i^* A E_i, \tag{3.24}$$

which is completely positive.

Similarly,

$$\sigma(A) = \sum_{i,j}^n E_i^* A E_j = \sum_{i=1}^n \sum_{j=1}^n E_i^* A E_j \tag{3.25}$$

$$= (\sum_{i=1}^n E_i^*) A (\sum_{j=1}^n E_j) = (\sum_{i=1}^n E_i)^* A (\sum_{j=1}^n E_j), \tag{3.26}$$

which proves that σ is completely positive.

3.6 (Choi) Let $\mathcal{A}$ be a C^*-algebra, let λ be a complex number with $|\lambda| = 1$, let U_λ be the unitary element of $M_n(\mathcal{A})$ that is diagonal with $u_{i,i} = \lambda^i 1$, and let $\mathrm{Diag} : M_n(\mathcal{A}) \to M_n(\mathcal{A})$ be defined by $\mathrm{Diag}((a_{i,j})) = (b_{i,j})$, where $b_{i,j} = 0$, for $i \ne j$ and $b_{i,i} = a_{i,i}$.

(i) Show that $U_\lambda^*(a_{i,j})U_\lambda = (\lambda^{j-i} a_{i,j})$.

(ii) By considering the non-trivial n-th roots of unity, show that the map $\Phi : M_n(\mathcal{A}) \to M_n(\mathcal{A})$ defined by $\Phi(A) = n\mathrm{Diag}(A) - A$ is completely positive.

(iii) Show that the map $\Phi : M_n \to M_n$ defined by $\Phi(A) = (n-1)\mathrm{Diag}(A) - A$ is not positive.

Solution:

(i) Recall that if $D = \mathrm{diag}(d_1, \ldots, d_n)$ then, $D(a_{i,j}) = (d_i a_{i,j})$ and $(a_{ij})D = (a_{i,j}d_j)$. Note that if $|\lambda| = 1$, then $\overline{\lambda} = \lambda^{-1}$. Thus,

$$U_\lambda^*(a_{i,j})U_\lambda = (\overline{\lambda^i} a_{i,j}\lambda^j) = (\lambda^{-i}\lambda^j a_{i,j}) = (\lambda^{j-i}a_{i,j}). \tag{3.27}$$

(ii) Let $A = (a_{i,j})$ and $\omega = e^{2\pi i/n}$, which is an n-th root of unity. Note that $\omega, \ldots, \omega^{n-1}$ are precisely the non-trivial n-th roots of unity. Note that

$$\sum_{k=0}^{n-1} \omega^{kl} = \begin{cases} n & \text{if } l = 0 \\ \dfrac{1 - e^{2\pi l i}}{1 - e^{2\pi l i/n}} = 0 & \text{if } 1 \le l \le n-1 \end{cases}. \tag{3.28}$$

We claim that $\sum_{k=1}^{n-1} U_{\omega^k}^* A U_{\omega^k} = n\mathrm{Diag}(A) - A$. Noting that U_1 is the identity matrix, we have,

$$A + \sum_{k=1}^{n-1} U_{\omega^k}^* A U_{\omega^k} = \sum_{k=0}^{n-1} U_{\omega^k}^* A U_{\omega^k} = \sum_{k=0}^{n-1} (\omega^{k(j-i)} a_{i,j}) \tag{3.29}$$

$$= (\sum_{k=0}^{n-1} \omega^{k(j-i)} a_{i,j}) = \mathrm{diag}(a_{1,1}, \ldots, a_{n,n}), \tag{3.30}$$

which proves our claim. Since any map of the form $A \mapsto X^*AX$ is completley positive and sums of completely positive maps are completely positive we see that $A \mapsto n\mathrm{Diag}(A) - A$ is completely positive.

(iii) Consider the matrix A which has 1 in every entry. Note that A is positive and that,

$$(n-1)\mathrm{Diag}(A) - A = \begin{bmatrix} n-2 & -1 & \cdots & -1 \\ -1 & n-2 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-2 \end{bmatrix}. \tag{3.31}$$

Now,

$$\left\langle \begin{bmatrix} n-2 & -1 & \cdots & -1 \\ -1 & n-2 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right\rangle = -n, \tag{3.32}$$

which shows that $(n-1)\mathrm{Diag}(A) - A$ is not positive, and hence is not completely positive.

3.7 Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras with unit and let $\varphi_1, \varphi_2 : \mathcal{A} \to \mathcal{B}$ be bounded linear maps with $\varphi_1 \pm \varphi_2$ completely positive. Prove that $\|\varphi_2\|_{cb} \le \|\varphi_1(\mathbf{1})\|$.

Solution: Define $\varphi_+ = \varphi_1 + \varphi_2$ and $\varphi_- = \varphi_1 - \varphi_2$. Note that $\varphi_1 = \frac{1}{2}(\varphi_+ + \varphi_-)$ and so φ_1 is completely positive. Let $A \in M_n(\mathcal{A})$ with $\|A\| \leq$ and note (by Lemma 3.1.i) that

$$\begin{bmatrix} \mathbf{1} & A \\ A^* & \mathbf{1} \end{bmatrix} \text{ and } \begin{bmatrix} \mathbf{1} & -A \\ -A^* & \mathbf{1} \end{bmatrix}, \tag{3.33}$$

are positive. Since φ_+ and φ_- are completely positive we have,

$$0 \leq (\varphi_+)_{2n}\left(\begin{bmatrix} \mathbf{1} & A \\ A^* & \mathbf{1} \end{bmatrix}\right) + (\varphi_-)_{2n}\begin{bmatrix} \mathbf{1} & -A \\ -A^* & \mathbf{1} \end{bmatrix} = 2\begin{bmatrix} (\varphi_1)_n(\mathbf{1}) & (\varphi_2)_n(A) \\ (\varphi_2)_n(A)^* & (\varphi_1)_n(\mathbf{1}) \end{bmatrix}. \tag{3.34}$$

Applying the result of exercise 3.2.iv we get $\|(\varphi_2)_n(A)\| \leq \|(\varphi_1)_n(\mathbf{1})\| = \|\varphi_1(\mathbf{1})\|$ since φ_1 is completely positive. Therefore $\|\varphi_2\|_{cb} \leq \|\varphi_1(\mathbf{1})\|$

3.8 Let $\mathcal{A}$ be a C^*-algebra with unit. Define $T_1, T_2 : M_n(\mathcal{A}) \to M_n(\mathcal{A})$ by $T_1((a_{i,j})) = (b_{i,j})$, where $b_{i,i} = \sum_{l=1}^n a_{l,l}$, $b_{i,j} = 0$, for $i \neq j$ and $T_2((a_{i,j})) = (c_{i,j})$ where $c_{i,j} = a_{j,i}$. Fix k and l, $k \neq l$, and define $U_{k,l}^{\pm}$ to be 1 in the (k,l)-entry, ± 1 in the (l,k)-entry and 0 elsewhere.

(i) Show that

$$T_1(A) - T_2(A) = \frac{1}{2}\sum_{k \neq l} U_{k,l}^{-*} A U_{k,l}^{-}. \tag{3.35}$$

(ii) Show that

$$T_1(A) + T_2(A) = \frac{1}{2}\sum_{k \neq l} U_{k,l}^{+*} A U_{k,l}^{+} + \mathrm{Diag}(A). \tag{3.36}$$

(iii) Deduce that $T_1 \pm T_2$ are completely positive and that $\|T_2\|_{cb} \leq n$.

(iv) By considering, $\mathcal{A} = \mathbb{C}$, show that $\|T_2\|_{cb} = n$.

Solution:

(i) Consider first the case $k = 1$ and note that

$$U_{1,l}^{-*} A U_{1,l}^{-} = \begin{bmatrix} a_{l,l} & \cdots & -a_{l,1} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ -a_{1,l} & \cdots & -a_{1,1} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}, \tag{3.37}$$

where $a_{l,l}$ is in the $(1,1)$-entry, and $a_{1,1}$ is in the (l,l)-entry. Therefore,

$$\sum_{l \neq 1} U_{1,l}^{-*} A U_{1,l}^{-} = \begin{bmatrix} a_{2,2} + \ldots + a_{n,n} & -a_{2,1} & \cdots & -a_{n,1} \\ -a_{1,2} & a_{1,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1,n} & 0 & \cdots & a_{1,1} \end{bmatrix}, \tag{3.38}$$

In general we have,

$$\sum_{l\neq k} U_{k,l}^{-*} A U_{k,l}^{-} = \begin{bmatrix} a_{k,k} & \cdots & -a_{k,1} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ -a_{1,k} & \cdots & \sum_{l\neq k} a_{l,l} & \cdots & -a_{n,k} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & -a_{k,n} & \cdots & a_{k,k} \end{bmatrix}. \tag{3.39}$$

Note that for $i \neq j$ the element $-a_{j,i}$ appears twice in the (i,j)-th place, once when $l = j$ and $k = i$ and once when $l = i$ and $k = j$. In the k-th diagonal entry we get $2\sum_{l\neq k} a_{l,l}$. Thus,

$$\begin{aligned} \sum_{k=1}^{n}\sum_{l\neq k} U_{k,l}^{-*} A U_{k,l}^{-} &= \begin{bmatrix} 2\sum_{l\neq 1} a_{l,l} & -2a_{2,1} & \cdots & -2a_{n,1} \\ -2a_{1,2} & 2\sum_{l\neq 2} a_{l,l} & \cdots & -2a_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ -2a_{1,n} & -2a_{2,n} & \cdots & 2\sum_{l\neq n} a_{l,l} \end{bmatrix} && (3.40) \\ &= 2(T_1(A) - T_2(A)) && (3.41) \\ & && (3.42) \end{aligned}$$

(ii) Using similar ideas to the those used in the previous part we have that,

$$\begin{aligned} \sum_{k=1}^{n}\sum_{l\neq k} U_{k,l}^{+*} A U_{k,l}^{+} &= \begin{bmatrix} 2\sum_{l\neq 1} a_{l,l} & 2a_{2,1} & \cdots & 2a_{n,1} \\ 2a_{1,2} & 2\sum_{l\neq 2} a_{l,l} & \cdots & 2a_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ 2a_{1,n} & 2a_{2,n} & \cdots & 2\sum_{l\neq n} a_{l,l} \end{bmatrix} && (3.43) \\ &= 2(T_1(A) + T_2(A) - \mathrm{Diag}(A)) && (3.44) \\ & && (3.45) \end{aligned}$$

(iii) Using the fact that maps of the type $A \to X^*AX$ and $A \to \mathrm{Diag}(A)$ are completely positive and that sums of completely positive maps are completely positive we see that $T_1 \pm T_2$ are completely positive. From exercise 3.7 we get that

$$\|T_2\|_{cb} \leq \|T_1(1)\| = \left\| \begin{bmatrix} n1 & 0 & \cdots & 0 \\ 0 & n1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n1 \end{bmatrix} \right\| = n. \tag{3.46}$$

(iv) Let $A = (E_{j,i})_{i,j=1}^{n} \in M_n(M_n)$, where $E_{i,j}$ are the matrix units in M_n. We proved in exercise 1.7 that A was unitary and that $\frac{1}{n}(E_{i,j})_{i,j=1}^{n} = \frac{1}{n}(T_2)_n(A)$ was a rank one projection. Thus, $\|(T_2)_n\| \geq \|(T_2)_n(A)\| = n$ which proves that $\|T_2\|_{cb} \geq n$. Combining this with the estimate from the previous part of this exercise we get $\|T_2\|_{cb} = n$.

3.9 Let $\mathcal{A}$ be a C^*-algebra and let $\mathcal{A}^{op}$ denote the set $\mathcal{A}$ with the same norm and $*$-operation, but with a multiplication defined by $a \circ b = ba$.

(i) Prove that $\mathcal{A}^{op}$ is a C^*-algebra.

(ii) Prove that M_2 and M_2^{op} are $*$-isomorphic via the transpose map.

(iii) Show that the identity map from $\mathcal{A}$ to $\mathcal{A}^{op}$ is always positive.

(iv) Prove that the identity map from M_2 to M_2^{op} is not 2-positive.

(v) (Walter) Let U, V, and X be elements of $\mathcal{A}$ with U, V unitary. Prove that

$$\begin{bmatrix} 1 & U & X \\ U^* & \mathrm{I} & V \\ X^* & V^* & 1 \end{bmatrix} \geq 0 \tag{3.47}$$

if and only if $X = UV$.

(vi) Prove that the identity map from $\mathcal{A}$ to $\mathcal{A}^{op}$ is completely positive if and only if $\mathcal{A}$ is commutative.

Solution:

i) It is straightforward to check that $\mathcal{A}$ is a $*$-algebra. The C^*-identity follows from

$$\|a^* \circ a\| = \|aa^*\| = \|(a^*)^*a^*\| = \|a^*\|^2 = \|a\|^2 . \tag{3.48}$$

ii) It is clear that $\pi : M_2 \to M_2^{op}$ defined by $\pi(A) = A^T$ is a linear map that preserves the $*$-operation. We check that this map is a homorphism:

$$\pi(AB) = (AB)^T = B^T A^T = A^T \circ B^T = \pi(A) \circ \pi(B). \tag{3.49}$$

iii) Suppose that A is positive in $\mathcal{A}$, then $A = B^*B$ for some B. Thus, $A = B \circ B^* = (B^*)^* \circ B^*$ which shows that A is positive in $\mathcal{A}^{op}$, and so the identity map is positive.

iv) Note that the map $\pi : M_2^{op} \to M_2$ is a $*$-isomorphism and is therefore completely positive. If the identity map $\mathrm{id} : M_2 \to M_2^{op}$ were 2-positive, then $\mathrm{id} \circ \pi : M_2 \to M_2$ would be 2-positive. Note that the matrix

$$A = \begin{bmatrix} E_{1,1} & E_{1,2} \\ E_{2,1} & E_{2,2} \end{bmatrix} \tag{3.50}$$

is positive in $M_2(M_2)$, but that

$$\mathrm{id} \circ \pi(A) = \begin{bmatrix} E_{1,1} & E_{2,1} \\ E_{1,2} & E_{2,2} \end{bmatrix}, \tag{3.51}$$

is not positive. This contradiction shows that id is not 2-positive.

(v) We present 2 proofs of this result.

Proof 1. Note that an element a of a C^*-algebra $\mathcal{A}$ is positive if and only if x^*ax is positive for all $x \in \mathcal{A}$ Using this fact we get,

$$\begin{bmatrix} 1 & U & X \\ U^* & 1 & V \\ X^* & V^* & 1 \end{bmatrix} \geq 0 \iff \begin{bmatrix} U^* & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & U & X \\ U^* & 1 & V \\ X^* & V^* & 1 \end{bmatrix}\begin{bmatrix} U & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \geq 0 \tag{3.52}$$

$$\iff \begin{bmatrix} 1 & 1 & U^*X \\ 1 & 1 & V \\ X^*U & V^* & 1 \end{bmatrix} \geq 0 \tag{3.53}$$

$$\iff \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & U^*X \\ 1 & 1 & V \\ X^*U & V^* & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \geq 0 \tag{3.54}$$

$$\iff \begin{bmatrix} 41 & 0 & U^*X + V \\ 0 & 0 & U^*X - V \\ X^*U + V^* & X^*U - V^* & 1 \end{bmatrix} \geq 0 \tag{3.55}$$

It follows that the lower-left 2×2 corner of this matrix must also be positive, and so

$$\begin{bmatrix} 0 & U^*X - V \\ X^*U - V & 1 \end{bmatrix} \geq 0, \tag{3.56}$$

which implies by exercise 3.2.i that $U^*X - V = 0$ or $X = UV$, since U is invertible.

Proof 2. Cholesky lemma: *Suppose that $P \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ and $A \in B(\mathcal{K}, \mathcal{H})$ where $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces. If P is positive and invertible, then the operator matrix*

$$\begin{bmatrix} P & A \\ A^* & B \end{bmatrix} \tag{3.57}$$

is positive if and only if $B - A^*P^{-1}A$ is positive.

Proof of the lemma: Assume that B is invertible .The matrix in (3.57) is positive if and only if

$$\begin{bmatrix} P^{-1/2} & 0 \\ 0 & B^{-1/2} \end{bmatrix} \begin{bmatrix} P & A \\ A^* & B \end{bmatrix} \begin{bmatrix} P^{-1/2} & 0 \\ 0 & B^{-1/2} \end{bmatrix} = \begin{bmatrix} 1 & P^{-1/2}AB^{-1/2} \\ B^{-1/2}A^*P^{-1/2} & 1 \end{bmatrix} \geq 0 \tag{3.58}$$

which by Lemma 3.1.i happens if and only if

$$B^{-1/2}A^*P^{-1/2}P^{-1/2}AB^{-1/2} \leq 1, \tag{3.59}$$

which is equivalent to

$$A^*P^{-1}A \leq B. \tag{3.60}$$

In the case where B is not invertible we consider the invertible operator $B + \varepsilon 1$, with $\varepsilon > 0$, and note that the matrix

$$\begin{bmatrix} P & A \\ A^* & B + \varepsilon 1 \end{bmatrix} \tag{3.61}$$

is positive if and only if $B + \varepsilon 1 - A^*P^{-1}A \geq 0$. We now have our result by letting $\varepsilon \to 0$.

If we apply this result to the matrix in (3.47) with $P = 1$, $B = \begin{bmatrix} 1 & V \\ V^* & 1 \end{bmatrix}$ $A = \begin{bmatrix} U & X \end{bmatrix}$ we get,

$$0 \leq \begin{bmatrix} 1 & V \\ V^* & 1 \end{bmatrix} - \begin{bmatrix} U^* \\ X^* \end{bmatrix} \begin{bmatrix} U & X \end{bmatrix} \tag{3.62}$$

$$= \begin{bmatrix} 1 - U^*U & V - U^*X \\ V^* - X^*U & 1 - X^*X \end{bmatrix} \tag{3.63}$$

$$= \begin{bmatrix} 0 & V - U^*X \\ V^* - X^*U & 1 - X^*X \end{bmatrix}. \tag{3.64}$$

Once again by exercise 3.2.i we have $V - U^*X = 0$ or $X = UV$.

(vi) Note that if $\mathcal{A}$ is commutative then $\mathcal{A}^{op} = \mathcal{A}$ and the identity map from a C^*-algebra to itself is completely positive.

For the converse we use the fact that the unitary elements span a C^*-algebra. Let U, V be two unitary elements in $\mathcal{A}$. If the identity map is completely positive then

$$\begin{bmatrix} 1 & U & UV \\ U^* & 1 & V \\ V^*U^* & V^* & 1 \end{bmatrix} \tag{3.65}$$

is positive in $\mathcal{A}^{op}$ and so $UV = U \circ V = VU$, by exercise 3.9.v. Thus, any two unitaries in $\mathcal{A}$ commute and so $\mathcal{A}$ is commutative.

4.1 Use Stinespring's representation theorem to prove that $\|V\|^2 = \|\varphi\|_{cb}$ when φ is completely positive. Also, use the representation theorem to prove that $\varphi(a)^*\varphi(a) \le \|\varphi(1)\|\,\varphi(a^*a)$

Solution: We first note by proposition 3.6 that $\|\varphi(1)\| = \|\varphi\| = \|\varphi\|_{cb}$. By Stinespring's theorem we have $\varphi(1) = V^*V$ and so $\|\varphi(1)\| = \|V^*V\| = \|V\|^2$. We also have,

$$\varphi(a)^*\varphi(a) = (V^*\pi(a)V)^*V^*\pi(a)V = V^*\pi(a)^*VV^*\pi(a)V \tag{4.1}$$
$$\le \|V\|^2\, V^*\pi(a)^*\pi(a)V = \|V\|^2\, V^*\pi(a^*a)V \tag{4.2}$$
$$= \|\varphi(1)\|\,\varphi(a^*a). \tag{4.3}$$

4.2 (Multiplicative Domains) In this exercise, we provide an alternative proof of theorem 3.19. Let $\mathcal{A}$ be a C^*-algebra with unit and let $\varphi : \mathcal{A} \to B(\mathcal{H})$ be completely positive, $\varphi(1) = 1$, with minimal Stinespring representation, $(\pi, V, \mathcal{K})$.

(i) Prove that $\varphi(a)^*\varphi(a) = \varphi(a^*a)$ if and only if $V\mathcal{H}$ is an invariant subspace for $\pi(a)$.

(ii) Use this to give an alternative proof that $\{a \in \mathcal{A} : \varphi(a)^*\varphi(a) = \varphi(a^*a)\} = \{a \in \mathcal{A} : \varphi(ba) = \varphi(b)\varphi(a) \text{ for all } b \in \mathcal{A}\}$. Recall that this set is the *right multiplicative domain* of φ.

(iii) Similarly show that $\varphi(a)^*\varphi(a) = \varphi(a^*a)$ and $\varphi(a)\varphi(a)^* = \varphi(aa^*)$ if and only if $V\mathcal{H}$ is a reducing subspace for $\pi(a)$. Deduce that the set of such elements is a C^*-subalgebra of $\mathcal{A}$. Recall that this subalgebra is the *multiplicative domain* of φ.

