MATRIX ANALYSIS HOMEWORK

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1. Due 9/25

We let S_n denote the group of all permutations of $\{1, ..., n\}$.

1. Compute the determinants of the elementary matrices: $U(k, l), D(k, \lambda)$, and $S(k, l; \lambda)$

2. Let $J_n \in M_n$ be the matrix of all 1's.

(i) Use Laplace's formula and induction to prove that $det(J_n) = 0$ for all n.

(ii) Use (i) to prove that there is an equal number of even and odd permutations in S_n.

3. Given a permutation $\sigma \in S_n$, let $P_\sigma, Q_\sigma \in M_n$ be the matrices defined by $Q_\sigma = \sum_{i=1}^n E_{i,\sigma(i)}$ and $P_\sigma = \sum_{j=1}^n E_{\sigma(j),j}$.

(i) Prove that $P_\sigma Q_\sigma = Q_\sigma P_\sigma = I_n$

(ii) Prove that $det(P_\sigma) = det(Q_\sigma) = sgn(\sigma)$.(So this can be used to define $sgn(\sigma)$.)

(iii) Given $A \in M_n$ describe $P_\sigma A, Q_\sigma A$, AP_σ, and AQ_σ.

(iv) Prove that $\sum_{\sigma \in S_n} P_\sigma = (n-1)! \cdot J_n$.

(v) If $\pi : \{1, ..., n\} \to \{1, ..., n\}$ is another permutation, prove that $P_\sigma P_\pi = P_{\sigma \circ \pi}$ and $Q_\sigma Q_\pi = Q_{\pi \circ \sigma}$ where $(f \circ g)(i) = f(g(i))$ is the composition of two functions.

(vi) Find the mistake in the Wiki page on permutation matrices!

4. A matrix $P \in M_n$ is called a **permutation matrix** provided that each row and column of P has exactly one 1 and the remaining entries are 0's. Prove that P is a permutation matrix if and only if there is a permutation σ such that $P = P_\sigma$(hence the name).

5. A matrix $T = (t_{i,j})$ is **upper**(resp., **lower**) **triangular** provided that $t_{i,j} = 0$ for $i > j$(resp., $i < j$).

Use induction to prove that if $T \in M_n$ is upper or lower triangular then $det(T) = \prod_{i=1}^n t_{i,i}$.

6. Let $B_n = (min\{i, j\}) \in M_n$.

Prove that $det(B_n) = 1$.

7. Let $C_n = (max\{i, j\}) \in M_n$. Find and prove a formula for $det(C_n)$.

8. Prove that $perm(B_n) \leq (n!)^2 \leq perm(C_n)$.

2. Due 10/2

1. Let $A \in M_{n_1+n_2}$ have block form $A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{bmatrix}$, where $A_{i,j} \in M_{n_i,n_j}$ and $A_{2,1} = 0$. It is "easy" to see, but messy to write down a proof that $det(A) = det(A_{1,1})det(A_{2,2})$. From this it follows that A is invertible $\iff det(A) \neq 0 \iff det(A_{1,1}) \neq 0$ and $det(A_{2,2}) \neq 0 \iff A_{1,1}$ and $A_{2,2}$ are invertible. The purpose of this exercise is to prove this fact *without* using determinants.

(i) Prove that A is invertible if and only if $A_{1,1}$ and $A_{2,2}$ are invertible by expressing A^{-1} in block form.

(ii) Prove, more generally, that $\sigma(A) = \sigma(A_{1,1}) \cup \sigma(A_{2,2})$.

2. Let $A \in M_n$.

(i) Let q be a polynomial such that $q(\lambda) \neq 0$ for all $\lambda \in \sigma(A)$. Prove that $q(A)$ is invertible.

(ii) Prove that for q as in (i) and any other polynomial p, we have that $p(A)q(A)^{-1} = q^{-1}(A)p(A)$.

(iii) Let p_1, p_2, q_1, q_2 be polynomials such that $q_i(\lambda) \neq 0$, for all $\lambda \in \sigma(A), i = 1, 2$ and such that $p_1 q_2 = p_2 q_1$. Prove that $p_1(A)q_1(A)^{-1} = p_2(A)q_2(A)^{-1}$.

(iv) For q as in (i) and p any polynomial, the rational function $r(x) = p(x)/q(x)$ is defined on $\sigma(A)$ and we set $r(A) = p(A)q(A)^{-1}$. Prove that $\sigma(r(A)) = \{r(\lambda) : \lambda \in \sigma(A)\}$.

3. Let $\lambda \in \mathbb{C}$ and let $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \in M_2$ and let $B = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \in M_3$.

(i) Prove that $\sigma(A) = \sigma(B) = \{\lambda\}$.

(ii) Prove formulas for the entries of A^n and B^n.

(ii) Let p, q be polynomials, with $q(\lambda) \neq 0$ and let $r(x) = p(x)/q(x)$. Give and prove explicit formulas for the entries of $r(A)$ and $r(B)$.

4. Let $1 \leq d < n$, and let $A = (a_{i,j}) \in M_n$ with $A = A^*$, $a_{i,i} = a_{j,j}, \forall i, j$ and $p_A(t) = t^{n-d}(t-1)^d$. (Such matrices do exist.)