Solution: Note that since φ is unital, V is an isometry and $V\mathcal{H}$ is a closed subspace of $\mathcal{K}$. The fact that $\varphi(a) = V^*\pi(a)V$ shows us that, relative to the decomposition $\mathcal{K} = V\mathcal{H} \oplus (V\mathcal{H})^\perp$, $\pi(a)$ is the operator matrix

$$\pi(a) = \begin{bmatrix} \varphi(a) & A(a) \\ B(a) & C(a) \end{bmatrix}. \tag{4.4}$$

(i) Observe that the invariance of $V\mathcal{H}$ under $\pi(a)$ is equivalent to the requirement that $B(a) = 0$. As π is a $*$-homomorphism we know that $\pi(a^*a) = \pi(a)^*\pi(a)$ or in terms of the $(1,1)$ entry of the corresponding operator matrices

$$\varphi(a^*a) = \varphi(a)^*\varphi(a) + B(a)^*B(a). \tag{4.5}$$

It is clear from this that $\varphi(a^*a) = \varphi(a)^*\varphi(a)$ if and only if $B(a)^*B(a) = 0$ if and only if $B(a) = 0$.

(ii) If $\varphi(ba) = \varphi(b)\varphi(a)$ for all $b \in \mathcal{A}$ then by setting $b = a^*$ and noting that φ is self-adjoint we see that $\varphi(a^*a) = \varphi(a)^*\varphi(a)$. Conversely assume that $\varphi(a^*a) = \varphi(a)^*\varphi(a)$ and that $b \in \mathcal{A}$. By equating the $(1,1)$ entries of $\pi(ba)$ and $\pi(b)\pi(a)$ we get,

$$\varphi(ba) = \varphi(b)(\varphi(a) + A(b)B(a) = \varphi(b)\varphi(a). \tag{4.6}$$

(iii) From part 4.2.(i). we have that $\varphi(a)^*\varphi(a) = \varphi(a^*a)$ and $\varphi(a)\varphi(a)^* = \varphi(aa^*)$ if and only if $V\mathcal{H}$ is invariant under both $\pi(a)$ and $\pi(a^*)$ if and only if $V\mathcal{H}$ is reducing for $\pi(a)$.

Note that a is in the multiplicative domain of φ if and only if $V\mathcal{H}$ is reducing which happens if and only if the $(2,1)$ entry of both $\pi(a)$ and $\pi(a^*)$ are 0. This is the collection of $\pi(a)$ that are diagonal, which is a C^*-algebra. Since the multiplicative domain is the inverse image of this set under π we see that it is a C^*-subalgebra of $\mathcal{A}$.

4.3 (Bimodule Maps) Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be C^*-algebras with unit and suppose that $\mathcal{C}$ is contained in both $\mathcal{A}$ and $\mathcal{B}$ with $\mathbf{1}_{\mathcal{C}} = \mathbf{1}_{\mathcal{A}}$ and $\mathbf{1}_{\mathcal{C}} = \mathbf{1}_{\mathcal{B}}$. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is called a $\mathcal{C}$-bimodule map if $\varphi(c_1 a c_2) = c_1 \varphi(a) c_2$ for all c_1, c_2 in $\mathcal{C}$. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be completely positive.

(i) If $\varphi(\mathbf{1}) = \mathbf{1}$, the prove that φ is a $\mathcal{C}$-bimodule map if and only if $\varphi(c) = c$ for all c in $\mathcal{C}$.

(ii) Prove, in general, that φ is a $\mathcal{C}$-bimodule map if and only if $\varphi(c) = c\varphi(\mathbf{1})$ for all c in $\mathcal{C}$. Moreover, in this case, $\varphi(\mathbf{1})$ commutes with $\mathcal{C}$.

Solution:

(i) See 4.3.(ii).

(ii) We give a proof of part (ii), and part (i) will follow as a special case. Assume first that φ is a $\mathcal{C}$-bimodule map. We have,

$$\varphi(c) = \varphi(c\mathbf{1}^2) = c\varphi(\mathbf{1})\mathbf{1} = c\varphi(\mathbf{1}). \tag{4.7}$$

Similarly $\varphi(c) = \varphi(\mathbf{1})c$.

For the converse assume $\varphi(c) = c\varphi(\mathbf{1})$ for all $c \in \mathcal{C}$. Note that by taking adjoints we get $\varphi(c^*) = \varphi(\mathbf{1})c^*$. Since $\mathcal{C}$ is self-adjoint it follows that $\varphi(c) = \varphi(\mathbf{1})c$ for all $c \in \mathcal{C}$. We assume that $\mathcal{B} = B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. Let π be a Stinespring dilation of φ on a Hilbert space $\mathcal{K}$ and let $V : \mathcal{H} \to \mathcal{K}$ be the associated linear operator. We will adopt the notation used in Theorem 4.1 (Stinespring's Dilation Theorem). We note that $\mathcal{C}$ is a C^*-subalgebra of $B(\mathcal{H})$ and begin by proving that $Vc = \pi(c)V$. Let $h \in \mathcal{H}$ and note that $Vc(h) = \mathbf{1} \otimes c(h) + \mathcal{N}$ and $\pi(c)V(h) = \pi(c)[\mathbf{1} \otimes h + \mathcal{N}] = c \otimes h + \mathcal{N}$. Thus equality will follow if we show that $\mathbf{1} \otimes c(h) - c \otimes h \in \mathcal{N}$. We have,

$$\begin{aligned}
\langle \mathbf{1} \otimes c(h) - c \otimes h, \mathbf{1} \otimes c(h) - c \otimes h \rangle &= \langle \mathbf{1} \otimes c(h), \mathbf{1} \otimes c(h) \rangle + \langle c \otimes h, c \otimes h \rangle && (4.8)\\
&\quad - \langle c \otimes h, \mathbf{1} \otimes ch \rangle - \langle \mathbf{1} \otimes c(h), c \otimes h \rangle && (4.9)\\
&= \langle \varphi(\mathbf{1})c(h), c(h) \rangle + \langle \varphi(c^*c)h, h \rangle && (4.10)\\
&\quad - \langle \varphi(c)h, c(h) \rangle - \langle \varphi(c^*)c(h), h \rangle && (4.11)\\
&= \langle \varphi(\mathbf{1})c(h), c(h) \rangle + \langle \varphi(\mathbf{1})c^*c(h), h \rangle && (4.12)\\
&\quad - \langle \varphi(\mathbf{1})c(h), c(h) \rangle - \langle \varphi(\mathbf{1})c^*c(h), h \rangle = 0 && (4.13)
\end{aligned}$$

Note that by taking adjoints we have $cV^* = V^*\pi(c)$ for all $c \in \mathcal{C}$. Now,

$$\begin{aligned}
\varphi(c_1 a c_2) &= V^*\pi(c_1 a c_2)V = V^*\pi(c_1)\pi(a)\pi(c_2)V && (4.14)\\
&= c_1 V^*\pi(a)V c_2 = c_1\varphi(a)c_2. && (4.15)
\end{aligned}$$

4.4 Let $\mathcal{D}_n$ be the C^*-subalgebra of diagonal matrices in M_n. Prove that a linear map $\varphi : M_n \to M_n$ is a $\mathcal{D}_n$-bimodule map if and only if φ is the Schur product map, S_T, for some matrix T.

Solution: Let J be the matrix all of whose entries are 1 and let $T = (t_{i,j}) = \varphi(J)$. Let $\{E_{i,j}\}_{i,j=1}^n$ be the set of matrix units for M_n. We have,

$$\varphi(E_{i,j}) = \varphi(E_{j,j} J E_{i,i}) = E_{j,j}\varphi(J)E_{i,i} = t_{i,j}E_{i,j} = T \circ E_{i,j}. \tag{4.16}$$

By linearity φ must be the Schur product map S_T.

5.1 Prove that if $(V_1, \mathcal{K}_1)$ and $(V_2, \mathcal{K}_2)$ are two minimal isometric dilations of a contraction operator T on $\mathcal{H}$, then there exists a unitary $U : \mathcal{K}_1 \to \mathcal{K}_2$ such that $Uh = h$ for all $h \in \mathcal{H}$ and $UV_1U^* = V_2$.

Solution: Since $(V_j, \mathcal{K}_j)$, $j = 1, 2$ are minimal isometric dilations we know that

$$\mathcal{H}_j = \{V_j^n h \, : \, h \in \mathcal{H}, n \geq 0\} \tag{5.1}$$

is dense in $\mathcal{K}_j$. Define a map $U : \mathcal{H}_1 \to \mathcal{H}_2$ by

$$U\left(\sum_{k=-N}^{N} V_1^k h_k\right) = \sum_{k=-N}^{N} V_2^k h_k. \tag{5.2}$$

We claim that U is an isometry. It follows from this that U has a unique extension to an isometry from $\mathcal{K}_1 \to \mathcal{K}_2$. We will denote this extension by U. As the range of U is dense in $\mathcal{K}_2$, U is unitary. Since V_j is a dilation of T on $\mathcal{K}_j$,

$$V_j^n = \begin{bmatrix} T^n & * \\ * & * \end{bmatrix}, \tag{5.3}$$

relative to the decomposition $\mathcal{K}_j = \mathcal{H} \oplus \mathcal{H}^\perp$. It follows that if $h, k \in \mathcal{H}$ then $\left\langle V_j^n h, k \right\rangle = \langle T^n h, k\rangle$. We have,

$$\begin{aligned}
\left\| U\left(\sum_{k=-N}^{N} V_1^k h_k\right)\right\|^2 &= \sum_{k,l=-N}^{N} \left\langle V_2^k h_k, V_2^l h_l \right\rangle && (5.4)\\
&= \sum_{k<l} \left\langle V_2^k h_k, V_2^l h_l \right\rangle + \sum_{k\geq l} \left\langle V_2^k h_k, V_2^l h_l \right\rangle && (5.5)\\
&= \sum_{k<l} \left\langle h_k, V_2^{l-k} h_l \right\rangle + \sum_{k\geq l} \left\langle V_2^{k-l} h_k, h_l \right\rangle && (5.6)\\
&= \sum_{k<l} \left\langle h_k, T^{l-k} h_l \right\rangle + \sum_{k\geq l} \left\langle T^{k-l} h_k, h_l \right\rangle && (5.7)\\
&= \sum_{k<l} \left\langle h_k, V_1^{l-k} h_l \right\rangle + \sum_{k\geq l} \left\langle V_1^{k-l} h_k, h_l \right\rangle && (5.8)\\
&= \left\| \left(\sum_{k=-N}^{N} V_1^k h_k\right)\right\|^2. && (5.9)
\end{aligned}$$

Now,

$$\begin{aligned}
UV_1\left(\sum_{k=-N}^{N} V_1^k h_k\right) &= \left(\sum_{k=-N}^{N} V_2^{k+1} h_k\right) && (5.10)\\
&= V_2\left(\sum_{k=-N}^{N} V_2^k h_k\right) && (5.11)\\
&= V_2 U\left(\sum_{k=-N}^{N} V_1^k h_k\right). && (5.12)
\end{aligned}$$

Therefore $UV_1 = V_2U$ on the dense subspace $\mathcal{H}_j$. It follows by a limit argument that $UV_1 = V_2U$ or $UV_1U^* = V_2$.

5.2 Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, let $A_n \in B(\mathcal{H}_1, \mathcal{H}_2)$ be a sequence of operators and let $A = (A_{i+j}) : \ell^2(\mathcal{H}_1) \to \ell^2(\mathcal{H}_2)$ be the corresponding Hankel operator. Prove the analogue of Nehari-Page in this setting.

Solution: Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $A = (A_{i+j})$ and let

$$\tilde{A}_n = \begin{bmatrix} 0 & 0 \\ A_n & 0 \end{bmatrix} \in B(\mathcal{H}). \tag{5.13}$$

By the Nehari-Page theorem the Hankel matrix $\tilde{A} = (\tilde{A}_{i+j})$ is bounded if and only if there exists

$$\tilde{A}_n = \begin{bmatrix} * & * \\ A_n & * \end{bmatrix} \tag{5.14}$$

for $n < 0$ such that $\|\tilde{B}\|_\infty = \sup_{0<r<1} \|\tilde{B}_r\|_\infty < \infty$ where $\tilde{B}_r(e^{i\theta}) = \sum_{n=-\infty}^{\infty} \tilde{A}_n r^{|n|} e^{in\theta}$. Further $\|(\tilde{A}_{i+j})\| = \|\tilde{B}\|_\infty$. Let $B = \sum_{n=-\infty}^{\infty} A_n e^{in\theta}$. Since B_r is a compression of $\tilde{B}_r$ we have that

$$\|B\| = \sup_{0<r<1} \|B_r\| \leq \sup_{0<r<1} \|\tilde{B}_r\| = \|\tilde{B}\| < \infty. \tag{5.15}$$

All that remains to be shown is that for this choice of A_n, $\|(A_{i+j})\| = \|B\|_\infty$. We have already proven that $\|B\|_\infty \leq \|\tilde{B}\|_\infty$. We can identify the Hilbert spaces $\ell^2(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and $\ell^2(\mathcal{H}_1) \oplus \ell^2(\mathcal{H}_2)$ by the map $(h_n^{(1)} \oplus h_n^{(2)}) \mapsto (h_n^{(1)}) \oplus (h_n^{(2)})$. The operator $\tilde{A}$ is identified with the operator $\begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}$ and so $\|\tilde{A}\| = \|A\|$. Since A is a compression of the multiplication operator M_B we always have $\|(A_{i+j})\| \leq \|M_B\| = \|B\|_\infty$.

5.3 (Caratheodory's Completion Theorem) Let $a_0, \ldots, a_n$ be in $\mathbb{C}$. Use commutant lifting to prove that

$$\left\| \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_n & \cdots & a_1 & a_0 \end{bmatrix} \right\| = \inf \left\| \sum_{j=0}^{n} a_j z^j + \sum_{j=n+1}^{\infty} b_j z^j \right\|_\infty, \tag{5.16}$$

where the infinum is over sequences $\{b_j\}$ such that the resulting power series is bounded on $\mathbb{D}$ and the ∞-norm is the supremum over $\mathbb{D}$. Moreover, there exists a sequence $\{b_j\}$ where the infimum is attained. Thus, a polynomial can be completed to a power series whose supremum over the disk is bounded by 1 by adding higher order terms if and only if the norm of the corresponding Toeplitz matrix is at most 1.

Deduce that the map

$$f \in H^\infty(\mathbb{T}) \mapsto \begin{bmatrix} \widehat{f}(0) & 0 & \cdots & 0 \\ \widehat{f}(1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \widehat{f}(n) & \cdots & \widehat{f}(1) & \widehat{f}(0) \end{bmatrix} \in M_{n+1}, \tag{5.17}$$

yields an isometric isomorphism of $H^\infty(\mathbb{T})/e^{i(n+1)\theta} H^\infty(\mathbb{T})$ into M_{n+1}. Generalize this to the case where $A_0, \ldots, A_n$ are operators in a (separable) Hilbert Space.

Solution: Denote the matrix on the left side of (5.16) by R and let

$$S = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \in M_{n+1}. \tag{5.18}$$

Note first that the minimal unitary dilation of S is the operator U on $L^2(\mathbb{T})$ given by $(Uf)(e^{i\theta}) = e^{i\theta} f(e^{i\theta})$. We have

$$SR = RS = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ a_0 & \ddots & & 0 \\ \vdots & \ddots & \ddots & 0 \\ a_{n-1} & \cdots & a_0 & 0 \end{bmatrix}, \tag{5.19}$$

and so by the commutant lifting theorem there is an operator V commuting with U, with $\|V\| = \|R\|$ such that $RS^m = P_{\mathcal{H}} V U^m|_{\mathcal{H}}$. Since V commutes with U, V is multiplication by $g \in L^\infty$. If $m < 0$, then,

$$\left\langle g, e^{im\theta} \right\rangle = \langle V1, U^m 1\rangle = \langle VU^{-m}1, 1\rangle = \langle P_{\mathcal{H}} VU^{-m}1, 1\rangle = \langle RS^{-m}1, 1\rangle = 0. \tag{5.20}$$

Therefore, $g(e^{i\theta}) = \sum_{k=0}^{\infty} a_k e^{ik\theta}$. Let T_g denote the Toeplitz operator associated with g acting on $H^2(\mathbb{T})$. We have,

$$\|R\| = \|V\| = \|T_g\| = \|g\|_{H^\infty(\mathbb{T})} = \|g\|_{H^\infty(\mathbb{D})} = \sup_{z\in\mathbb{D}} \left| \sum_{j=0}^{\infty} a_j z^j \right|. \tag{5.21}$$

Let $\{b_j\}$ be a sequence of complex numbers such that $h(z) = \sum_{j=0}^{n} a_j z^j + \sum_{j=n+1}^{\infty} b_j z^j$ defines a bounded function on $\mathbb{D}$. Let T_h be the corresponding Toeplitz operator on $H^2(\mathbb{D})$ and note that R is the compression of T_h to the subspace spanned by $1, z, \ldots, z^n$. Therefore $\|R\| \leq \|T_h\|$. Combining this with (5.21) we see that,

$$\|R\| = \inf \left\| \sum_{j=0}^{n} a_j z^j + \sum_{j=n+1}^{\infty} b_j z^j \right\|_\infty. \tag{5.22}$$

Let $f, g \in H^\infty(\mathbb{T})$, let $\Phi : H^\infty(\mathbb{T}) \to M_{n+1}$ be the map

$$f \in H^\infty(\mathbb{T}) \mapsto \begin{bmatrix} \widehat{f}(0) & 0 & \cdots & 0 \\ \widehat{f}(1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \widehat{f}(n) & \cdots & \widehat{f}(1) & \widehat{f}(0) \end{bmatrix} \in M_{n+1}. \tag{5.23}$$

Since $\widehat{f+g}(k) = \widehat{f}(k) + \widehat{g}(k)$ and $\widehat{\alpha f}(k) = \alpha \widehat{f}(k)$ we see that Φ is linear. To check that Φ is in fact a homomorphism we take note of the fact that $\widehat{fg}(k) = \sum_{j=0}^{k} \widehat{f}(j)\widehat{g}(k-j)$ and compute,

$$\Phi(f)\Phi(g) = \begin{bmatrix} \widehat{f}(0) & 0 & \cdots & 0 \\ \widehat{f}(1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \widehat{f}(n) & \cdots & \widehat{f}(1) & \widehat{f}(0) \end{bmatrix} \begin{bmatrix} \widehat{g}(0) & 0 & \cdots & 0 \\ \widehat{g}(1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \widehat{g}(n) & \cdots & \widehat{g}(1) & \widehat{g}(0) \end{bmatrix} \tag{5.24}$$

$$= \begin{bmatrix} \widehat{f}(0)\widehat{g}(0) & 0 & \cdots & 0 \\ \widehat{f}(1)\widehat{g}(0) + \widehat{f}(0)\widehat{g}(1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \widehat{f}(n)\widehat{g}(0) + \ldots + \widehat{f}(0)\widehat{g}(n) & \cdots & \widehat{f}(1)\widehat{g}(0) + \widehat{f}(0)\widehat{g}(1) & \widehat{f}(0)\widehat{g}(0) \end{bmatrix} \tag{5.25}$$

$$= \Phi(fg) \tag{5.26}$$

The kernel of Φ is the set of functions $f \in H^\infty(\mathbb{T})$ such that $\widehat{f}(0) = \ldots = \widehat{f}(n) = 0$, which is precisely $e^{i(n+1)\theta}H^\infty(\mathbb{T})$. Therefore, $H^\infty(\mathbb{T})/e^{i(n+1)\theta}H^\infty(\mathbb{T})$ is isomorphic, via $\tilde{\Phi}(f + e^{i(n+1)\theta}H^\infty(\mathbb{T})) = \Phi(f)$, to a subalgebra of M_{n+1}.

Finally,

$$\left\|\tilde{\Phi}(f)\right\| = \inf\left\{\|f+g\|_\infty : g \in e^{i(n+1)\theta}H^\infty(\mathbb{T})\right\} \tag{5.27}$$

$$= \inf\left\{\left\|\sum_{j=0}^{n} \widehat{f}(j)z^j + \sum_{j=n+1}^{\infty} b_j z^j\right\|_\infty : \sum_{j=n+1}^{\infty} b_j z^j \in H^\infty(\mathbb{D})\right\} \tag{5.28}$$

$$= \left\|\begin{bmatrix} \widehat{f}(0) & 0 & \cdots & 0 \\ \widehat{f}(1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \widehat{f}(n) & \cdots & \widehat{f}(1) & \widehat{f}(0) \end{bmatrix}\right\| \tag{5.29}$$

5.4 Let $\{T_1, \ldots, T_n\}$ be contractions on a Hilbert Space $\mathcal{H}$ (possibly non-commuting). Prove that there exits a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and unitaries $\{U_1, \ldots, U_n\}$ on $\mathcal{K}$ such that

$$T_{i_1}^{k_1} \ldots T_{i_m}^{k_m} = P_\mathcal{H} U_{i_1}^{k_1} \ldots U_{i_m}^{k_m}|_\mathcal{H}, \tag{5.30}$$

where $m, k_1, \ldots, k_m$ are arbitrary non-negative integers, and $1 \le i_l \le n$, for $l = 1, \ldots, n$.

Solution: We begin by recalling the construction used in Bz.-Nagy's dilation theorem. Given a contraction T on a Hilbert space we construct an isometric dilation V on $\ell^2(\mathcal{H})$ by defining

$$V(h_1, h_2, \ldots) = (Th_1, (\mathbf{1} - T^*T)^{1/2}h_1, h_2, h_3, \ldots). \tag{5.31}$$

We can dilate an isometry V on a Hilbert space $\mathcal{H}$ to a unitary U on $\mathcal{H} \oplus \mathcal{H}$ by

$$U = \begin{bmatrix} V & \mathbf{1} - V^*V \\ 0 & V^* \end{bmatrix}. \tag{5.32}$$

If we combine these two constructions we see that a contraction T on $\mathcal{H}$ can be dilated to a unitary U on $\ell^2(\mathcal{H}) \oplus \ell^2(\mathcal{H})$.

Given a collection of contractions $T_1, \ldots, T_n$ dilate them as above to $U_1, \ldots, U_n$. Let $h \in \mathcal{H}$ and note that,

$$U_j^k((0,0,\ldots) \oplus (h,0,\ldots)) = (0,0,\ldots) \oplus (T_j^k h, *, *, \ldots). \tag{5.33}$$

It follows from this that,

$$U_{i_1}^{k_1} \ldots U_{i_m}^{k_m}((0,0,\ldots) \oplus (h,0,\ldots)) = (0,0,\ldots) \oplus (T_{i_1}^{k_1} \ldots T_{i_m}^{k_m} h, *, *, \ldots). \tag{5.34}$$

Therefore $T_{i_1}^{k_1} \ldots T_{i_m}^{k_m} h = P_{\mathcal{H}} U_{i_1}^{k_1} \ldots U_{i_m}^{k_m} h$.

5.5 (Schaeffer) Let T be a contraction on a Hilbert space $\mathcal{H}$, let $\ell^2_{\mathbf{Z}}(\mathcal{H}) = \sum_{n=-\infty}^{\infty} \oplus \mathcal{H}$ denote the Hilbert space formed as a direct sum of copies of $\mathcal{H}$ indexed by the integers $\mathbb{Z}$. Define an operator matrix $U = (U_{i,j})$ by setting $U_{0,0} = T$, $U_{0,1} = (\mathbf{1} - TT^*)^{1/2}$, $U_{-1,0} = (\mathbf{1} - T^*T)^{1/2}$, $U_{-1,1} = T^*$, $U_{n,n+1} = \mathbf{1}$, for $n \geq 1$ or $n \leq -2$ and $U_{i,j} = 0$ for all other pairs (i,j). Prove that U defines a unitary operator on $\ell^2_{\mathbf{Z}}(\mathcal{H})$ and that if we identify $\mathcal{H}$ with the 0-th copy of $\mathcal{H}$ in $\ell^2_{\mathbf{Z}}(\mathcal{H})$, then $T^n = P_{\mathcal{H}} U^n|_{\mathcal{H}}$ for all non-negative integers n.

Solution: We can check by a direct calcuation that all the diagonal entries of UU^* and U^*U are $\mathbf{1}$, and that all other entries, except the $(-1,0)$ and $(0,-1)$ entries are 0. We find that both the $(-1,0)$ entry of UU^* and the $(0,-1)$ entry of U^*U are $(\mathbf{1}-T^*T)^{1/2}T^* - T^*(\mathbf{1}-TT^*)^{1/2}$ while both the $(0,-1)$ entry of UU^* and $(-1,0)$ entry of U^*U are $T(\mathbf{1}-T^*T)^{1/2} - (\mathbf{1}-TT^*)^{1/2}T$.

Thus to prove that U is unitary we need to check that $T(\mathbf{1} - T^*T)^{1/2} = (\mathbf{1} - TT^*)^{1/2}T$. Since T is a contraction $\mathbf{1} - T^*T$ and $\mathbf{1} - TT^*$ are positive operators whose spectrum is contained in the interval $[0,1]$. Let $f : [0,1] \to \mathbb{R}$ denote the function $f(t) = \sqrt{1-t}$. Choose a sequence p_n of polynomials defined on the interval $[0,1]$ such that $\|p_n - f\|_\infty \to 0$ as $n \to \infty$. Note then that $Tp_n(T^*T) = p_n(TT^*)T$ and so via the continuous functional calculus,

$$T(\mathbf{1} - T^*T)^{1/2} = \lim_{n\to\infty} Tp_n(T^*T) = \lim_{n\to\infty} p_n(TT^*)T = (\mathbf{1} - TT^*)^{1/2}T. \tag{5.35}$$

CHAPTER 6: COMPLETELY POSITIVE MAPS INTO M_n

6.1 Let $\mathcal{A}$ be any unital C^*-algebra. Give an example of a linear functional, $s : M_n(\mathcal{A}) \to \mathbb{C}$, such that s is unital and positive, but such that the associated linear map , $\varphi_s : \mathcal{A} \to M_n$ has norm n.

Solution: We first consider the case when $\mathcal{A} = \mathbb{C}$ and the linear functional $s : M_n \to \mathbb{C}$ defined by

$$s((a_{i,j})) = \frac{1}{n} \sum_{i,j=1}^{n} a_{i,j}. \tag{6.1}$$

We have seen in chapter 3 that this functional is positive and unital. We have,

$$\varphi_s(a)_{i,j} = ns(a \otimes E_{i,j}) = a. \tag{6.2}$$

Thus,

$$\varphi_s(a) = a \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}, \tag{6.3}$$

which implies,

$$\|\varphi_s(a)\| = a \|J\| = an, \tag{6.4}$$

and so $\|\varphi_s\| = n$.

For a general C^*-algebra $\mathcal{A}$ choose a state s_0 of this algebra and define a linear functional s by

$$s((a_{i,j})) = \frac{1}{n} \sum_{i,j=1}^{n} s_0(a_{i,j}). \tag{6.5}$$

Recall that the map $\sigma : M_n(\mathcal{A}) \to \mathcal{A}$ defined by

$$\sigma((a_{i,j})) = \sum_{i,j=1}^{n} a_{i,j}, \tag{6.6}$$

is completely positive. With this notation $s = \frac{1}{n}(s_0 \circ \sigma)$ which, being the composition of two positive maps, is positive. We have

$$s(1) = \frac{1}{n} \sum_{i,j=1}^{n} s_0(\delta_{i,j} 1) = 1. \tag{6.7}$$

$\varphi_s(a)_{i,j} = ns(a \otimes E_{i,j}) = s_0(a)$ and so

$$\varphi_s(a) = \begin{bmatrix} s_0(a) & \cdots & s_0(a) \\ \vdots & \ddots & \vdots \\ s_0(a) & \cdots & s_0(a) \end{bmatrix}. \tag{6.8}$$

It follows that $\|\varphi_s(a)\| = s_0(a)n$. Since s_0 is a state $\|s_0\| = 1$ and so $\|\varphi_s\| \leq n$. By setting $a = 1$ we get that $\|\varphi_s\| = n$.

6.2 Let $\varphi : \mathcal{S} \to M_n$ be positive and set $\varphi(1) = P$. Let Q be the projection onto the range of P and let R be positive with $(1-Q)R = 0$, $RPR = Q$. Let $\psi : \mathcal{S} \to M_n$ be any positive, unital map and set $\varphi'(a) = R\varphi(a)R + (1-Q)\psi(a)(1-Q)$.

(i) Show that φ' is a unital, positive map.

(ii) Show that is $(a_{i,j})$ is positive in $M_n(\mathcal{S})$, but $\varphi_k((a_{i,j}))$ is not positive, then $\varphi'_k((a_{i,j}))$ is not positive either.

(iii) Deduce the equivalence of i) and ii) in Theorem 6.6.

Solution:

(i) Suppose that $a \geq 0$. Since φ and ψ are positive, $\varphi(a)$ and $\psi(a)$ are positive. Since R and $(1-Q)$ are self-adjoint $R\varphi(a)R$ and $(1-Q)\psi(a)(1-Q)$ are positive and so $\varphi'(a)$ is positive. We have,

$$\begin{aligned} \varphi'(1) &= R\varphi(1)R + (1-Q)\psi(1)(1-Q) && (6.9) \\ &= RPR + (1-Q)^2 = Q + (1-Q) = 1 && (6.10) \end{aligned}$$

(ii) We claim that range $\varphi(a) \subseteq$ range P for all $a \in \mathcal{S}$. Indeed if $A, B \in B(\mathcal{H})$ are two positive operators such that $A \leq B$, then $\ker B \subseteq \ker A$. To see this assume that $Bh = 0$ and note that this implies that $\langle Ah, h\rangle = 0$. Thus,

$$0 = \langle Ah, h\rangle = \left\langle A^{1/2}h, A^{1/2}h\right\rangle = \left\|A^{1/2}h\right\|^2. \tag{6.11}$$

Hence, $Ah = A^{1/2}(A^{1/2}h) = 0$.

The fact that φ is positive implies that $\varphi(a) \leq \|a\|\, P$ for $a \geq 0$. It follows that $\ker\varphi(a) \supseteq \ker P$, which in turn implies on taking orthogonal complements that range $\varphi(a) \subseteq$ range P. Therefore, since φ is linear and every element of $\mathcal{S}$ is a sum of at most 4 positive elements, range $\varphi(x) \subseteq$ range P for all $x \in \mathcal{S}$. If $z \in \mathbb{C}^n$, then we may write $z = x + y$ with $x \in$ range P and $y \in \ker P$. Since range $\varphi(a) \subseteq$ range P we have

$$\langle\varphi(a)z, z\rangle = \langle\varphi(a)x, x\rangle\,. \tag{6.12}$$

Let $(a_{i,j}) \in M_k(\mathcal{S})$ be positive and assume that $\varphi_k((a_{i,j}))$ is not positive. By our observation in (6.12) we can choose $x_1, \ldots, x_n \in$ range P such that,

$$\sum_{i,j=1}^{k} \langle\varphi(a_{i,j})x_j, x_i\rangle \ngeq 0. \tag{6.13}$$

Set $z_j = PRx_j \in$ range P,

$$\begin{aligned} \sum_{i,j=1}^{k} \langle\varphi'(a_{i,j})z_j, z_i\rangle &= \sum_{i,j=1}^{k} \langle R\varphi(a_{i,j})Rz_j, z_i\rangle + \sum_{i,j=1}^{k} \langle((1-Q)\psi(a_{i,j})(1-Q)z_j, z_i\rangle && (6.14) \\ &= \sum_{i,j=1}^{k} \langle\varphi(a_{i,j})Rz_j, Rz_i\rangle + \sum_{i,j=1}^{k} \langle\psi(a_{i,j})(1-Q)z_j, Qz_i\rangle && (6.15) \\ &= \sum_{i,j=1}^{k} \langle\varphi(a_{i,j})Qx_j, Qx_i\rangle + \sum_{i,j=1}^{k} \langle\psi(a_{i,j})(1-Q)z_j, Qz_i\rangle && (6.16) \\ &= \sum_{i,j=1}^{k} \langle\varphi(a_{i,j})x_j, x_i\rangle \ngeq 0. && (6.17) \end{aligned}$$

(iii) Clearly (i) implies (ii) in Theorem 6.6. Now assume that every unital, posiitve map $\varphi : \mathcal{S} \longrightarrow M_n$ is completely positive and that $\psi : \mathcal{S} \longrightarrow M_n$ is a positive map. By the previous part of this exercise we can construct a unital, positive map ψ' such that ψ is k-positive whenever ψ' is k-positive. By our hypothesis ψ' is completely positive and so ψ is completely positive.

6.3 Assume that the equivalent conditions of Theorem 6.8 are not met. Show that then there always exists a unital, positive map $\varphi : \mathcal{S} \longrightarrow M_2$ which is not contractive.

Solution: If the equivalent conditons of theorem 6.8 are not met then there exists an element x of norm at most 1 and a positive map φ such that $\|\varphi(x)\| > \|\varphi(1)\|$. It follows that

$$\begin{bmatrix} \varphi(1) & \varphi(x) \\ \varphi(x)^* & \varphi(1) \end{bmatrix}, \tag{6.18}$$

is not positive. Construct a positive unital map φ' from φ as in exercise 6.2.(i) and note that

$$\begin{bmatrix} \varphi'(1) & \varphi'(x) \\ \varphi'(x)^* & \varphi'(1) \end{bmatrix}, \tag{6.19}$$

is not positive by exercise 6.2.(ii). Hence, $\|\varphi'(x)\| > \|\varphi'(1)\| = \|1\| = 1$. Thus φ' is unital and positive but not contractive.

6.5 Let $\mathcal{S}$ be an operator system. Prove that the following are equivalent:

(i) For every C^*-algebra $\mathcal{B}$, every positive $\varphi : \mathcal{S} \longrightarrow \mathcal{B}$ is n-positive.

(ii) $\mathcal{S}^+ \otimes M_n^+$ is dense in $M_n(\mathcal{S})^+$.

Solution: (i) implies (ii): Let $\mathcal{B} = M_n$ and note then that any positive map φ from $\mathcal{S}$ into M_n is n-positive. It follows from theorem 6.1 that φ is completely positive and so by Theorem 6.6 $\mathcal{S}^+ \otimes M_n^+$ is dense in $M_n(\mathcal{S})^+$.

(ii) implies (i): We may assume that $\mathcal{B} = B(\mathcal{H})$. By theorem 6.6 we know that any positive map $\varphi : \mathcal{S} \longrightarrow M_n$ is completely positive. Let $(a_{i,j})$ be positive in $M_n(\mathcal{S})$. We need to check that $\varphi_n((a_{i,j}))$ is positive in $B(\mathcal{H}^{(n)})$. Let $h_1, \ldots, h_n \in \mathcal{H}$ and let $\mathcal{F}$ be the m-dimensional subspace spanned by $\{h_1, \ldots, h_n\}$. Let $\psi : \mathcal{S} \longrightarrow B(\mathcal{F})$ be the compression of φ to $\mathcal{F}$. If we identify M_m with $B(\mathcal{F})$ and note that M_m can be identified with the subalgebra of M_n via

$$A \in M_m \mapsto \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \in M_n, \tag{6.20}$$

then we see that ψ is n-positive by ii). Therefore,

$$\sum_{i,j=1}^{n} \langle \varphi(a_{i,j})h_j, h_i \rangle = \sum_{i,j=1}^{n} \langle \psi(a_{i,j})h_j, h_i \rangle \geq 0. \tag{6.21}$$

6.6 Use corollary 6.7 to give an alternate proof of the fact that every positive map with domain $C(X)$ is completely positive.

Solution: Let $F \in M_n(C(X))^+$. Since X is compact we may choose a finite cover of open sets $\{U_j\}_{j=1}^m$ for X and sequence of points $x_j \in U_j$ such that $\|F(x) - F(x_j)\| < \varepsilon$ for all $x \in U_j$. Let p_j be a partition of unity subordinate to the cover U_j. Note that if $p_j(x) \neq 0$, then $\|F(x) - F(x_j)\| < \varepsilon$. It follows that,

$$\left\| F(x) - (\sum_{j=1}^{m} p_j \otimes F(x_j))(x) \right\| = \left\| F(x) - \sum_{j=1}^{m} p_j(x) F(x_j) \right\| \tag{6.22}$$

$$= \left\| \sum_{j=1}^{m} p_j(x)(F(x) - F(x_j)) \right\| \tag{6.23}$$

$$\leq \sum_{j=1}^{m} \|p_j(x)(F(x) - F(x_j))\| \tag{6.24}$$

$$= \sum_{j=1}^{m} p_j(x) \|F(x) - F(x_j)\| < \varepsilon \tag{6.25}$$

We have shown that $C(X)^+ \otimes M_n^+$ is dense in $M_n(C(X))^+$ and so by corollary 6.7 every positive map with domain $C(X)$ is completely positive.

7.1 Let X and Y be Banach spaces and for $f \in X^*$, $g \in Y^*$, define $L_{f,g} \in B(X,Y^*)$ by $L_{f,g}(x) = f(x)g$. By considering these operators prove that the map $j : X \otimes Y \to B(X,Y^*)^*$ defined in this chapter has the following properties:

(i) j is linear,

(ii) $\|j(x \otimes y)\| = \|x\| \, \|y\|$,

(iii) j is one-to-one.

Since j is one-to-one, the identification of $X \otimes Y$ with $j(X \otimes Y)$ endows $X \otimes Y$ with a norm (rather than just a seminorm). Conclude that Z, of Lemma 7.1, can be identified with the completion of $X \otimes Y$ with respect to this norm.

A norm on the tensor product of two normed spaces that satisifes $\|x \otimes y\| = \|x\| \, \|y\|$ is called a cross-norm.

Solution:

(i) We define $j : X \otimes Y \to B(X,Y^*)^*$ as $j(x \otimes y)(L) = L(x)(y)$ and extend this map linearly, which guarantees that j is linear.

(ii) Suppose that $L \in B(X,Y^*)$. We have,

$$|j(x \otimes y)(L)| = |L(x)(y)| \leq \|L(x)\| \, \|y\| \leq \|x\| \, \|y\| \,. \tag{7.1}$$

It follows from this that $\|j(x \otimes y)\| \leq \|x\| \, \|y\|$. To prove the reverse inequality choose linear functionals $f \in X^*$, $g \in Y^*$ such that $f(x) = \|x\|$, $g(y) = \|y\|$. Note that,

$$j(x \otimes y)(L_{f,g}) = f(x)g(y) = \|x\| \, \|y\| \,, \tag{7.2}$$

and so $\|j(x \otimes y\| = \|x\| \, \|y\|$.

(iii) Let $v \in X \otimes Y$ and suppose that $j(v) = 0$. We can write, $v = \sum_{i=1}^m x_i \otimes y_i$ where the set $\{x_1, \ldots, x_m\}$ is linearly independent. Choose linear functionals $f_j \in X^*$ such that $f_j(x_i) = \delta_{i,j}$. Let $g \in Y^*$ and consider,

$$0 = j(v)(L_{f_j,g}) = \sum_{i=1}^{m} f_j(x_i)g(y_i) = g(y_j). \tag{7.3}$$

Thus $g(y_j) = 0$ for all $g \in Y^*$ which implies that $y_j = 0$ for all $j = 1, \ldots, m$. Hence $v = 0$ and j is one-to-one.

7.3 Let $\mathcal{A}$ be an operator algebra contained in the C^*-algebra $\mathcal{B}$, let $\rho : \mathcal{A} \to B(\mathcal{H})$ be a completely contractive, unital homomorphism, and let $\pi_i : \mathcal{B} \to B(\mathcal{K}_i), i = 1,2$, define minimal $\mathcal{B}$-dilations of ρ. Define completely positive maps, $\varphi_i : \mathcal{B} \to B(\mathcal{H})$, by $\varphi_i(b) = P_{\mathcal{H}}\pi_i(b)|_{\mathcal{H}}, i = 1,2$.

(i) Show that there exists a unitary $U : \mathcal{K}_1 \to \mathcal{K}_2$ with $Uh = h$ for h in $\mathcal{H}$, and $U^*\pi_2(b)U = \pi_1(b)$ if and only if $\varphi_1 = \varphi_2$. Such dilations are called *unitarily equivalent*.

(ii) Show that there is a one-to-one correspondence between unitarily equivalent, minimal $\mathcal{B}$ dilations of ρ and completely positive extensions of ρ to $\mathcal{B}$.

(iii) Show that the set of completely positive extensions of ρ is a compact, convex set in the BW-topology on $CP(\mathcal{B}, \mathcal{H})$.

Solution:

(i) Assume first that such a unitary does exist. Since $Uh = h$ for all $h \in \mathcal{H}$ we see that $U^*h = h$ for all $h \in \mathcal{H}$ and so $P_{\mathcal{H}}U^* = P_{\mathcal{H}}$ We have

$$\begin{aligned}\varphi_1(b)(h) &= P_{\mathcal{H}}\pi_1(b)(h) = P_{\mathcal{H}}U^*\pi_2(b)Uh && (7.4)\\ &= P_{\mathcal{H}}U^*\pi_2(b)h = P_{\mathcal{H}}\pi_2(b)(h) && (7.5)\\ &= \varphi_2(b)(h), && (7.6)\end{aligned}$$

which shows that $\varphi_1 = \varphi_2$.
Assume that $\varphi_1 = \varphi_2 = \varphi$. Define $V_i : \mathcal{H} \to \mathcal{K}_i$ by $V_i h = h$. If we decompose $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$, and $h \in \mathcal{H}, h' \in \mathcal{H}^{\perp}$, then $V_i^* : \mathcal{K} \to \mathcal{H}$ is given by $V_i^*(h + h') = h$. Therefore $V_i^*\pi_i(b)V_i = \varphi_i(b)$ for any $b \in \mathcal{B}$. Since $\pi(\mathcal{B})\mathcal{H}$ is dense in $\mathcal{K}_i$ it follows that $(\pi_i, \mathcal{K}_i, V_i)$ are minimal Stinespring dilations of φ. Thus there exists a unitary $U : \mathcal{K}_1 \to \mathcal{K}_2$ such that $U^*\pi_2(b)U = \pi_1(b)$.

(ii) Given a minimal $\mathcal{B}$-dilation we define an extension $\varphi : \mathcal{B} \to B(\mathcal{H})$ of ρ by $\varphi(b) = P_{\mathcal{H}}\pi(b)|_{\mathcal{H}}$. The preceeding part of this exercise shows that unitarily equivalent $\mathcal{B}$-dilations give rise to the same completely positive extension.

(iii) Let φ_1, φ_2 be two completely positive extensions of ρ. For $0 \le t \le 1$ the map $\varphi = t\varphi_1 + (1-t)\varphi_2$ is completely positive. If $a \in \mathcal{A}$, then

$$\varphi(a) = t\varphi_1(a) + (1-t)\varphi_2(a) = t\rho(a) + (1-t)\rho(a) = \rho(a). \tag{7.7}$$

Let $\mathcal{E}$ denote the set of completely positive extensions of ρ and note that this set is a subset of the $CP(\mathcal{B}, \mathcal{H}, 1)$ which is compact in the BW-topology. It is enough to show that $\mathcal{E}$ is closed in the BW-topology. Let $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ be a net in $\mathcal{E}$ and assume that $\varphi_\lambda \to \varphi$. Since $CP(\mathcal{B}, \mathcal{H}, 1)$ is closed φ is completely positive. As φ_λ is completely positive and unital, φ_λ is contractive. From Lemma 7.3 we have that

$$\langle \varphi_\lambda(b)x, y\rangle \to \langle \varphi(b)x, y\rangle\,, \tag{7.8}$$

for all $x, y \in \mathcal{H}$ and $b \in \mathcal{B}$. If $a \in \mathcal{A}$, then

$$\langle \rho(a)x, y\rangle = \langle \varphi_\lambda(a)(x), y\rangle \to \langle \varphi(a)x, y\rangle\,. \tag{7.9}$$

Therefore $\langle \rho(a)x, y\rangle = \langle \varphi(a)x, y\rangle$ for all $x, y \in \mathcal{H}$. Hence, $\varphi(a) = \rho(a)$ and $\varphi \in \mathcal{E}$.

7.4 (Extension of Bimodule Maps) Let $\mathcal{A}, \mathcal{C}$ be C^*-algebras, let $\mathcal{S}$ be an operator system, and suppose that $\mathcal{C} \subseteq \mathcal{S} \subseteq \mathcal{A}$. If $\mathcal{C} \subseteq B(\mathcal{H})$, then $\varphi : \mathcal{S} \to B(\mathcal{H})$ is $\mathcal{C}$-bimodule map provided $\varphi(c_1ac_2) = c_1\varphi(a)c_2$. Prove that if $\varphi : \mathcal{S} \to B(\mathcal{H})$ is a completely positive $\mathcal{C}$-bimodule map, then every completely positive extension of φ to $\mathcal{A}$ is also a $\mathcal{C}$-bimodule map.

Solution: To check that a completely positive extension ψ of φ is $\mathcal{C}$-bimodule it is enough by exercise 4.3.ii to check that $\psi(c) = c\psi(1)$. Since ψ is an extension of φ this is the same as saying that $\varphi(c) = c\varphi(1)$. The fact that φ is $\mathcal{C}$-bimodule gives,

$$\varphi(c) = \varphi(c1^2) = c\varphi(1)1 = c\varphi(1). \tag{7.10}$$

7.5 Let $\mathcal{B} \subseteq B(\mathcal{H})$ be a unital C^*-algebra. Prove that $\mathcal{B}$ is injective if and only if there exists a completely positive map $\varphi : B(\mathcal{H}) \to \mathcal{B}$ such that $\varphi(b) = b$ for all b in $\mathcal{B}$. Show that φ is necessarily a $\mathcal{B}$-bimodule map. A map with the above properties is called a *completely positive conditional expectaton.*

Solution: Assume that $\mathcal{B}$ is injective. Note that the identity map $I : \mathcal{B} \to \mathcal{B}$ is completely positive and that $\mathcal{B}$ is an operator system contained in $B(\mathcal{H})$. Since $\mathcal{B}$ is injective I has a completely positive extension $\varphi : B(\mathcal{H}) \to \mathcal{B}$. As φ extends I, $\varphi(b) = I(b) = b$ for all $b \in \mathcal{B}$.

For the converse assume that there is a completely positive map $\varphi : B(\mathcal{H}) \to \mathcal{B}$ such that $\varphi(b) = b$ for all $b \in \mathcal{B}$. Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{S}$ be an operator system contained in $\mathcal{A}$ and suppose that $\psi : \mathcal{S} \to \mathcal{B}$ is completely positive. Let $j : \mathcal{B} \to B(\mathcal{H})$ denote the inclusion map and note that $j \circ \psi : \mathcal{S} \to B(\mathcal{H})$ is completely positive. Since $B(\mathcal{H})$ is injective this map extends to a completely positive map $\theta : \mathcal{A} \to B(\mathcal{H})$. Thus $\varphi \circ \theta : \mathcal{A} \to \mathcal{B}$ is completely positive. We need to check that $\varphi \circ \theta$ extends ψ. For $x \in \mathcal{S}$ we have,

$$\varphi \circ \theta(x) = \varphi(j(\psi(x))) = \varphi(\psi(x)) = \psi(x), \tag{7.11}$$

since $\varphi(b) = b$ for all $b \in \mathcal{B}$ and $\psi(x) \in \mathcal{B}$.

CHAPTER 8: COMPLETELY BOUNDED MAPS

8.1 Show that $\text{Re}\,(\varphi_n) = (\text{Re}\,\varphi)_n$ and that $(\varphi_n)^* = (\varphi^*)_n$.

Solution: Let $(a_{i,j})_{i,j=1}^n \in M_n(\mathcal{M})$. We have,

$$\begin{aligned}
(\varphi^*)_n((a_{i,j})) &= (\varphi^*(a_{i,j})) = (\varphi(a_{i,j}^*)^*) = (\varphi(a_{j,i}^*))^* && (8.1)\\
&= \varphi_n((a_{j,i}^*))^* = \varphi_n((a_{i,j})^*)^* && (8.2)\\
&= (\varphi_n)^*((a_{i,j})) && (8.3)
\end{aligned}$$

Using this we get,

$$\begin{aligned}
2(\text{Re}\,\varphi_n) &= \varphi_n + (\varphi_n)^* && (8.4)\\
&= \varphi_n + (\varphi^*)_n && (8.5)\\
&= (\varphi + \varphi^*)_n && (8.6)\\
&= 2(\text{Re}\,\varphi)_n && (8.7)
\end{aligned}$$

8.2 Let $\varphi : \mathcal{M} \to \mathcal{B}$, let H and K be in M_n, and let A be in $M_n(\mathcal{M})$. Prove that $\varphi_n(HAK) = H\varphi_n(A)K$. Thus, $\varphi_n : M_n(\mathcal{M}) \to M_n(\mathcal{B})$ is an M_n-bimodule map.

Solution: Let $H = (\lambda_{i,j})$, $K = (\mu_{i,j})$ and $A = (a_{i,j})$. Note that the (i,j) entry of HAK is

$$\sum_{l=1}^{n}\sum_{k=1}^{n} \lambda_{i,k} a_{k,l} \mu_{l,j}. \qquad (8.8)$$

It follows that,

$$\begin{aligned}
\varphi_n(HAK) &= \varphi_n\left(\left(\sum_{k,l=1}^{n} \lambda_{i,k} a_{k,l} \mu_{l,j}\right)_{i,j=1}^{n}\right) && (8.9)\\
&= \left(\varphi\left(\sum_{k,l=1}^{n} \lambda_{i,k} a_{k,l} \mu_{l,j}\right)\right)_{i,j=1}^{n} && (8.10)\\
&= (\sum_{k,l=1}^{n} \lambda_{i,k}\varphi(a_{k,l})\mu_{l,j}) && (8.11)\\
&= H(\varphi(a_{i,j}))K = H\varphi_n(A)K. && (8.12)
\end{aligned}$$

8.3 Verify the claim of Theorem 8.4.

Solution: Let $\{E_{i,j}\}_{i,j=1}^2$ denote the matrix units in $M_2 \subseteq M_2(\mathcal{A})$. Let $\mathcal{K} = \text{range}\,\pi_1(E_{1,1})$. Note since π_1 is a $*$-homomorphism that $\pi_1(E_{1,1})$ is a projection and thus $\mathcal{K}$ is closed. We claim that $\mathcal{K}_1 \cong \mathcal{K} \oplus \mathcal{K}$. Define $U : \mathcal{K} \oplus \mathcal{K} \to \mathcal{K}_1$ by,

$$U(x \oplus y) = x + \pi_1(E_{2,1})y. \qquad (8.13)$$

It is straightforward that U is linear. We now prove that it is an isometry.

$$\begin{aligned}
\|U(x \oplus y)\|^2 &= \langle x, x\rangle + \langle \pi_1(E_{2,1})y, \pi_1(E_{2,1})y\rangle + 2\text{Re}\,\langle x, \pi_1(E_{2,1})y\rangle && (8.14)\\
&= \|x\|^2 + \langle \pi_1(E_{1,2}E_{2,1})y, y\rangle + 2\text{Re}\,\langle \pi_1(E_{1,1})x, \pi_1(E_{2,1})y\rangle && (8.15)\\
&= \|x\|^2 + \langle \pi_1(E_{1,1})y, y\rangle + 2\text{Re}\,\langle \pi_1(E_{1,2}E_{1,1})x, y\rangle && (8.16)\\
&= \|x\|^2 + \|y\|^2 = \|x \oplus y\|^2. && (8.17)
\end{aligned}$$

Now let $z \in \mathcal{K}_1$ and note that

$$
\begin{aligned}
z = I_{\mathcal{K}_1} z &= \pi(E_{1,1})z + \pi(E_{2,2})z && (8.18)\\
&= \pi(E_{1,1})z + \pi(E_{2,1})\pi(E_{1,1})\pi(E_{1,2})z && (8.19)\\
&= U(\pi(E_{1,1})z \oplus \pi(E_{1,1})\pi(E_{1,2})z). && (8.20)
\end{aligned}
$$

This shows that U is surjective and is therefore a unitary.

Define $P_j : \mathcal{K} \oplus \mathcal{K} \to \mathcal{K}$ by $P_j(x_1 \oplus x_2) = x_j$ and $\pi : \mathcal{A} \to B(\mathcal{K})$ by

$$\pi(a) = P_1\pi_1(a)P_1^*. \tag{8.21}$$

Note that $P_i^* P_i = I_{\mathcal{K}}$. It follows that π is a unital, linear map and we check that it is a $*$-homomorphism. Let $a, b \in \mathcal{A}$,

$$\pi(ab) = P_i\pi_1(ab)P_i^* = P_i(\pi_1(a)\pi_1(b))P_i^* = P_i\pi_1(a)P_i^*P_i\pi_1(b)P_i^* = \pi(a)\pi(b). \tag{8.22}$$

and,

$$\pi(a^*) = P_i\pi_1(a)^*P_i^* = (P_i^*\pi_1(a)P_i)^* = \pi(a)^* \tag{8.23}$$

To complete the verification we must show that the (i, j)-th entry of $\pi_1((a_{i,j}))$ is $\pi(a_{i,j})$. Note that the (i, j)-th entry is $P_i\pi_1((a_{i,j}))P_j^*$. Now,

$$
\begin{aligned}
P_i\pi_1((a_{i,j}))P_j^* &= \sum_{k,l=1}^{2} P_i\pi_1(a_{k,l} \otimes E_{k,l})P_j^* && (8.24)\\
&= \sum_{k,l=1}^{2} P_i\pi_1(a_{k,l} \otimes (E_{k,1}E_{1,1}E_{1,l}))P_j^* && (8.25)\\
&= \sum_{k,l=1}^{2} P_i\pi_1(E_{k,1})\pi_1(a_{k,l} \otimes E_{1,1})\pi_1(E_{1,l}) && (8.26)\\
&= P_i\pi_1(E_{i,1})\pi_1(a_{i,j} \otimes E_{1,1})\pi_1(E_{1,j})P_j^* && (8.27)\\
&= P_1\pi_1(a_{i,j}E_{1,1})P_1^* = \pi(a_{i,j}). && (8.28)
\end{aligned}
$$

8.4 Show that if φ is completely bounded, and $\varphi(a) = V_1^*\pi(a)V_2$ is the representation of theorem 8.4 with $\|V_1\| = \|V_2\|$, then setting $\varphi_i(a) = V_i^*\pi(a)V_i$, yields the map Φ of Theorem 8.3.

Solution: Note that

$$\|\varphi\|_{cb} = \|V_1\|\,\|V_2\| = \|V_i\|^2 = \|V_i^*V_i\| = \|V_i^*\pi(1)V_i\| \leq \|\varphi_i(1)\|. \tag{8.29}$$

Also,

$$
\begin{aligned}
\|\varphi_i(a)\| &= \|V_i^*\pi(a)V_i\| \leq \|V_i^*\|\,\|\pi(a)\|\,\|V_i\| \leq \|V_i^*\|\,\|V_i\|\,\|a\| && (8.30)\\
&= \|V_i\|^2\,\|a\| = \|\varphi\|_{cb}\,\|a\| && (8.31)
\end{aligned}
$$

Therefore $\|\varphi\|_{cb} = \|\varphi_i\|_{cb}$. If $\|\varphi\|_{cb} = 1$, then we may choose V_i, $i = 1, 2$, to be isometries. Therefore, $\varphi_i(1) = V_i\pi(1)V_i = V_i^*V_i = 1$.

We compute,

$$\varphi^*(a) = \varphi(a^*)^* = (V_1^*\pi(a^*)V_2)^* = V_2^*\pi(a)V_1. \tag{8.32}$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}$. We have,

$$\begin{aligned} \Phi(A) &= \begin{bmatrix} \varphi_1(a) & \varphi(b) \\ \varphi^*(c) & \varphi_2(d) \end{bmatrix} & (8.33) \\ &= \begin{bmatrix} V_1^*\pi(a)V_1 & V_1^*\pi(b)V_2 \\ V_2^*\pi(c)V_1 & V_2^*\pi(d)V_2 \end{bmatrix} & (8.34) \\ &= \begin{bmatrix} V_1^* & 0 \\ 0 & V_2^* \end{bmatrix}\begin{bmatrix} \pi(a) & \pi(b) \\ \pi(c) & \pi(d) \end{bmatrix}\begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} & (8.35) \\ &= V^*\pi_2(A)V. & (8.36) \end{aligned}$$

This shows that Φ is completely positive, since π is a $*$-homomorphism

8.5 Prove that the conclusions of Theorems 8.2, 8.3 and 8.5 still hold when the range is changed from $B(\mathcal{H})$ to an arbitrary injective C^*-algebra.

Solution: Throughout this solution $\mathcal{B}$ will denote an injective C^*-algebra, which is represented on $\mathcal{H}$. By exercise 7.5 there exists a completely positive map $\theta : B(\mathcal{H}) \to \mathcal{B}$ such that $\theta(b) = b$ for all $b \in \mathcal{B}$. *Theorem 8.2* Let $\varphi : \mathcal{M} \to \mathcal{B}$ be completely bounded. Extend φ, by Wittstock's theorem, to a completely bounded map $\psi : \mathcal{A} \to B(\mathcal{H})$. Notice that the map $\theta \circ \psi : \mathcal{A} \to \mathcal{B}$ is completely bounded. Let $a \in \mathcal{M}$, note that $\varphi(a) \in \mathcal{B}$ and so

$$\theta \circ \psi(a) = \theta \circ \varphi(a) = \varphi(a). \tag{8.37}$$

Thus, $\theta \circ \psi$ is a completely bounded extension of φ. Since θ is unital and completely positive we have $\|\theta\| = \|\theta\|_{cb} = 1$. Using this fact we get,

$$\|\varphi\|_{cb} \leq \|\theta \circ \psi\|_{cb} \leq \|\theta\|_{cb} \|\psi\|_{cb} = \|\varphi\|_{cb}. \tag{8.38}$$

Theorem 8.3 Let $\mathcal{A}$ be a C^*-algebra with unit, let $\psi : \mathcal{A} \to \mathcal{B}$ be completely bounded. Then there exists completely positive maps $\psi_i : \mathcal{A} \to \mathcal{B}$ with $\|\psi_i\|_{cb} = \|\psi\|_{cb}$, $i = 1, 2$, such that the map $\Psi : M_2(\mathcal{A}) \to M_2(\mathcal{B})$ given by

$$\Psi \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \psi_1(a) & \psi(b) \\ \psi^*(c) & \psi_2(d) \end{bmatrix} \tag{8.39}$$

is completely positive. Moreover, if $\|\psi\|_{cb} = 1$, then we may take $\psi_i(1) = 1$, $i = 1, 2$.

We may assume that $\|\psi\|_{cb} = 1$. Let $\theta : B(\mathcal{H}) \to \mathcal{B}$ be a completely positive projection. Construct with ψ is place of φ the maps φ_i, $i = 1, 2$ and Φ as in Theorem 8.3. Let $\psi_i = \theta \circ \varphi_i$. As θ fixes $\mathcal{B}$ we have that $\theta \circ \psi = \psi$ and $\theta \circ \psi^* = \psi^*$. It follows that and $\Psi = \theta_2 \circ \Phi$. As composites of completely positive (completely bounded) maps are completely positive (completely bounded) we see that ψ_i, $i = 1, 2$ are completely bounded and Ψ is completely positive. θ is unital and so $\psi_i(1) = \theta(\varphi(1)) = 1$. Together with,

$$\|\psi_i\|_{cb} = \|\theta \circ \varphi_i\|_{cb} \leq \|\theta\|_{cb} \|\varphi_i\|_{cb} \leq 1, \tag{8.40}$$

this proves $\|\psi_i\| = 1 = \|\psi\|_{cb}$.

Theorem 8.5 Let $\mathcal{A}$ be a C^*-algebra with unit, and let $\varphi : \mathcal{A} \to \mathcal{B}$ be completely bounded. then there exists a completely positive map $\psi : \mathcal{A} \to \mathcal{B}$ with $\|\psi\| \leq \|\varphi\|_{cb}$ such that $\psi \pm \operatorname{Re} \varphi$ and $\psi \pm \operatorname{Im} \varphi$ are all completely positive. In particular, the completely bounded maps are the linear span of the completely positive maps.

Since $\mathcal{B} \subseteq B(\mathcal{H})$ construct as in Theorem 8.5 a map $\rho : \mathcal{A} \to B(\mathcal{H})$ such that $\|\rho\|_{cb} \leq \|\varphi\|_{cb}$ and set $\psi = \theta \circ \rho : \mathcal{A} \to \mathcal{B}$. We have

$$\|\psi\|_{cb} \leq \|\theta\|_{cb} \|\rho\|_{cb} \leq \|\varphi\|_{cb} . \tag{8.41}$$

As θ is a projection onto $\mathcal{B}$, $\theta \circ \varphi = \varphi$. Notice that,

$$(\theta \circ \varphi^*)(a) = \theta(\varphi(a^*)^*) = \theta(\varphi(a^*))^* = ((\theta \circ \varphi)(a^*))^* = (\theta \circ \varphi)^*(a). \tag{8.42}$$

Consequently,

$$\theta \circ (\rho \pm \operatorname{Im} \varphi) = (\theta \circ \rho) \pm \operatorname{Im} (\theta \circ \varphi) = \psi \pm \operatorname{Im} \varphi, \tag{8.43}$$

is completely positive. Similarly, $\psi \pm \operatorname{Re} \varphi$ is completely positive.

8.6 Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be C^*-algebras with unit, with $\mathcal{C}$ contained in both $\mathcal{A}$ and $\mathcal{B}$, and $\mathbf{1}_{\mathcal{A}} = \mathbf{1}_{\mathcal{B}} = \mathbf{1}_{\mathcal{C}}$. Let $\mathcal{M} \subseteq \mathcal{A}$ be a subspace such that $c_1 \mathcal{M} c_2 \subseteq \mathcal{M}$ for all c_1, c_2 in $\mathcal{C}$, and set

$$\mathcal{S} = \left\{ \begin{bmatrix} c_1 & a \\ b^* & c_2 \end{bmatrix} : a, b \in \mathcal{M}, c_1, c_2 \in \mathcal{C} \right\}. \tag{8.44}$$

(i) Prove that if $\varphi : \mathcal{M} \to \mathcal{B}$ is a completely contractive $\mathcal{C}$-bimodule map, then $\Phi : \mathcal{S} \to M_2(\mathcal{B})$ defined by

$$\Phi\left(\begin{bmatrix} c_1 & a \\ b^* & c_2 \end{bmatrix} \right) = \begin{bmatrix} c_1 & \varphi(a) \\ \varphi(b)^* & c_2 \end{bmatrix}, \tag{8.45}$$

is completely positive.

(ii) Prove that if $\mathcal{B}$ is injective, then the conclusions of Theorems 8.2, 8.3 and 8.5 still hold with the additonal assumption that the maps be $\mathcal{C}$-bimodule maps.

Solution:

(i) Let

$$(S_{i,j}) = \begin{bmatrix} c_{i,j}^{(1)} & a_{i,j} \\ b_{i,j}^* & c_{i,j}^{(2)} \end{bmatrix} \in M_n(\mathcal{S}). \tag{8.46}$$

After a canonical shuffle $(S_{i,j})$ is the matrix

$$\begin{bmatrix} H & A \\ B^* & K \end{bmatrix}, \tag{8.47}$$

where $H = (c_{i,j}^{(1)})$, $K = (c_{i,j}^{(2)})$, $A = (b_{i,j})$ and $B = (b_{j,i})$.Similarly after a canonical shuffle $\Phi_n((S_{i,j}))$ is

$$\begin{bmatrix} H & \varphi_n(A) \\ \varphi_n(B)^* & K \end{bmatrix}. \tag{8.48}$$

If the matrix in (8.47) is positive, then $A = B$ and H, K are positive. Let $\varepsilon > 0$ and note that the matrices $H_\varepsilon = H + \varepsilon I$ and $K_\varepsilon = K + \varepsilon I$ are positive and invertible. We have,

$$\begin{bmatrix} H_\varepsilon^{-1/2} & 0 \\ 0 & K_\varepsilon^{-1/2} \end{bmatrix} \begin{bmatrix} H_\varepsilon & A \\ A^* & K_\varepsilon \end{bmatrix} \begin{bmatrix} H_\varepsilon^{-1/2} & 0 \\ 0 & K_\varepsilon^{-1/2} \end{bmatrix} = \begin{bmatrix} I & H_\varepsilon^{-1/2} A K_\varepsilon^{-1/2} \\ K_\varepsilon^{-1/2} A^* H_\varepsilon^{-1/2} & I \end{bmatrix} \tag{8.49}$$

The matrix on the left of (8.49) is positive and by Lemma 3.1 we have that

$$\left\| H_\varepsilon^{-1/2} A K_\varepsilon^{-1/2} \right\| \leq 1. \tag{8.50}$$

As φ is $\mathcal{C}$-bimodule, $\varphi_n(H_\varepsilon^{-1/2} A K_\varepsilon^{-1/2}) = H_\varepsilon^{-1/2}\varphi_n(A) K_\varepsilon^{-1/2}$. A similar calculation to (8.49) shows that $\Phi_n((S_{i,j}))$ is positive if and only if

$$\begin{bmatrix} I & H_\varepsilon^{-1/2}\varphi_n(A)K_\varepsilon^{-1/2} \\ K_\varepsilon^{-1/2}\varphi_n(A)^* H_\varepsilon^{-1/2} & I \end{bmatrix} \geq 0. \tag{8.51}$$

From the assumption that φ is completely contractive we get

$$\left\| H_\varepsilon^{-1/2}\varphi_n(A)K_\varepsilon^{-1/2} \right\| = \left\| \varphi_n(H_\varepsilon^{-1/2} A K_\varepsilon^{-1/2}) \right\| \leq 1, \tag{8.52}$$

which is equivalent to the matrix in (8.51) being positive.

(ii) We check that the maps considered in the proofs of Theorems 8.2, 8.3, and 8.5 are $\mathcal{C}$-bimodule.
Theorem 8.2 Recall that ψ was defined by

$$\Psi\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} * & \psi(a) \\ * & * \end{bmatrix}, \tag{8.53}$$

where Ψ is the extension of Φ described in part i). Being an extension of Φ, Ψ fixes $\mathcal{C} \oplus \mathcal{C}$, and so by Exercise 4.3, Ψ is $\mathcal{C} \oplus \mathcal{C}$-bimodule. If $c_1, c_2 \in \mathcal{C}$, then

$$\begin{aligned} \Psi\left(\begin{bmatrix} 0 & c_1 a c_2 \\ 0 & 0 \end{bmatrix}\right) &= \Psi\left(\begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & c_2 \end{bmatrix}\right) && (8.54) \\ &= \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix}\Psi\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & 0 \\ 0 & c_2 \end{bmatrix}, && (8.55) \end{aligned}$$

from which we get $\psi(c_1 a c_2) = c_1\psi(a)c_2$.
Theorem 8.3 A matrix factoring similar to the one used above shows that φ_i, for $i = 1, 2$, is a $\mathcal{C}$-bimodule map.
Theorem 8.5 We will prove that φ^* is $\mathcal{C}$-bimodule. The claims of theorem 8.5 will then follow as before.

$$\begin{aligned} \varphi^*(c_1 a c_2) &= \varphi((c_1 a c_2)^*)^* = \varphi(c_2^* a^* c_1^*)^* && (8.56) \\ &= (c_2^*\varphi(a^*)c_1^*)^* = c_1\varphi(a^*)^* c_2 && (8.57) \\ &= c_1\varphi^*(a)c_2. && (8.58) \end{aligned}$$

8.7 Let $A = (a_{i,j})_{i,j=1}^{\infty}$. Prove that the following are equivalent:

(i) $S_A : B(\ell^2) \to B(\ell^2)$ is positive.
(ii) $S_A : B(\ell^2) \to B(\ell^2)$ is completely positive.
(iii) There exists a Hilbert space $\mathcal{H}$ and a bounded sequence of vectors $\{x_i\}$ in $\mathcal{H}$ such that $a_{i,j} = \langle x_j, x_i \rangle$.

Solution: Let $P_n : \ell^2 \to \ell^2$ denote the projection onto the first n coordinates. Let $A_n = P_n A P_n$ which is the matrix equal to A in the $n \times n$ top left corner and 0 otherwise. Observe that $A_n \to_{\mathrm{WOT}} A$ as $n \to \infty$. Note that $A_n * T = (P_n A P_n) * T = P_n(A * T)P_n = (A * T)_n$ for any $T \in B(\ell^2)$. By Theorem 3.7 the following three statements are equivalent

(a) S_{A_n} is positive for all $n \geq 1$.

(b) $A_n \geq 0$ for all $n \geq 1$.

(c) S_{A_n} is completely positive for all $n \geq 1$.

(i) implies (ii). If $T \geq 0$, then

$$S_{A_n}(T) = A_n * T = A_n * T_n = (A * T)_n = (S_A(T))_n \geq 0. \tag{8.59}$$

Therefore S_{A_n} is positive and by the observation above S_{A_n} is completely positive. Assume $T = (T_{i,j})_{i,j=1}^{m} \geq 0$ and let $x_j \in \ell^2$, $j = 1, \ldots, m$.

$$\left\langle (S_A)_m(T_{i,j}) \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \right\rangle = \sum_{i,j=1}^{m} \langle (A * T_{i,j}) x_j, x_i \rangle \tag{8.60}$$

$$= \sum_{i,j=1}^{m} \lim_{n \to \infty} \langle (A * T_{i,j})_n x_j, x_i \rangle \tag{8.61}$$

$$= \lim_{n \to \infty} \sum_{i,j=1}^{m} \langle (A * T_{i,j}) x_j, x_i \rangle \tag{8.62}$$

$$= \lim_{n \to \infty} \sum_{i,j=1}^{m} \left\langle (S_{A_n})_m(T_{i,j}) \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \right\rangle \geq 0 \tag{8.63}$$

(ii) implies (iii). Let $S_A : B(\ell^2) \to B(\ell^2)$ be completely positive. Let $(\pi, \mathcal{H}, V)$ be a minimal Stinespring representation of S_A. Let $E_k = E_{1,k}$ and note that $E_i^* E_k = E_{i,j}$. For each $j \in \mathbb{N}$, define $x_k = \pi(E_k) V e_k \in \mathcal{H}$. Note that $\|x_k\| \leq \|V\|$. Now,

$$\begin{aligned}
\langle x_j, x_i \rangle &= \langle \pi(E_j) V e_j, \pi(E_i) V e_i \rangle && (8.64)\\
&= \langle V^* \pi(E_i)^* \pi(E_j) V e_j, e_i \rangle && (8.65)\\
&= \langle V^* \pi(E_{i,j}) V e_j, e_i \rangle && (8.66)\\
&= \langle S_A(E_{i,j}) e_j, e_i \rangle && (8.67)\\
&= \langle a_{i,j} E_{i,j} e_j, e_i \rangle && (8.68)\\
&= a_{i,j} && (8.69)
\end{aligned}$$

(iii) implies (i). Let $\|x_k\| \leq M$ for all $k \geq 1$, $T = (t_{i,j})_{i,j=1}^{\infty} \in B(\ell^2)$ and $h, k \in \ell^2$. It is enough to prove that $\|(S_A(T))_n\|$ is bounded independent of n (which establishes the fact that $S_A(T)$ is an element of $B(\ell^2)$) and $A_n \geq 0$. The latter claim implies, for a positive operator T, that

$$\langle S_A(T)h, h \rangle = \lim_{n \to \infty} \langle (S_A(T))_n h, h \rangle = \lim_{n \to \infty} \langle S_{A_n}(T)h, h \rangle \geq 0. \tag{8.70}$$

Now,

$$
\begin{aligned}
|\langle (S_A(T))_n h, k\rangle| &= |\langle (A_n * T_n)h, k\rangle| && (8.71)\\
&= \left|\sum_{i,j=1}^{n} a_{i,j} t_{i,j} h_j \overline{k_i}\right| && (8.72)\\
&= \left|\sum_{i,j=1}^{n} \langle x_j, x_i\rangle\, t_{i,j} h_j \overline{k_i}\right| && (8.73)\\
&= \left|\left\langle (t_{i,j} I_{\mathcal{H}}) \begin{bmatrix} h_1 x_1 \\ \vdots \\ h_n x_n \end{bmatrix}, \begin{bmatrix} k_1 x_1 \\ \vdots \\ k_n x_n \end{bmatrix} \right\rangle\right| && (8.74)\\
&\le \|(t_{i,j} I_{\mathcal{H}})\|_{B(\mathcal{H}^{(n)})} \left(\sum_{i=1}^{n} \|h_i x_i\|^2\right)^{1/2} \left(\sum_{i=1}^{n} \|k_i x_i\|^2\right)^{1/2} && (8.75)\\
&= \|T_n\| \left(\sum_{i=1}^{n} |h_i|^2 \|x_i\|^2\right)^{1/2} \left(\sum_{i=1}^{n} |k_i|^2 \|x_i\|^2\right)^{1/2} && (8.76)\\
&\le \|T\|\, M^2\, \|h\|\, \|k\|\,. && (8.77)
\end{aligned}
$$

Therefore, $\|(S_A(T))_n\| \le \|T\|\, M^2$.

Also,

$$
\begin{aligned}
\langle A_n h, h\rangle_{\ell^2} &= \sum_{i,j=1}^{n} a_{i,j} h_j \overline{h_i} && (8.78)\\
&= \sum_{i,j=1}^{n} \langle x_j, x_i\rangle\, h_j \overline{h_i} && (8.79)\\
&= \sum_{i,j=1}^{n} \langle h_j x_j, h_i x_i\rangle && (8.80)\\
&= \left\langle \begin{bmatrix} h_1 x_1 \\ \vdots \\ h_n x_n \end{bmatrix}, \begin{bmatrix} h_1 x_1 \\ \vdots \\ h_n x_n \end{bmatrix} \right\rangle_{\mathcal{H}^{(n)}} \ge 0 && (8.81)
\end{aligned}
$$

9.7 (Sz.-Nagy) This exercise gives a more direct proof of Corollary 9.4. Let T be an invertible operator on $\mathcal{H}$ such that $\|T^n\| \leq M$ for all integers n, and let glim be a Banach generalized limit [34].

(i) Show that $\langle x, y\rangle_1 = \text{glim}\,\langle T^n x, T^n y\rangle$ defines a new inner product on $\mathcal{H}$ and that $M^{-2}\langle x,x\rangle \leq \langle x,x\rangle_1 \leq M^2\langle x,x\rangle$.

(ii) Show that T is a unitary transformation on $(\mathcal{H}, \langle,\rangle_1)$.

(iii) Prove that there exists a similarity on $\mathcal{H}$, with $\|S^{-1}\|\|S\| \leq M^2$, such that $S^{-1}TS$ is unitary.

Solution: We begin by noting that glim : $\ell^\infty \to \mathbb{C}$ is a positive linear functional, $\text{glim}((\alpha_n)_{n=1}^\infty) = \lim_{n\to\infty}\alpha_n$ whenever $(\alpha_n)_{n=1}^\infty$ is convergent and $\text{glim}((\alpha_n)_{n=1}^\infty) = \text{glim}((\alpha_{n+1})_{n=1}^\infty)$ for any $(\alpha_n)_{n=1}^\infty \in \ell^\infty$.

(i) We have by the Cauchy-Schwarz inequality that,

$$\langle T^n x, T^n x\rangle \leq \|T^n\|^2\langle x,x\rangle \leq M^2\langle x,x\rangle, \tag{9.1}$$

and,

$$\langle x,x\rangle = \left\langle T^{-n}T^n x, T^{-n}T^n x\right\rangle \leq \left\|T^{-n}\right\|^2\langle T^n x, T^n x\rangle \leq M^2\langle T^n x, T^n x\rangle. \tag{9.2}$$

This shows that $\langle T^n x, T^n x\rangle$ is an ℓ^∞ sequence and so $\langle,\rangle_1$ is well-defined. Since glim is positive and linear $\langle,\rangle_1$ is a semi-inner product on $\mathcal{H}$. Applying glim to the inequalities in (9.1) and (9.2) establishes $M^{-2}\langle x,x\rangle \leq \langle x,x\rangle_1 \leq M^2\langle x,x\rangle$, and so $\langle,\rangle_1$ is an inner product.

(ii) As T is invertible, our claim is established if we prove that T is an isometry. We have,

$$\|Tx\|_1^2 = \langle Tx, Tx\rangle_1 = \text{glim}\left\langle T^{n+1}x, T^{n+1}x\right\rangle = \text{glim}\,\langle T^n x, T^n x\rangle = \langle x,x\rangle_1 = \|x\|_1^2. \tag{9.3}$$

(iii) Using the fact that $M^{-1}\|x\| \leq \|x\|_1 \leq M\|x\|$ we see that $R : (\mathcal{H}, \langle,\rangle) \to (\mathcal{H}, \langle,\rangle_1)$ defined by $R(h) = h$ is a bounded invertible operator with $\|R\| \leq M$ and $\left\|R^{-1}\right\| \leq M$. Therefore the Hilbert spaces $(\mathcal{H}, \langle,\rangle)$ and $(\mathcal{H}, \langle,\rangle_1)$ have the same dimension. It follows that there is a unitary $U : (\mathcal{H}, \langle,\rangle) \to (\mathcal{H}, \langle,\rangle_1)$. The operator,

$$U^*RTR^{-1}U = (R^{-1}U)^{-1}TR^{-1}U, \tag{9.4}$$

is unitary and similar to T, via the similarity $S = R^{-1}U$. Since U and U^* are unitary we have $\|S\| = \left\|R^{-1}\right\| \leq M$ and $\left\|S^{-1}\right\| = \|R\| \leq M$. Hence $\|S\|\|S^{-1}\| \leq M^2$.

9.10 (Sz.-Nagy-Foias) An operator T in $B(\mathcal{H})$ is said to belong to class C_ρ if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$, and a unitary on $\mathcal{K}$, such that

$$T^n = \rho P_{\mathcal{H}}U^n|_{\mathcal{H}}, \tag{9.5}$$

for all positive integers n. Prove that such a T is completely polynomially bounded and that there exists an invertible operator S such that $S^{-1}TS$ is a contraction with $\|S^{-1}\|\|S\| \leq 2\rho - 1$ when $\rho \geq 1$.

Solution: By Theorem 9.8 it is enough to prove that $\varphi : \mathcal{P}(\mathbb{D}) \to B(\mathcal{H})$ defined by $\varphi(p) = p(T)$ is completely bounded with $\|\varphi\|_{cb} \leq 2\rho - 1$. Let $p(z) = \alpha_0 + \alpha_1 z + \ldots + \alpha_m z^m$. We have,

$$\begin{aligned}
p(T) &= \alpha_0 I + \alpha_1 T + \ldots + \alpha_m T^m && (9.6)\\
&= \alpha_0 I + \rho P_{\mathcal{H}}(\alpha_1 U + \ldots + \alpha_m U^m)|_{\mathcal{H}} && (9.7)\\
&= \alpha_0 I - \rho\alpha_0 I + \rho\alpha_0 I + \rho P_{\mathcal{H}}(\alpha_1 U + \ldots + \alpha_m U^m)|_{\mathcal{H}} && (9.8)\\
&= \alpha_0(1-\rho)I + \rho P_{\mathcal{H}}p(U)|_{\mathcal{H}}. && (9.9)
\end{aligned}$$

It follows by the spectral mapping theorem since $p(U)$ is normal that,

$$\|p(T)\| \leq (\rho - 1)\,|p(0)| + \rho\,\|p(U)\| \leq (\rho - 1)\,\|p\|_\infty + \rho\,\|p\|_\infty = (2\rho - 1)\,\|p\|_\infty . \tag{9.10}$$

Let $(p_{i,j}) \in M_n(\mathcal{P}(\mathbb{D}))$. We have,

$$\|\varphi_n((p_{i,j}))\| = \|(p_{i,j}(T))\| \leq (\rho - 1)\,\|(p_{i,j}(0))\| + \rho\,\|(p_{i,j}(U))\| . \tag{9.11}$$

Since U is a unitary we can apply von Neumann's inequality for matrices (Corollary 3.12),

$$\|(p_{i,j}(U))\|_{B(\mathcal{K}^{(n)})} \leq \|(p_{i,j})\|_{M_n(C(\mathbb{T}))} , \tag{9.12}$$

to (9.11) to get,

$$\|\varphi_n\| \leq 2\rho - 1. \tag{9.13}$$

9.12 let T be an operator with real spectrum. Prove that T is similar to a self-adjoint operator if and only if the Cayley transform of T, $C = (T+i)(T-i)^{-1}$, has the property that for some constant M, $\|C^n\| \leq M$ for all integers n.

Solution: Assume first that $\|C^n\| \leq M$ for all integers n. By Sz.-Nagy's theorem $U = SCS^{-1}$ is unitary for some invertible operator S. We claim that STS^{-1} is self-adjoint. Note that,

$$\sigma(C) = \sigma(U) = \left\{ \frac{t+i}{t-i} : t \in \sigma(T) \right\}, \tag{9.14}$$

and that $1 \notin \sigma(U)$. From $U = SCS^{-1} = S(T+i)(T-i)^{-1}S^{-1}$ it follows that $US(T-i)S^{-1} = S(T+i)S^{-1}$. On rearranging this equation we get $(U-1)STS^{-1} = i(U+1)$. As $(U-1)$ is invertible $STS^{-1} = i(U-1)^{-1}(U+1)$. Therefore,

$$\begin{aligned}
(STS^{-1})^* &= -i(U+1)^*((U-1)^*)^{-1} && (9.15)\\
&= -i(U^{-1}+1)(U^{-1}-1)^{-1} && (9.16)\\
&= -i(1+U)U^{-1}((1-U)U^{-1})^{-1} && (9.17)\\
&= -i(1+U)U^{-1}U(1-U)^{-1} && (9.18)\\
&= i(U-1)^{-1}(U+1) && (9.19)\\
&= STS^{-1}. && (9.20)
\end{aligned}$$

To prove the converse suppose that $T = SRS^{-1}$ where R is self-adjoint and let B denote the Cayley transform of R. The adjoint of B is given by,

$$\begin{aligned}
B^* &= ((R-i)^*)^{-1}(R+i)^* && (9.21)\\
&= (R+i)^{-1}(R-i) && (9.22)\\
&= ((R-i)^{-1}(R+i))^{-1} && (9.23)\\
&= ((R+i)(R-i)^{-1})^{-1} && (9.24)\\
&= B^{-1}, && (9.25)
\end{aligned}$$

and so B is unitary. Further,

$$\begin{aligned}
B &= (S^{-1}TS+i)(S^{-1}TS-i)^{-1} && (9.26)\\
&= S^{-1}(T+i)S[S^{-1}(T-i)S]^{-1} && (9.27)\\
&= S^{-1}(T+i)(T-i)^{-1}S && (9.28)\\
&= S^{-1}CS. && (9.29)
\end{aligned}$$

Hence,

$$\|C^n\| = \|S^{-1}B^nS\| \leq \|S^{-1}\|\,\|B^n\|\,\|S\| = \|S^{-1}\|\|S\|. \tag{9.30}$$

9.15 Let T be in $B(\mathcal{H})$ with $\sigma(T)$ contained in the open unit disk. Prove that $P = \sum_{k=0}^{\infty} T^{*k}T^k$ is a norm convergent series with $\|P^{1/2}TP^{-1/2}\| \leq 1$.

Solution: Let $R > 0$ be chosen so that $r(T) < R < 1$. By the spectral radius formula there exists $N \in \mathbb{N}$ such that $\|T^n\|^{1/n} \leq R$ for all $n \geq N$. Therefore $\|T^n\| = \|T^{*n}\| \leq R^n$ for all $n \geq N$ and so $\|T^{*n}T^n\| \leq R^{2n}$ for all $n \geq N$. It follows that the series is absolutely convergent in $B(\mathcal{H})$ and consequently norm convergent. Since P is the sum of positive operators and $P \geq I$ we see that P is positive and invertible.

Note that,

$$\begin{aligned} \|P^{1/2}TP^{-1/2}\| \leq 1 \quad &\Longleftrightarrow \quad (P^{1/2}TP^{-1/2})^*(P^{1/2}TP^{-1/2}) \leq I && (9.31)\\ &\Longleftrightarrow \quad P^{-1/2}T^*PTP^{-1/2} \leq I && (9.32)\\ &\Longleftrightarrow \quad T^*PT \leq P. && (9.33) \end{aligned}$$

We establish this last inequality,

$$T^*PT = \sum_{k=0}^{\infty} T^*(T^{*k}T^k)T = \sum_{k=0}^{\infty} T^{*k+1}T^{k+1} = \sum_{k=1}^{\infty} T^{*k}T^k = P - I \leq P. \tag{9.34}$$

10.1 Let

$$J_\lambda = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \ddots & 0 \\ \vdots & \ddots & & \ddots & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{bmatrix} \in M_k \tag{10.1}$$

be an elementary Jordan block.

(i) Show that the $(1, k)$ entry of J_λ^m is the $(k-1)$-th derivative of $z^m/(k-1)!$ evaluated at λ. Deduce that J_λ is power bounded if and only if $|\lambda| < 1$.

(ii) Prove that if $|\lambda| < 1$ then J_λ is similar to a contraction.

(iii) Let $T \in M_n$ be power bounded. Prove that T is similar to a contraction.

Solution:

(i) We write $J_\lambda = \lambda I + N$ and note that N is nilpotent of order k. Therefore,

$$\begin{aligned} J_\lambda^m &= (\lambda I + N)^m && (10.2)\\ &= \sum_{j=0}^{\min\{k-1,m\}} \binom{m}{j} \lambda^{m-j} N^j && (10.3)\\ & && (10.4) \end{aligned}$$

If $m \le k-1$ then the $(1, k)$ entry is 0 which is also the $(k-1)$th derivative of $z^m/(k-1)!$. For $m \ge k$ The $(1, k)$ entry is given by,

$$\binom{m}{k-1} \lambda^{m-k+1} = \frac{m(m-1)\dots(m-k+2)}{(k-1)!} \lambda^{m-k+1} = \left.\frac{d^{k-1}}{dz^{k-1}}\right|_{z=\lambda} \frac{z^m}{(k-1)!}. \tag{10.5}$$

If $\lambda \ge 1$ then

$$\|J_\lambda^m\| \ge \binom{m}{k-1} |\lambda|^{m-k+1} \to \infty \text{ as } m \to \infty. \tag{10.6}$$

Therefore, J_λ is not power bounded. Conversely if $|\lambda| < 1$, then for $m \ge k-1$

$$\|J_\lambda^m\| \le \sum_{j=0}^{k-1} \binom{m}{j-1} |\lambda|^{m-j} \le km^k |\lambda|^{m-j} \to 0 \text{ as } m \to \infty. \tag{10.7}$$

(ii) Let

$$D_r = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r^{k-1}. \end{bmatrix} \tag{10.8}$$

If $r \neq 0$, then

$$D_r^{-1} J_\lambda D_r = \begin{bmatrix} \lambda & r & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \ddots & 0 \\ \vdots & \ddots & & \ddots & r \\ 0 & \cdots & 0 & 0 & \lambda \end{bmatrix} \tag{10.9}$$

and so

$$\|D_{r^{-1}} J_\lambda D_r\| \leq |\lambda| + r \leq 1, \tag{10.10}$$

for any $0 < r \leq 1 - |\lambda|$.

(iii) As T is power bounded there exists a constant M such that $\|T^n\| \leq M$ for all $n \geq 0$. Let J denote the Jordan form of T and suppose that $J = \text{diag}(J_{\lambda_1}, \ldots, J_{\lambda_l})$, where J_{λ_i} is an elementary Jordan block. Suppose that $R^{-1}TR = J$. We have that,

$$J^n = R^{-1}T^nR = \text{diag}(J_{\lambda_1}^n, \ldots, J_{\lambda_l}^n) \tag{10.11}$$

and,

$$\left\|J_{\lambda_i}^n\right\| \leq \left\|R^{-1}T^nR\right\| \leq M \left\|R^{-1}\right\| \|R\| . \tag{10.12}$$

Hence J_{λ_i} is power bounded. It follows by part (i) that $|\lambda_i| < 1$ for $i = 1, \ldots, l$ and so by part (ii) there exist similarities $S_1, \ldots, S_l$ such that $S_i^{-1} J_{\lambda_i} S_i$ is a contraction. Let $D = \text{diag}(S_1, \ldots, S_l)$, $S = RD$ and note that

$$\left\|S^{-1}TS\right\| = \left\|D^{-1}R^{-1}TRD\right\| = \left\|D^{-1}JD\right\| = \max_{i=1,\ldots,l} \left\|S_i^{-1} J_{\lambda_i} S_i\right\| \leq 1. \tag{10.13}$$

10.3 i) Show that g_1, f_1, p_1, q_1 satisfy 1)-4) of Theorem 10.8.

ii) Show that if g_m, f_m, p_m, q_m satisfy 1)-4), then $g_{m+1}, f_{m+1}, p_{m+1}, q_{m+1}$ satisfy 1)-4).

Solution:

i) We have $g_1 \equiv q_1 \equiv 1$, $p_1(z) = -\overline{h_1} z^{k_1}$ which has degree at most k_1 and $f_1(z) = h_1 z^{k_1}$ which implies $\Gamma(f_1) = (h_1, 0, \ldots)$.

ii) We compute

$$F_{m+1}(e^{i\theta}) = F_m(e^{i\theta}) B_{m+1}(e^{i\theta}) \tag{10.14}$$

$$= \begin{bmatrix} g_m(e^{i\theta}) & p_m(e^{-i\theta}) \\ f_m(e^{i\theta}) & q_m(e^{-i\theta}) \end{bmatrix} \begin{bmatrix} 1 & -\overline{h}_{m+1} e^{-ik_{m+1}\theta} \\ h_{m+1} e^{ik_{m+1}\theta} & 1 \end{bmatrix} \tag{10.15}$$

$$= \begin{bmatrix} g_m(e^{i\theta}) + h_{m+1} e^{ik_{m+1}\theta} p_m(e^{-i\theta}) & p_m(e^{-i\theta}) - \overline{h}_{m+1} e^{-ik_{m+1}\theta} g_m(e^{i\theta}) \\ f_m(e^{i\theta}) + h_{m+1} e^{ik_{m+1}\theta} q_m(e^{-i\theta}) & q_m(e^{-i\theta}) - \overline{h}_{m+1} e^{-ik_{m+1}\theta} f_m(e^{i\theta}) \end{bmatrix} \tag{10.16}$$

We will deduce properties 1) and 4), the remaining deductions are similar. We have,

$$g_{m+1}(e^{i\theta}) = g_m(e^{i\theta}) + h_{m+1} e^{ik_{m+1}\theta} p_m(e^{-i\theta}). \tag{10.17}$$

By assumption the degree of g_m and p_m is at most k_m but $p_m(e^{i\theta})$ is a polynomial in negative powers of $e^{i\theta}$ and so the degree of the second term in (10.17) is at most k_{m+1}. The only contribution to the constant term in g_{m+1} is from g_m, since $k_{m+1} > k_m$. It follows that $g_m(0) = 1$. Now consider

$$f_{m+1}(e^{i\theta}) = f_m(e^{i\theta}) + h_{m+1} e^{ik_{m+1}\theta} q_m(e^{-i\theta}). \tag{10.18}$$

Since $k_{m+1} > 2k_m + 1$ the smallest exponent that appears in $h_{m+1}e^{ik_{m+1}\theta}q_m(e^{-i\theta})$ is at least $k_{m+1} - k_m > k_m + 1$. In addition the constant term of q_m is 1 and so the coefficient of $z^{k_{m+1}}$ is h_{m+1}. Therefore, the coefficients of z^k, $k = 0, \dots, k_m$, in f_m and f_{m+1} are the same and the coefficient of $z^{k_{m+1}}$ in f_{m+1} is h_{m+1}. Hence, $\Gamma(f_{m+1}) = (h_1, \dots, h_m, h_{m+1}, 0, \dots)$.

10.4 Let Γ be the map of Theorem 10.8. Prove that $\dot{\Gamma} : A(\mathbb{D})/\ker\Gamma \to \ell^2$ is a Banach space isomorphism with $\|\dot{\Gamma}\| \le 1, \|\dot{\Gamma}^{-1}\| \le \sqrt{e}$.

Solution: By standard results on quotient spaces and using the fact that Γ is contractive and onto we have that $\dot{\Gamma}$ is a bijective, contractive map. From the proof of Theorem 10.8 we see that given $h \in \ell^2$ there exists a $f \in A(\mathbb{D})$ such that $\Gamma(f) = h$ and $\|f\| \le \sqrt{e}\,\|h\|$. Thus,

$$\|\dot{\Gamma}^{-1}(h)\| = \|f + \ker\Gamma\| \le \|f\| \le \sqrt{e}\,\|h\|\,, \tag{10.19}$$

and so $\|\dot{\Gamma}^{-1}\| \le \sqrt{e}$

10.5 Let $C_1, \dots, C_n$ denote the $2^n \times 2^n$ CAR matrices, let $E_{1,1}, \dots, E_{n,1}$ be the standard matrix units in M_n and let $\Phi(\lambda_1 E_{1,1} + \dots + \lambda_n E_{n,1}) = \lambda_1 C_1 + \dots + \lambda_n C_n$, so that Φ is an isometry. Prove that $\|\Phi\|_{cb} \ge \sqrt{n}/2$.

Solution: Let $C = \lambda_1 C_1 + \dots + \lambda_n C_n$. Let $P = CC^*$ and $Q = C^*C$. Note that

$$C^2 = \sum_{1\le i<j\le n} \lambda_i\lambda_j(C_iC_j + C_jC_i) + \sum_{j=1}^{n} \lambda_j^2 C_j^2 = 0. \tag{10.20}$$

From this we get $QP = C^*C^2C^* = 0$. Moreover,

$$P + Q = CC^* + C^*C = \sum_{i,j=1}^{n} \lambda_i\overline{\lambda_j}(C_iC_j^* + C_i^*C_j) = \sum_{i=1}^{n} |\lambda_i|^2\, I. \tag{10.21}$$

It follows that $\|P\| = \|Q\| = \|P + Q\| = \sum_{i=1}^{n} |\lambda_i|^2$. Therefore Φ is an isometry.

Recall that $C_iC_i^* = I_2^{\otimes(i-1)} \otimes E_{2,2} \otimes I_2^{\otimes(n-i)}$, $i = 1, \dots, n$. Therefore $C_iC_i^*$ is a diagonal matrix with ones and zeroes on the diagonal. Inspection reveals that the pattern of ones and zeros is 2^{i-1} zeroes followed by 2^{i-1} ones and so on down the diagonal. Therefore the matrix $\sum_{i=1}^{n} C_iC_i^*$ is diagonal with n being the largest diagonal entry. Hence, $\|\sum_{i=1}^{n} C_iC_i^*\| = n$.

Let $A \in M_n(M_n)$ be the matrix.

$$\begin{bmatrix} E_{1,1} & \dots & E_{n,1} \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}. \tag{10.22}$$

By permuting the columns of A we see that A is unitarily equivalent to the matrix which has 1's in the first n diagonal entries and is zero otherwise. Therefore $\|A\| = 1$. Now,

$$\|\Phi_n(A)\Phi_n(A)^*\| = \left\| \begin{bmatrix} C_1 & \dots & C_n \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} C_1^* & \dots & 0 \\ \vdots & & \vdots \\ C_n^* & \dots & 0 \end{bmatrix} \right\| \tag{10.23}$$

$$= \left\| \sum_{i=1}^{n} C_iC_i^* \right\| = n. \tag{10.24}$$

Hence $\|\Phi\|_{cb} \ge \|\Phi_n\| \ge \sqrt{n}$.

Chapter 11: Applications to K-spectral Sets.

There are no written solutions to this chapter

Chapter 12: Tensor Products

There are no written solutions to this chapter

13.1 Let V be a matrix normed space, let $X \in M_{n,m}(V)$ and let $Y \in M_{p,q}(V)$ be a matrix obtained from X by introducing finitely many rows and columns of 0's. Prove that $\|X\|_{m,n} = \|Y\|_{p,q}$.

Solution: Let

$$A = \left[\begin{array}{c|c} I_m & 0 \end{array}\right] \in M_{m,p}, B = \left[\begin{array}{c} I_n \\ 0 \end{array}\right] \in M_{q,n}. \tag{13.1}$$

We see then that $X = AYB$ and so $\|X\|_{m,n} \le \|A\| \|Y\|_{p,q} \|B\| \le \|Y\|_{p,q}$. Now setting

$$A = \left[\begin{array}{c} I_m \\ 0 \end{array}\right] \in M_{p,m}, B = \left[\begin{array}{c|c} I_n & 0 \end{array}\right] \in M_{n,q}, \tag{13.2}$$

we see that $AXB = Y$ and so $\|Y\|_{p,q} \le \|A\| \|X\|_{m,n} \|B\| = \|X\|_{m,n}$. Hence $\|X\|_{m,n} = \|Y\|_{p,q}$.

13.2 Let V be a vector space and assume that we are given a sequence of norms, $\|\cdot\|_n$ on $M_n(V)$ satisfying:

(i) $\|AXB\|_n \le \|A\| \|X\|_n \|B\|$ for $X \in M_n(V)$, $A \in M_n$ and $B \in M_n$.
(ii) for $X \in M_n(V)$, $\|X \oplus 0\|_{m+n} = \|X\|_n$, where 0 denotes an $m \times m$ matrix of 0's.

For $X \in M_{m,n}(V)$ set $\|X\|_{m+n} = \left\|\widehat{X}\right\|_l$ where $l = \max\{m,n\}$ and $\widehat{X}$ is the matrix obtained by adding sufficiently many rows or columns to X to make it square. Prove that $(V, \|\cdot\|_{m,n})$ is a matrix normed space.

These alternate axioms are often given as the axioms for a matrix normed space and, consequently, no mention is given of the norms of rectangular matrices.

Solution: Let $A \in M_{p,m}$, $X \in M_{m,n}$ and $B \in M_{n,q}$. Let $s = \max\{m,n,p,q\}$ and inflate A, B, X by introducing rows and columns of zeros to make them $s \times s$ matrices. Let $l = \max\{p,q\} \le s$, $k = \max\{m,n\}$. Now,

$$\begin{aligned}
\|AXB\|_{p,q} &= \left\|\widehat{AXB}\right\|_l & (13.3)\\
&= \left\|\begin{bmatrix} AXB & 0 \\ 0 & 0 \end{bmatrix}\right\| & (13.4)\\
&\le \left\|\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right\| \left\|\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\right\|_s \left\|\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}\right\| & (13.5)\\
&= \|A\| \|\widehat{X} \oplus 0\|_s \|B\| & (13.6)\\
&= \|A\| \|\widehat{X}\|_k \|B\| & (13.7)\\
&= \|A\| \|X\|_{m,n} \|B\|. & (13.8)
\end{aligned}$$

13.3 Let V be an operator space and let W be a closed subspace and let $\pi : V \to V/W$ denote the quotient map $\pi(v) = v + W$. Prove that if we define norms on $M_{m,n}(V/W)$ by setting

$$\|(\pi(v_{i,j}))\|_{m,n} = \inf\{\|v_{i,j} + w_{i,j}\|_{m,n} : w_{i,j} \in W\} \tag{13.9}$$

then V/W is an operator space.

Solution: Let $A \in M_{p,m}$ $X \in M_{m,n}$ and $B \in M_{n,q}$. Let $\varepsilon > 0$ and choose Z such that $\|X\|_{m,n} + \varepsilon > \|Z\|_{m,n}$ with $\pi_{m,n}(Z) = X$. It is a straighforward calculation to check that $\pi_{p,q}(AZB) = A\pi_{m,n}(Z)B = AXB$. Therefore,

$$\begin{aligned}
\|AXB\|_{p,q} &\leq \|AZB\|_{p,q} && (13.10)\\
&\leq \|A\| \, \|Z\|_{m,n} \, \|B\| && (13.11)\\
&\leq \|A\| \, \|X\|_{m,n} \, \|B\| + \varepsilon \, \|A\| \, \|B\| \, . && (13.12)
\end{aligned}$$

As the choice of ε was arbitrary we see that $\|AXB\|_{p,q} \leq \|A\| \, \|X\|_{m,n} \, \|B\|$.

We now check that this is an L^∞-matrix normed space. To do this we need only check that

$$\|X \oplus Y\|_{m+p,n+q} \leq \max\{\|X\|_{m,n} \, , \|Y\|_{p,q}\} \tag{13.13}$$

where $X \in M_{m,n}(V/W)$ and $Y \in M_{p,q}(V/W)$. Choose R, Z such that $\pi_{m,n}(R) = X$, $\pi_{p,q}(Z) = Y$ and $\|X\|_{m,n} + \varepsilon > \|Z\|_{m,n}$, $\|Y\|_{p,q} + \varepsilon > \|R\|_{p,q}$. Note that $\pi_{m+p,n+q}(R \oplus Z) = X \oplus Y$. We have,

$$\begin{aligned}
\|X \oplus Y\|_{m+p,n+q} &\leq \|R \oplus Z\|_{m+p,n+q} && (13.14)\\
&= \max\{\|R\|_{m,n} \, , \|Z\|_{p,q}\} && (13.15)\\
&\leq \max\{\|X\|_{m,n} \, , \|Y\|_{p,q}\} + \varepsilon. && (13.16)
\end{aligned}$$

By letting $\varepsilon \longrightarrow 0$ we get our result.

13.5 Verify the claims of Proposition 13.3.

Solution: Given an a matrix ordered $*$-vector space $\mathcal{S}$ with an Archimedean matrix order unit e we define

$$\|x\| = \inf\left\{r : \begin{bmatrix} re & x \\ x^* & re \end{bmatrix} \in \mathcal{C}_2\right\}. \tag{13.17}$$

Denote by S_x the set on the right hand side of (13.17). We have already seen that $\|x\| \geq 0$ and that $\|x\| = 0$ if and only if $x = 0$. We will now prove that $\|\lambda x\| = |\lambda| \, \|x\|$, $\|x + y\| \leq \|x\| + \|y\|$ and $\|x^*\| = \|x\|$. We may assume that $\lambda \neq 0$. Note that,

$$\begin{bmatrix} \lambda^{1/2} & 0 \\ 0 & \overline{\lambda}^{1/2} \end{bmatrix} \begin{bmatrix} re & x \\ x^* & re \end{bmatrix} \begin{bmatrix} \overline{\lambda}^{1/2} & 0 \\ 0 & \lambda^{1/2} \end{bmatrix} = \begin{bmatrix} |\lambda| \, re & \lambda x \\ \overline{\lambda} x^* & |\lambda| \, re \end{bmatrix}. \tag{13.18}$$

It follows that,

$$\begin{bmatrix} re & x \\ x^* & re \end{bmatrix} \in \mathcal{C}_2 \iff \begin{bmatrix} |\lambda| \, re & \lambda x \\ \overline{\lambda} x^* & |\lambda| \, re \end{bmatrix} \in \mathcal{C}_2. \tag{13.19}$$

Hence,

$$\begin{aligned}
\|\lambda x\| &= \inf\left\{r : \begin{bmatrix} re & \lambda x \\ \overline{\lambda} x^* & re \end{bmatrix} \in \mathcal{C}_2\right\} && (13.20)\\
&= \inf\left\{r : \begin{bmatrix} |\lambda|^{-1} re & x \\ x^* & |\lambda|^{-1} re \end{bmatrix} \in \mathcal{C}_2\right\} && (13.21)\\
&= \inf\left\{|\lambda| \, s : \begin{bmatrix} se & x \\ x^* & se \end{bmatrix} \in \mathcal{C}_2\right\} && (13.22)\\
&= |\lambda| \, \|x\| \, . && (13.23)
\end{aligned}$$

Let $x, y \in \mathcal{S}$ and note that if $r \in S_x$ and $s \in S_y$, then $r + s \in S_{x+y}$. It follows that, $\inf S_{x+y} \leq r + s$ for every $r \in S_x$ and $s \in S_y$. Therefore $\|x + y\| = \inf S_{x+y} \leq \inf S_x + \inf S_y = \|x\| + \|y\|$.

Finally we see that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} re & x \\ x^* & re \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} re & x^* \\ x & re \end{bmatrix}. \tag{13.24}$$

From which we get $\|x\| = \|x^*\|$.

14.1 Let V be a normed space, X an operator space and let $\varphi : X \to \mathrm{MIN}(V)$, $\psi : \mathrm{MAX}(V) \to X$ be linear maps. Prove that $\|\varphi\| = \|\varphi\|_{cb}$ and $\|\psi\| = \|\psi\|_{cb}$. Deduce that $\mathrm{MIN}(V)$ and $\mathrm{MAX}(V)$ are homogeneous.

Solution: Let $(x_{i,j}) \in M_n(X)$ with $\|(x_{i,j})\| \le 1$. Recall that if g is a linear functional on an operator space, then g is completely bounded and $\|g\|_{cb} = \|g\|$. Consider,

$$\begin{aligned}
\|\varphi_n((x_{i,j}))\| &= \sup\{\|(f(\varphi(x_{i,j})))\| \,:\, f \in V_1^*\} && (14.1)\\
&= \sup\{\|(f \circ \varphi)_n((x_{i,j}))\| \,:\, f \in V_1^*\} && (14.2)\\
&\le \sup \|(f \circ \varphi)_n\| && (14.3)\\
&= \|f \circ \varphi\| && (14.4)\\
&\le \|f\|\,\|\varphi\| && (14.5)\\
&\le \|\varphi\| && (14.6)
\end{aligned}$$

We may assume that $\|\psi\| \le 1$. We begin by noting that

$$\|(v_{i,j})\|_{M_n(\mathrm{MAX}(V))} = \sup\{\|(\varphi(v_{i,j}))\| \,:\, \varphi : V \to B(\mathcal{H}), \|\varphi\| \le 1\}. \quad (14.7)$$

To prove this, note that the equation in (14.7) defines an operator space norm on V which is larger than $\|\|_{\mathrm{MAX}(V)}$. These must be equal, since $\|\|_{\mathrm{MAX}(V)}$ is the largest operator space norm on V. By Ruan's theorem there exists a complete isometry $\rho : X \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Note that the map $\rho \circ \psi : V \to B(\mathcal{H})$ is contractive. Therefore,

$$\begin{aligned}
\|(\psi(v_{i,j}))\|_X &= \|\rho_n(\psi(v_{i,j}))\| && (14.8)\\
&= \|((\rho \circ \psi)(v_{i,j}))\| && (14.9)\\
&\le \|(v_{i,j})\|_{M_n(\mathrm{MAX}(V))}\,. && (14.10)
\end{aligned}$$

Hence $\|\psi_n\| \le 1$ and $\|\psi\| = \|\psi\|_{cb}$

14.2 Let V be a normed space, X an operator space and let $\varphi : \mathrm{MIN}(V) \to X$. Prove that $\|\varphi\|_{cb} \le \alpha(V)\,\|\varphi\|$.

Solution: We may assume $\alpha(V) < \infty$. Define $j : \mathrm{MIN}(V) \to \mathrm{MAX}(V)$ by $j(v) = v$, and $\psi : \mathrm{MAX}(V) \to X$ by $\psi(x) = \varphi(x)$. Note,

$$\|\psi(v)\| = \|\varphi(v)\| \le \|\varphi\|\,\|v\|_{\mathrm{MIN}(V)} \le \|\varphi\|\,\|v\|_{\mathrm{MAX}(V)}\,. \quad (14.11)$$

Therefore, ψ is bounded and $\|\psi\| \le \|\varphi\|$. By Exercise 14.1 ψ is completely bounded and $\|\psi\|_{cb} = \|\psi\|$. Since $\varphi = \psi \circ j$ we see that φ is completely bounded. Now,

$$\|\varphi\|_{cb} = \|\psi \circ j\|_{cb} \le \|\psi\|_{cb}\,\|j\|_{cb} = \alpha(V)\,\|\psi\| \le \alpha(V)\,\|\varphi\|\,. \quad (14.12)$$

14.3 (Zhang) Let $\mathbb{F}_n$ denote the free group on n generators

$$\{u_1^{(n)}, \ldots, u_n^{(n)}\}. \quad (14.13)$$

Prove that the maps $\varphi : \mathrm{MAX}(\ell_n^1) \to C_u^*(\mathbb{F}_n)$ and $\psi : \mathrm{MAX}(\ell_n^1) \to C_u^*(\mathbb{F}_{n-1})$ given by

$$\varphi((\lambda_1, \ldots, \lambda_n)) = \lambda_1 u_1^{(n)} + \ldots + \lambda_n u_n^{(n)}, \quad (14.14)$$

and

$$\psi((\lambda_1,\dots,\lambda_n)) = \lambda_1 u_1^{(n-1)} + \dots + \lambda_{n-1} u_{n-1}^{(n-1)} + \lambda_n I, \tag{14.15}$$

are complete isometries.

Solution: Let $e_1,\dots,e_n \in \ell_n^1$ denote the standard basis, let $\lambda = (\lambda_1,\dots,\lambda_n) \in \ell_n^1$ and let $(v_{i,j}) = \sum_{k=1}^n A_k \otimes e_k \in M_m(\text{MAX}(\ell_n^1))$. We have $\varphi_m((v_{i,j})) = \sum_{k=1}^n A_k \otimes u_k^{(n)}$. Assume that $\rho : \ell_n^1 \to B(\mathcal{H})$. From,

$$\|\rho(\lambda)\| = \left\| \sum_{k=1}^n \lambda_k \rho(e_k) \right\| \le \sum_{k=1}^n |\lambda_k| \, \|\rho(e_k)\| , \tag{14.16}$$

it follows that ρ is contractive if and only if $\rho(e_k)$ is a contraction on $\mathcal{H}$ for $k = 1,\dots,n$. By definition, and the observation just made, we have,

$$\left\| \sum_{k=1}^n A_k \otimes e_k \right\|_{\text{MAX}(\ell_n^1)} = \sup \left\{ \left\| \sum_{k=1}^n A_k \otimes \rho(e_k) \right\|_{B(\mathcal{H}^{(m)})} : \rho : \ell_n^1 \to B(\mathcal{H}), \|\rho\| \le 1 \right\} \tag{14.17}$$

$$= \sup \left\{ \left\| \sum_{k=1}^n A_k \otimes T_k \right\|_{B(\mathcal{H}^{(m)})} : T_1,\dots,T_n \in B(\mathcal{H}), \|T_k\| \le 1 \right\} \tag{14.18}$$

Now,

$$\left\| \sum_{k=1}^n A_k \otimes u_k^{(n)} \right\|_{C^*(\mathbb{F}_n)} = \sup \left\{ \left\| \sum_{k=1}^n A_k \otimes U_k \right\| : U_1,\dots,U_n \in U(\mathcal{H}), \mathcal{H} \text{ a Hilbert space} \right\} \tag{14.19}$$

where $U(\mathcal{H})$ denotes the unitary group of $B(\mathcal{H})$. If $\rho : \ell_n^1 \to B(\mathcal{H})$ then $\rho(e_k)$ is a contraction and so the supremum on the right side of (14.19) is smaller than the right side of (14.17).

By exercise 5.4, given any n contractions $T_1,\dots,T_n$ on $\mathcal{H}$ we can dilate $T_1,\dots,T_n$ to n unitaries $U_1,\dots,U_n$ on $\mathcal{K}$. Therefore,

$$\left\| \sum_{k=1}^n A_k \otimes T_k \right\|_{B(\mathcal{H}^{(n)})} \le \left\| \sum_{k=1}^n A_k \otimes U_k \right\|_{B(\mathcal{K}^{(n)})} . \tag{14.20}$$

Hence, (14.17) and (14.19) are equal.

If $(v_{i,j}) = \sum_{k=1}^n A_k \otimes e_k \in M_m(\text{MAX}(\ell_n^1))$, then $\psi_m((v_{i,j})) = \sum_{k=1}^{n-1} A_k \otimes u_k^{(n-1)} + A_n \otimes I$. The norm of ψ_m is given by,

$$\left\| \sum_{k=1}^{n-1} A_k \otimes u_k^{(n-1)} + A_n \otimes I \right\|_{C^*(\mathbb{F}_{n-1})} \tag{14.21}$$

By the universal property of $C^*(\mathbb{F}_n)$ there is a $*$-homomorphism $\pi : C^*(\mathbb{F}_n) \to C^*(\mathbb{F}_{n-1})$ such that $u_k^{(n)} \mapsto u_{k-1}^{(n-1)}$ for $k = 1,\dots,n-1$ and $u_n^{(n)} \mapsto 1$. Similarly there is a $*$-homomorphism $\sigma : C^*(\mathbb{F}_{n-1})$ to $C^*(\mathbb{F}_n)$ such that$u_k^{(n-1)} \to u_n^{(n)*} u_k^{(n)}$. We have,

$$(\pi \circ \sigma)(u_k^{(n-1)}) = \pi(u_n^{(n)*} u_k^{(n)}) = u_k^{(n-1)}. \tag{14.22}$$

Therefore σ is one-one and a $*$-isomorphism. Now,

$$\left\|\sum_{k=1}^{n-1} A_k \otimes u_k^{(n-1)} + A_n \otimes I\right\|_{C^*(\mathbb{F}_{n-1})} = \left\|\sum_{k=1}^{n-1} A_k \otimes u_n^{(n)*} u_k^{(n)} + A_n \otimes u_n^{(n)} u_n\right\|_{C^*(\mathbb{F}_n)} \tag{14.23}$$

$$= \left\|\sum_{k=1}^{n} (I_n \otimes u_n^{(n)})(A_k \otimes u_k^{(n)})\right\|_{C^*(\mathbb{F}_n)} \tag{14.24}$$

$$= \left\|(I_n \otimes u_n^{(n)}) \sum_{k=1}^{n} A_k \otimes u_k^{(n)}\right\|_{C^*(\mathbb{F}_n)} \tag{14.25}$$

$$= \left\|\sum_{k=1}^{n} A_k \otimes u_k^{(n)}\right\|_{C^*(\mathbb{F}_n)}. \tag{14.26}$$

(14.26) is due to the fact that $I_n \otimes u_n^{(n)}$ is a unitary in $M_m(C^*(\mathbb{F}_n))$.

14.4 Prove that the maps $\varphi : \mathrm{MIN}(\ell_n^1) \to C(\mathbb{T}^n)$ and $\psi : \mathrm{MIN}(\ell_n^1) \to C(\mathbb{T}^{n-1})$ given by

$$\varphi((\lambda_1, \ldots, \lambda_n)) = \lambda_1 z_1 + \ldots + \lambda_n z_n \tag{14.27}$$

and

$$\psi((\lambda_1, \ldots, \lambda_n)) = \lambda_1 z_1 + \ldots + \lambda_{n-1} z_{n-1} + \lambda_n \tag{14.28}$$

are complete isomteries.

Solution: Let $(v_{i,j}) \in M_m(\ell_n^1)$, $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \ell_m^2$ and $z = (z_1, \ldots, z_n) \in \mathbb{T}^n$. If $\mu = (\mu_1, \ldots, \mu_n) \in \ell_n^1$, then

$$\|\mu\|_{\ell_n^1} = \sup_{z_1, \ldots, z_n \in \mathbb{T}} |\mu_1 z_1 + \ldots + \mu_n z_n|. \tag{14.29}$$

It is a consequence of the triangle inequality that the right hand side of (14.29) is smaller than the left hand side. To prove the reverse inequality we simply choose $z_j \in \mathbb{T}$, $j = 1, \ldots, n$ such that $\mu_j z_j = |\lambda_j|$ for $j = 1, \ldots, n$. To see that φ is a complete isometry we compute,

$$\|(\varphi(v_{i,j}))\|_{M_m(C(\mathbb{T}^n))} = \sup_{z \in \mathbb{T}^n} \|(\varphi(v_{i,j})z)\|_{M_m} \tag{14.30}$$

$$= \sup_{z \in \mathbb{T}^n} \sup_{\|\alpha\|_2 \leq 1, \|\beta\|_2 \leq 1} |\langle(\varphi(v_{i,j})z)\alpha, \beta\rangle| \tag{14.31}$$

$$= \sup_{z \in \mathbb{T}^n} \sup_{\|\alpha\|_2 \leq 1, \|\beta\|_2 \leq 1} \left|\sum_{i,j=1}^{m} \varphi(v_{i,j})(z)\overline{\alpha_i}\beta_j\right| \tag{14.32}$$

$$= \sup_{z \in \mathbb{T}^n} \sup_{\|\alpha\|_2 \leq 1, \|\beta\|_2 \leq 1} \left|\sum_{i,j=1}^{m} \varphi(v_{i,j})(z)\alpha_i\beta_j\right| \tag{14.33}$$

$$= \sup_{\|\alpha\|_2 \leq 1, \|\beta\|_2 \leq 1} \sup_{z_1, \ldots, z_n \in \mathbb{T}} \left|\sum_{k=1}^{n} \left(\sum_{i,j=1}^{m} \alpha_i \lambda_{i,j}^{(k)} \beta_j\right) z_k\right| \tag{14.34}$$

$$= \sup_{\|\alpha\|_2 \leq 1, \|\beta\|_2 \leq 1} \left\|\sum_{i,j=1}^{m} \alpha_i v_{i,j} \beta_j\right\|_{\ell_n^1} \tag{14.35}$$

$$= \|(v_{i,j})\|_{M_m(\mathrm{MIN}(\ell_n^1))}. \tag{14.36}$$

To establish that ψ is a complete isometry we need only prove the following analogue of (14.29):

$$\|\mu\|_{\ell_n^1} = \sup_{z_1,\ldots,z_{n-1}\in\mathbb{T}} |\mu_1 z_1 + \ldots + \mu_{n-1}z_{n-1} + \mu_n| \,. \tag{14.37}$$

If we choose $w \in \mathbb{T}$ such that $w\mu_n = |\mu_n|$ and for $j = 1, \ldots, n-1$, pick $z_j \in \mathbb{T}$ such that $z_j\mu_j = w^{-1}|\mu_j|$, then

$$\begin{aligned} \|\mu\|_{\ell_n^1} &= |z_1 w\mu_1 + \ldots + z_{n-1}w\mu_{n-1}| & (14.38)\\ &= |w|\,|z_1\mu_1 + \ldots + z_{n-1}\mu_{n-1} + \mu_n| & (14.39)\\ &= |z_1\mu_1 + \ldots + z_{n-1}\mu_{n-1} + \mu_n| \,. & (14.40) \end{aligned}$$

This implies (14.37).

14.5 Prove Proposition 14.7.

Solution: We endow $M_{m,n}(CB(E,F))$ with the norm it inherits through its identification with $CB(E, M_{m,n}(F))$. Let $\Phi = (\varphi_{i,j}) \in M_{m,n}(CB(E,F))$ and let $X = (x_{k,l}) \in M_r(E)$. Let $A \in M_{p,m}$ and $B \in M_{n,q}$.We have,

$$\begin{aligned} (A\Phi B)_r((x_{k,l})) &= ((A\Phi B)(x_{k,l})) & (14.41)\\ &= (A\Phi(x_{k,l})B) & (14.42)\\ &= \underbrace{(A\oplus\ldots\oplus A)}_{r\text{ times}}\,\Phi_r((x_{k,l}))\,\underbrace{(B\oplus\ldots\oplus B)}_{r\text{ times}}. & (14.43) \end{aligned}$$

Hence,

$$\|(A\Phi B)_r)((x_{k,l}))\| \le \|A\|\,\|\Phi_r((x_{k,l}))\|\,\|B\| \,. \tag{14.44}$$

It follows,

$$\begin{aligned} \|A\Phi B\|_{CB(E,F)} &= \|A\Phi B\|_{cb} & (14.45)\\ &= \sup_{r\ge1}\|(A\Phi B)_r\| & (14.46)\\ &= \sup_{r\ge1}\|A\|\,\|\Phi_r\|\,\|B\| & (14.47)\\ &= \|A\|\,\|\Phi\|_{cb}\,\|B\| & (14.48)\\ &= \|A\|\,\|\Phi\|_{CB(E,F)}\,\|B\| \,. & (14.49) \end{aligned}$$

Let $\Phi \in M_{m,n}(CB(E,F))$ and $\Psi \in M_{p,q}(CB(E,F))$. Now,

$$\begin{aligned} (\Phi\oplus\Psi)_r((x_{k,l})) &= ((\Phi\oplus\Psi)(x_{k,l})) & (14.50)\\ &= \left(\begin{bmatrix} \Phi(x_{k,l}) & 0 \\ 0 & \Psi(x_{k,l}) \end{bmatrix}\right). & (14.51) \end{aligned}$$

A permutation of the rows of this last matrix shows that it has the same norm as

$$\begin{bmatrix} \Phi_r((x_{k,l})) & 0 \\ 0 & \Psi_r((x_{k,l})) \end{bmatrix}. \tag{14.52}$$

Hence,

$$\begin{aligned}
\|\Phi \oplus \Psi\|_{CB(E,F)} &= \|\Phi \oplus \Psi\|_{cb} && (14.53)\\
&= \sup_{r\geq 1} \|(\Phi \oplus \Psi)_r\| && (14.54)\\
&= \sup_{r\geq 1} \max\{\|\Phi_r\|, \|\Psi_r\|\} && (14.55)\\
&= \max\{\sup_{r\geq 1} \|\Phi_r\|, \sup_{r\geq 1} \|\Psi_r\|\} && (14.56)\\
&= \max\{\|\Phi\|_{CB(E,F)}, \|\Psi\|_{CB(E,F)}\}. && (14.57)
\end{aligned}$$

Thus, $CB(E,F)$ is an operator space.

14.8 Let $\gamma : C_n \to R_n$ be the map $\gamma(\sum_{i=1}^n \lambda_i E_{i,1}) = \sum_{i=1}^n \lambda_i E_{1,i}$. Prove that γ is an isometry and $\|\gamma\|_{cb} = \|\gamma^{-1}\|_{cb} = \sqrt{n}$.

Solution: As γ is the transpose map γ is an isometry. Let $C = (C_{i,j}) \in M_m(C_n)$ where

$$C_{i,j} = \left[\begin{array}{c|c} \lambda_1^{(i,j)} & \\ \vdots & 0 \\ \lambda_n^{(i,j)} & \end{array}\right]. \qquad (14.58)$$

Let

$$R_{i,j} = \gamma(C_{i,j}) = \left[\begin{array}{ccc} \lambda_1^{(i,j)} & \dots & \lambda_n^{(i,j)} \\ \hline & 0 & \end{array}\right] \qquad (14.59)$$

By a canonical shuffle we see that

$$\|(C_{i,j})\| = \left\|\left[\begin{array}{c|c} A_1 & \\ \dots & 0 \\ A_n & \end{array}\right]\right\|, \qquad (14.60)$$

where $(A_k)_{i,j} = \lambda_k^{(i,j)}$ and $A_k \in M_m$. Similarly

$$\|(R_{i,j})\| = \left\|\left[\begin{array}{ccc} A_1 & \cdots & A_n \\ \hline & 0 & \end{array}\right]\right\|. \qquad (14.61)$$

Now,

$$\begin{aligned}
\left\|\left[\begin{array}{c|c} A_1 & \\ \dots & 0 \\ A_n & \end{array}\right]\right\|^2 &= \left\|\left[\begin{array}{c|c} A_1 & \\ \dots & 0 \\ A_n & \end{array}\right]^* \left[\begin{array}{c|c} A_1 & \\ \dots & 0 \\ A_n & \end{array}\right]\right\| && (14.62)\\
&= \left\|\begin{bmatrix} \sum_{i=1}^n A_i^* A_i & 0 \\ 0 & 0 \end{bmatrix}\right\| && (14.63)\\
&= \left\|\sum_{i=1}^n A_i^* A_i\right\|, && (14.64)
\end{aligned}$$

and,

$$\left\| \left[\begin{array}{ccc} A_1 & \cdots & A_n \\ \hline & 0 & \end{array} \right] \right\|^2 = \left\| \left[\begin{array}{ccc} A_1 & \cdots & A_n \\ \hline & 0 & \end{array} \right] \left[\begin{array}{c|c} A_1^* & \\ \cdots & 0 \\ A_n^* & \end{array} \right] \right\| \tag{14.65}$$

$$= \left\| \begin{bmatrix} \sum_{i=1}^n A_i A_i^* & 0 \\ 0 & 0 \end{bmatrix} \right\| \tag{14.66}$$

$$= \left\| \sum_{i=1}^n A_i A_i^* \right\|, \tag{14.67}$$

Since $\|A_k A_k^*\| \leq \|\sum_{i=1}^n A_i A_i^*\|$ we see that,

$$\left\| \sum_{i=1}^n A_i^* A_i \right\| \leq \sum_{i=1}^n \|A_i\|^2 \tag{14.68}$$

$$= \sum_{i=1}^n \|A_i A_i^*\| \tag{14.69}$$

$$\leq n \left\| \sum_{i=1}^n A_i A_i^* \right\| \tag{14.70}$$

A similar argument shows that $\|\sum_{i=1}^n A_i A_i^*\| \leq n \,\|\sum_{i=1}^n A_i^* A_i\|$. Therefore

$$\|\gamma_m\| \leq \sqrt{n} \text{ and } \|\gamma_m^{-1}\| \leq \sqrt{n} \tag{14.71}$$

for all $m \geq 1$.

To prove equality in (14.71) we consider $E \in M_n(C_n)$ given by

$$E = \left[\begin{array}{c|c} e_1 & \\ \vdots & 0 \\ e_n & \end{array} \right]. \tag{14.72}$$

by permuting rows we see that $\|E\| = \sqrt{n}$. Note that a permutation of the rows of $\gamma_n(E)$ brings it to the form

$$\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}. \tag{14.73}$$

Therefore $\|\gamma_n(E)\| = 1$. It follows that $\|\gamma_n^{-1}\| \geq \sqrt{n}$. By considering

$$E = \left[\begin{array}{ccc} e_1^T & \cdots & e_n^T \\ \hline & 0 & \end{array} \right], \tag{14.74}$$

we get $\|\gamma_n\| \geq \sqrt{n}$. Combining these with the inequalities in (14.71) we get

$$\|\gamma\|_{cb} = \|\gamma^{-1}\|_{cb} = \sqrt{n}. \tag{14.75}$$

15.1 Verify that an operator space I is injective in $\mathcal{O}_1$ if and only if every completely bounded map into I has a completely bounded extension of the same completely bounded norm.

Solution: Let $\varphi : E \to I$ be a completely bounded map and let $E \subseteq F$. We may assume that $\varphi \neq 0$. If every completely bounded map into I has an extension of the same completely bounded norm, then trivially every completely contractive map into I has a a completely contractive extension.

For the converse let $\tilde{\varphi} = \varphi / \|\varphi\|_{cb}$. Then $\|\tilde{\varphi}\|_{cb} = 1$ and since I is injective in $\mathcal{O}_1$, there exists $\tilde{\psi} : F \to I$ such that $\|\tilde{\psi}\|_{cb} = 1$. Let $\psi = \|\varphi\|_{cb} \tilde{\psi}$. It is straightforward that ψ extends φ and $\|\psi\|_{cb} = \|\varphi\|_{cb} \|\tilde{\psi}\|_{cb} = \|\varphi\|_{cb}$.

15.2 Let $E \subseteq B(\mathcal{H})$ be an operator space. Prove that E is injective if and only if there exists a completely contractive map $\varphi : B(\mathcal{H}) \to E$ such that $\varphi(e) = e$ for an $e \in E$.

Solution: Assume first that E is injective. Extend the identity map on E to a map $\varphi : B(\mathcal{H}) \to E$. The map φ is completely contractive, because the identity is a complete isometry. If $e \in E$, then

$$\varphi(e) = \mathrm{id}_E(e) = e. \tag{15.1}$$

For the converse let $F \subseteq G$ be operator spaces and let $\psi : F \to E$ be a complete contraction. By composing with the inclusion map $j : E \to B(\mathcal{H})$ we get a complete contraction $j \circ \psi : F \to B(\mathcal{H})$. By Wittstock's theorem there exists a complete contraction $\rho : G \to B(\mathcal{H})$ such that $\rho|_F = j \circ \psi$. The map $\varphi \circ \rho : G \to E$ is completely contractive and

$$(\varphi \circ \rho)(f) = \varphi(j(\psi(f))) = \psi(f), \tag{15.2}$$

since ρ extends $j \circ \psi$ and φ fixes E.

15.3 Let $M \subseteq B(\mathcal{H})$ be an operator space and let $\varphi : B(\mathcal{H}) \to B(\mathcal{H})$ be an M-projection with p_φ a minimal M-seminorm. Assume that $u \in B(\mathcal{H})$ is a unitary that commutes with M. Prove that if $\gamma(x) = u^*\varphi(x)u$ then p_γ is a minimal M-seminorm and γ is a projection onto $u^*\varphi(B(\mathcal{H}))u$. Prove that if $\psi(x) = \varphi(u^*xu)$ then p_ψ is a minimal M-seminorm and ψ is a (possibly different) projection onto $\varphi(B(\mathcal{H}))$.

Solution: If $x \in M$, then

$$\gamma(x) = u^*\varphi(x)u = u^*xu = u^*ux = x. \tag{15.3}$$

This shows that γ fixes M. Assume that p is a seminorm such that $p \leq p_\gamma$. Note that

$$p_\gamma(x) = \|u^*\varphi(x)u\| = \|\varphi(x)\| = p_\varphi(x), \tag{15.4}$$

since u is unitary. Now,

$$p(x) \leq p_\gamma(x) = p_\varphi(x), \tag{15.5}$$

and so by the minimailty of φ we have that $p = p_\varphi = p_\gamma$. This shows that p_γ is a minimal M-seminorm and from theorem 15.4 that γ is an M-projection. The range of γ is clearly a subset of $u^*\varphi(B(\mathcal{H}))u$. The fact that,

$$\gamma(B(\mathcal{H})) = u^*\varphi(B(\mathcal{H}))u, \tag{15.6}$$

follows directly from $\gamma(x) = u^*\varphi(x)u$.

Next consider the map ψ. If $x \in M$, then,

$$\psi(x) = \varphi(u^*xu) = \varphi(u^*ux) = \varphi(x) = x, \tag{15.7}$$

and so ψ is an M-map. Assume that $p \le p_\psi$ and define a $\tilde{p}(x) = p(uxu^*)$. Now,

$$\tilde{p}(x) = p(uxu^*) \le \|\psi(uxu^*)\| = \|\varphi(x)\| = p_\varphi(x). \tag{15.8}$$

From the minimality of p_φ we have that $\tilde{p} = p_\varphi$. Hence,

$$p(x) = \tilde{p}(u^*xu) = \|\varphi(u^*xu)\| = \|\psi(x)\| = p_\psi(x). \tag{15.9}$$

Once again by Theorem 15.4 ψ is a minimal M-projection

15.5 Let $\varphi : B(\mathcal{H}) \to B(\mathcal{H})$ be a unital completely positive map that fixes the compacts. Prove that φ is necessarily the identity map.

Solution: As every operator is a linear combination of at most 4 positive operators, it is enough to prove that φ fixes every positive operator. In fact it is enough to show that φ fixes every invertible positive operator. To see this note that $P + \varepsilon I$ is invertible for $\varepsilon > 0$ and $P \ge 0$. If $\varphi(P + \varepsilon I) = P + \varepsilon I$, then $\varphi(P) = P$, since φ is unital.

We will need a Cholesky-type decomposition for positive operators on a separable Hilbert space which we now describe.

Cholesky Decomposition Let $P = (p_{i,j})_{i,j=1}^\infty \in B(\ell^2)$ be positive and invertible.

$$P = \begin{bmatrix} p_{1,1} & A \\ A^* & B \end{bmatrix}, \tag{15.10}$$

with $p_{1,1} > 0$. By the Cholesky decomposition described in exercise 3.9 we see that the operator

$$P_1 = \begin{bmatrix} 0 & 0 \\ 0 & B - p_{1,1}^{-1}A^*A \end{bmatrix} = P - \begin{bmatrix} p_{1,1}^{-1/2} & 0 \\ A^* & 0 \end{bmatrix} \begin{bmatrix} p_{1,1}^{-1/2} & A \\ 0 & 0 \end{bmatrix} \ge 0 \tag{15.11}$$

Let $R_1 = \begin{bmatrix} p_{1,1}^{-1/2} & A \\ 0 & 0 \end{bmatrix}$. If we now repeat this process we get a sequence R_k of rank one operators such that

$$0 \le P - \sum_{k=1}^{n} R_k^* R_k \le \begin{bmatrix} 0 & 0 \\ 0 & B_n \end{bmatrix}, \tag{15.12}$$

where B_n denotes the compression of P to the subspace

$$\mathcal{M}_n := \{(x_k)_{k=1}^\infty \,:\, x_1 = \ldots = x_n = 0\}. \tag{15.13}$$

Therefore,

$$\lim_{n\to\infty} \sum_{k=1}^{n} R_k^* R_k = P, \tag{15.14}$$

in the SOT.

Let P be positive. Then there exists a sequence of operators $\{R_k\}_{k=1}^\infty$ such that R_k is rank one, $K_n = \sum_{k=1}^n R_k^* R_k \le P$ for all $n \ge 1$ and $\sum_{k=1}^\infty R_k^* R_k = P$ in the strong operator topology.

Since φ fixes the compacts we see that $K_n = \varphi(K_n) \le \varphi(P)$. Hence,

$$\langle Px, x\rangle = \lim_{n\to\infty} \langle K_n x, x\rangle \le \langle \varphi(P)x, x\rangle, \tag{15.15}$$

which says that $P \leq \varphi(P)$. By choosing a constant α such that $\alpha I - P$ is positive and using the arguments above we see that

$$\varphi(\alpha I - P) \geq \alpha I - P. \tag{15.16}$$

As φ is unital we get $\varphi(P) \leq P$.

We now deal with the non-separable case. Let x be in $\mathcal{H}$. Let $\mathcal{M}$ be the closed linear span of $\{P^n x : n \geq 0\}$, which is a reducing subspace for P. With respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$ the matrix of P has the form

$$P = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \tag{15.17}$$

with $A, B \geq 0$. By the Cholesky decomposition we see that there is a sequence $\{K_n\}$ of positive compact operators on $\mathcal{M}$ which are bounded above by A and converge to A in the SOT. We have,

$$\begin{bmatrix} K_n & 0 \\ 0 & 0 \end{bmatrix} = \varphi \begin{bmatrix} K_n & 0 \\ 0 & 0 \end{bmatrix} \leq \varphi \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \varphi(P). \tag{15.18}$$

Therefore,

$$\langle Px, x \rangle = \langle Ax, x \rangle = \lim_{n \to \infty} \langle K_n x, x \rangle \leq \langle \varphi(P)x, x \rangle. \tag{15.19}$$

Arguing as in the separable case we see that $\varphi(P) = P$.

15.6 Let $\mathcal{S} \subseteq B(\mathcal{H})$ be an operator system which contains the compacts. Prove that $I(\mathcal{S}) = B(\mathcal{H})$.

Solution: Recall that $I(\mathcal{S})$ is the image of a completely positive map $\varphi : B(\mathcal{H}) \to B(\mathcal{H})$ which fixes $\mathcal{S}$. This map therefore fixes the compacts and is unital. By the previous exercise, φ must be the identity map. Therefore, $I(\mathcal{S}) = \varphi(B(\mathcal{H})) = B(\mathcal{H})$.

15.7 Let $\mathcal{A} \subseteq B(\ell^2)$ be the algebra of upper triangular operators. Prove that $C_e^*(\mathcal{A}) = B(\ell^2)$.

Solution: Let P be a positive invertible operator. Note that by the Cholesky decomposition described in exercise 15.5, there exists a sequence of operators $K_n = \sum_{k=1}^{n} R_k^* R_k$ such that $K_n \to P$ in the SOT. Therefore $P = U^*U$ where U is an upper-triangular operator whose rows are the R_k's. It follows that U is bounded and so the C^*-algebra generated by the upper-triangular operators contains every positive invertible operator. Therefore, $C^*(\mathcal{A}) = B(\ell^2)$. By Hamana's theorem (Theorem 15.16) there exists a onto $*$-homomorphism $\pi : B(\ell^2) \to C_e^*(\mathcal{A})$ which fixes $\mathcal{A}$. If π is not one-one, the kernel of π, which is a two-sided ideal in $B(\ell^2)$ must contain $K(\mathcal{H})$, the ideal of compact operators, and so $\pi(K(\mathcal{H})) = \{0\}$. Since $\mathcal{A}$ contains non-zero compact operators we see that π is a $*$-isomorphism. We have

$$\pi(P) = \pi(U^*U) = \pi(U)^*\pi(U) = U^*U = P. \tag{15.20}$$

Thus, π fixes all the positive invertible operators and must therefore be the identity map. Hence, $C_e^*(\mathcal{A}) = B(\ell^2)$.

15.8 Prove that $\partial_S A(\mathbb{D}) = \mathbb{T}$.

Solution: As $A(\mathbb{D}) \subseteq C(\mathbb{T})$ we see that $\partial_S A(\mathbb{D}) \subseteq \mathbb{T}$. We know that the restriction map $r : C(\mathbb{T}) \to C(\partial_S A(\mathbb{D}))$ given by $r(f) = f|_{\partial_S A(\mathbb{D})}$ is isometric on $A(\mathbb{D})$. For each $z \in \mathbb{T}$, let $f_z(e^{i\theta}) = z + e^{i\theta}$. It is clear that $\|f_z\|_{C(\mathbb{T})} = 2$ and that this is attained at $e^{i\theta} = z$. Also, $|f_z(e^{i\theta})| < 2$ at all other points on the circle. Therefore, r cannot be an isometry on $A(\mathbb{D})$ unless $\partial_S A(\mathbb{D}) = \mathbb{T}$.

15.9 Prove that if $\mathcal{A} \subseteq M_n$, then $C_e^*(\mathcal{A})$ is finite-dimensional.

Solution: By Hamana's theorem there exists an onto, $*$-homomorphism $\pi : C^*(\mathcal{A}) \to C_e^*(\mathcal{A})$. Being a subspace of M_n, $C^*(\mathcal{A})$ is finite-dimensional and $C^*(\mathcal{A})/\ker(\pi)$ is $*$-isomorphic to $C_e^*(\mathcal{A})$. Hence, the C^*-envelope of $\mathcal{A}$ is finite-dimensional.

15.12 Let $\{u_1, \ldots, u_n\}$ denote the unitaries in $C^*(\mathbb{F}_n)$ and let M denote the $(n+1)$-dimensional subspace spanned by these unitaries and the identity I. If $\mathcal{A} \subseteq M_2(C^*(\mathbb{F}_n))$ denotes the $(n+3)$-dimensional operator algebra

$$\mathcal{A} = \left\{ \begin{bmatrix} \lambda I & x \\ 0 & \mu I \end{bmatrix} : \lambda, \mu \in \mathbb{C}, x \in M \right\}, \tag{15.21}$$

then prove that $C_e^*(\mathcal{A}) = M_2(C^*(\mathbb{F}_n))$. Thus, a finite-dimensional operator algebra can have an infinite-dimensional C^*-envelope.

Solution: By Hamana's theorem there exists an onto $*$-homomorphism from $C^*(\mathcal{A}) \to C_e^*(\mathcal{A})$ such that $\pi(a) = a$. We will show that the C^*-algebra generated by $\mathcal{A}$ is all of $M_2(C^*(\mathbb{F}_n))$ and that this homomorphism is one-one.

We see that

$$\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}^* = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \in C^*(\mathcal{A}). \tag{15.22}$$

If u is one of the generating unitaries, then

$$\begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & u \end{bmatrix} \in C^*(\mathcal{A}). \tag{15.23}$$

Therefore, $M_2(C^*(\mathbb{F}_n)) = C^*(\mathcal{A})$.

We now check that π is one-one??

By matrix factorings similar to the ones used above we can check that π fixes all of $M_2(C^*(\mathbb{F}_n))$. Therefore, $M_2(C^*(\mathbb{F}_n)) = C^*(\mathcal{A})$.

16.1 Let C_n denote n-dimensional column Hilbert space. Prove that $I(S_{C_n}) \cong M_{n+1}$ in such a way that

$$I_{11}(C_n) = M_l(C_n) \cong M_n \text{ and } I_{22} = M_r(C_n) \cong \mathbb{C}. \tag{16.1}$$

Solution: Since C_n inherits its operator space structure from $M_{n,1}$, the operator system

$$S_{C_n} := \left\{ \begin{bmatrix} \lambda I_n & x \\ y^* & \mu \end{bmatrix} : \lambda, \mu \in \mathbb{C}, x, y \in C_n \right\}, \tag{16.2}$$

is naturally identified as an operator system in M_{n+1}. M_{n+1} is an injective C^*-algebra containing S_{C_n} and we claim that it is minimal. Suppose that E is injective and $S_{C_n} \subseteq E \subseteq M_{n+1}$. Then there is a completely positive map $\varphi : M_{n+1} \to E \subset M_{n+1}$ which fixes S_{C_n}. We claim that the only completely positive map $\varphi : M_{n+1} \to M_{n+1}$ which fixes S_{C_n} is the identity map on M_{n+1}. To prove this we will show that φ is the Schur multiplier induced by the matrix of all 1's.

If φ fixes S_{C_n}, then φ is unital and fixes the matrix units $E_{j,n+1}$ and $E_{n+1,j}$ for all $j = 1, \ldots, n+1$. Let $1 \leq j \leq n+1$ and note that

$$\begin{aligned} \varphi(E_{j,j}) &= \varphi(E_{j,n+1}E_{n+1,j}) && (16.3)\\ &= \varphi(E^*_{n+1,j}E_{n+1,j}) && (16.4)\\ &\geq \varphi(E_{n+1})^*\varphi(E_{n+1,j}) && (16.5)\\ &= E^*_{n+1,j}E_{n+1,j} = E_{j,j}. && (16.6) \end{aligned}$$

Now,

$$\varphi(I - E_{j,j}) = \varphi(\sum_{k \neq j} E_{k,k}) \geq \sum_{k \neq j} E_{k,k} = I - E_{j,j}. \tag{16.7}$$

It follows that $\varphi(E_{j,j}) \leq E_{j,j}$. We have established that φ fixes the diagonal matrices and is consequently a Schur multiplier. Let J be the $(n+1) \times (n+1)$ matrix all of whose entries are 1. The matrix which induces the multiplier is given by $\varphi(J)$. Let $R = \sum_{j=1}^{n+1} E_{n+1,j}$. We have,

$$\varphi(J) = \varphi(R^*R) \geq \varphi(R)^*\varphi(R) = R^*R = J. \tag{16.8}$$

From the fact that φ in unital it follows that the diagonal entries of $\varphi(J)$ are all 1. The matrix $\varphi(J) - J$ is positive and has zeros on the diagonal and must therefore be 0. Hence $\varphi(J) = J$ and so φ is the identity map. Having established that $I(S_{C_n}) \cong M_{n+1}$ we see that $I_{11}(C_n)$ is the $n \times n$ top left corner of M_{n+1} which is M_n. Similarly $I_{22}(C_n)$ is the bottom right 1×1 corner which we identify with $\mathbb{C}$.

16.2 Prove, by showing that the Blecher-Ruan-Sinclair axioms are met, that the matrix-normed algebra $(\mathcal{P}_n, \|\cdot\|_{u,k})$, introduced in Chapter 5, is a unital operator algebra.

Solution: Note that the universal operator algebra has matrix norms defined only for square matrices. It is easy to check that $\|\cdot\|_k$ satisfies the conditions of exercise 13.2 and so $\mathcal{P}_n$ is a matrix normed space. We now claim that it is an L^∞ matrix normed space. If $\{T_1, \ldots, T_n\}$ are n commuting contractions on a Hilbert space $\mathcal{H}$ then note that the operator matrices $(p_{i,j}(T_1, \ldots, T_n))_{i,j=1}^m$ are elements of the concrete operator algebra $B(\mathcal{H}^{(m)})$.

If $A, B \in M_k$ and $X = (p_{i,j}) \in M_k(\mathcal{P}_n)$, then

$$\begin{aligned} \|AXB\| &= \sup \left\| \left(\sum_{l,m=1}^{k} a_{i,l} p_{l,m}(T_1, \ldots, T_n) b_{m,j} \right) \right\| & (16.9) \\ &\leq \sup \|(a_{i,j}I)\| \, \|(p_{i,j}((T_1, \ldots, T_n))\| \, \|(b_{i,j}I)\| & (16.10) \\ &= \|A\| \, \|(p_{i,j})\|_{u,k} \, \|B\| & (16.11) \end{aligned}$$

which shows that $(\mathcal{P}_n, \|\cdot\|_{u,k})$ is a matrix normed space. Let $(p_{i,j}) \in M_k(\mathcal{P}_n)$ and $(q_{s,t}) \in M_m(\mathcal{P}_n)$.

$$\begin{aligned} \|(p_{i,j}) \oplus (q_{s,t})(T_1, \ldots, T_n)\| &= \|(p_{i,j}(T_1, \ldots, T_n)) \oplus (q_{s,t}(T_1, \ldots, T_n))\| & (16.12) \\ &= \max\{\|(p_{i,j}(T_1, \ldots, T_n))\| \, , \|(q_{s,t}(T_1, \ldots, T_n))\|\}. & (16.13) \end{aligned}$$

where the last equality follows from the fact that we are in a concrete operator algebra, as $(p_{i,j}(T_1, \ldots, T_n))$ and $(q_{s,t}(T_1, \ldots, T_n))$ are operators on a Hilbert space. It follows that,

$$\begin{aligned} \|(p_{i,j}) \oplus (q_{s,t})\|_{u,k+m} &= \sup \|(p_{i,j}) \oplus (q_{s,t})(T_1, \ldots, T_n)\| & (16.14) \\ &= \sup\{\max\{(p_{i,j}(T_1, \ldots, T_n)), (q_{s,t}(T_1, \ldots, T_n))\}\} & (16.15) \\ &= \max\{\sup\{(p_{i,j}(T_1, \ldots, T_n))\}, \sup\{(q_{s,t}(T_1, \ldots, T_n))\}\} & (16.16) \\ &= \max\{\|(p_{i,j})\|_{u,k} \, , \|(q_{s,t})\|_{u,m}\}. & (16.17) \end{aligned}$$

It remains to show that the multiplication on $M_k(\mathcal{P}_n)$ is contractive. We have,

$$\begin{aligned} \|(p_{i,j})(q_{i,j})\|_{u,k} &= \sup \|(p_{i,j}(T_1, \ldots, T_n))(q_{i,j}(T_1, \ldots, T_n))\| & (16.18) \\ &\leq \sup \|(p_{i,j}(T_1, \ldots, T_n))\| \, \|(q_{i,j}(T_1, \ldots, T_n))\| & (16.19) \\ &\leq \|(p_{i,j})\|_{u,k} \, \|(q_{i,j})\|_{u,k} & (16.20) \end{aligned}$$

Finally the polynomial that is identically 1 is the unit for $(\mathcal{P}_n, \|\cdot\|_{u,k})$.

16.3 Let $\mathcal{A}$ be a unital operator algebra and let J be a non-trivial 2-sided ideal in $\mathcal{A}$. Prove that the algebra $\mathcal{A}/J$ equipped with the quotient operator space structure is an operator algebra.

Solution: We have already seen that $\mathcal{A}/J$ is an operator space and the unit is $\mathbf{1} + J$. It is enough to show that the multiplication is contractive. Let $(a_{i,j}), (b_{i,j}) \in M_m(\mathcal{A}/J)$. Choose $(x_{i,j}), (y_{i,j}) \in J$ such that $\|(a_{i,j}) + (x_{i,j})\|_m \leq \|(a_{i,j} + J)\|_m + \varepsilon$ and $\|(b_{i,j}) + (y_{i,j})\|_m \leq \|(b_{i,j} + J)\|_m + \varepsilon$. It follows that,

$$\begin{aligned} (\|(a_{i,j} + J)\|_m + \varepsilon)(\|(b_{i,j} + J)\|_m + \varepsilon) &\geq \|(a_{i,j}) + (x_{i,j})\|_m \, \|(b_{i,j}) + (y_{i,j})\|_m & (16.21) \\ &\geq \|((a_{i,j}) + (x_{i,j}))((b_{i,j}) + (y_{i,j}))\|_m & (16.22) \\ &= \left\| (a_{i,j})(b_{i,j}) + \left(\sum_{k=1}^{m} x_{i,k} b_{k,j} + a_{i,k} y_{k,j} + x_{i,k} y_{k,j} \right) \right\|_m & (16.23) \\ &\geq \|(a_{i,j})(b_{i,j}) + J\|_m & (16.24) \end{aligned}$$

By letting $\varepsilon \to 0$ we get our result.