(i) Find the values of the symmetric functions, $S_k, 1 \leq k \leq n$ for this matrix.

(ii) Find the common value of the diagonal entries.

(iii) Find the sum of the squares of all the strictly upper triangular entries of A, i.e., $\sum_{1 \leq i < j \leq n} |a_{i,j}|^2$.

3. Due 10/9

1. Let $x, y \in \mathbb{F}^n \sim M_{n,1}$, $B \in M_n^{-1}$. Prove that $det(B - xy^*) = det(B)(1 - \langle B^{-1}x, y\rangle)$. HInt: Consider $\begin{pmatrix} 1 & y^* \\ x & B \end{pmatrix}$.

2. Suppose that $A \in M_n, n \geq 2$ is singular and that $\sigma(A)\backslash\{0\}$ is non-empty. Let $r = min\{|\lambda| : \lambda \in \sigma(A),\ \lambda \neq 0\}$.

(i) Prove that for $0 < |z| < r$ $A + zI_n \in M_n^{-1}$.

(ii) Prove that for any $A = (a_{i,j}) \in M_n$ and any $\epsilon > 0$, there is $B = (b_{i,j}) \in M_n^{-1}$ with $\sum_{i,j=1}^n |a_{i,j} - b_{i,j}|^2 < \epsilon^2$. This is often referred to as the "density of the invertibles".

3. Give an example of a matrix $A \in M_n$ such that λ is an e-value of A of geometric multiplicity 1, and such that for any $|S| \geq 1$, we have that λ is an e-value of $A[S, S]$.(This shows that $b \implies a$ is false in the persistence of e-values theorem.)

4. Let $A \in M_{m,n}$. Prove that A is rank 1 iff there exist non-zero $x \in M_{m,1}$, $y \in M_{n,1}$ such that $A = xy^*$.

5. Let $A = xy^*$ as in 4, with $n = m \geq 2$.

(i) Show that if x, y are linearly independent, then $\sigma(xy^*) = \{0\}$.

(ii) Show that if $y = \alpha x, \alpha \neq 0$ then $\sigma(xy^*) = \{0, \alpha\|x\|_2^2\}$.

4. Due 10/16

1. Let $V : \mathbb{F}^n \to \mathbb{F}^m$ be an isometry. Prove that $p_{VV^*}(t) = t^{m-n}(t-1)^n$.

2. Prove that the n vectors $v_j = (cos(2\pi j/n), sin(2\pi j/n)),\ 0 \le j \le n-1$ are a uniform Parseval frame for $\mathbb{R}^2$ and that in the case $n = 3$ they are an equiangular uniform Parseval frame.

3. Let $\{v_1, ..., v_m\} \subseteq \mathbb{F}^n$ be an equiangular uniform Parseval frame for $\mathbb{F}^n (m \ge n)$. Compute the value of the constant $|\langle v_i, v_j\rangle|, i \ne j$.

4. Let $A \in M_n$ and let $\{v_1, ..., v_k\}$ be eigenvectors of A corresponding to distinct, i.e., $\lambda_i \ne \lambda_j, i \ne j$ eigenvalues $\{\lambda_1, ..., \lambda_k\}$. Prove that $\{v_1, ..., v_k\}$ are linearly independent.

5. Due 10/30

Given subspaces $V, W \subseteq \mathbb{F}^n$ we set $V + W = \{v + w : v \in V, w \in W\}$.

1. Show that if $V, W \subseteq \mathbb{F}^n$ are subspaces such that $V + W = \mathbb{F}^n$ and $V \cap W = (0)$, then there is a well-defined linear map $L : \mathbb{F}^n \to \mathbb{F}^n$ such that $L(v + w) = v$. We call L the **projection onto** V **along** W. Show that $L^2 = L$, and conclude that if A is the matrix of this linear map, then $A^2 = A$.

A linear map $L : \mathbb{F}^n \to \mathbb{F}^n$ is called a **projection** iff there is a pair of subspaces V, W such that L is the projection onto V along W.

2. A matrix $A \in M_n$ is called **idempotent** if $A^2 = A$. Let A be idempotent.

(i) Prove that $\mathcal{N}(A) \cap \mathcal{R}(A) = (0)$ and that $\mathcal{N}(A) + \mathcal{R}(A) = \mathbb{F}^n$.

(ii) Prove that L_A is the projection onto $\mathcal{R}(A)$ along $\mathcal{N}(A)$.

(iii) Prove that $dim(\mathcal{R}(A)) = Tr(A)$.

3. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$, which is idempotent. Describe the set of vector such that $L_A x = (1, 0)$.

4. We call L as in 1, an **orthogonal projection** iff $V \perp W$. Prove that L is an orthogonal projection iff the matrix of L, A satisfies $A = A^2 = A^*$.

5. Prove that every projection is similar to an orthogonal projection.

6. Due 11/6

The purpose of these exercises is to use our theorems to derive some results in *spectral graph theory*.

By a graph on n vertices, we mean a pair $G = (V, E)$ where for now $V = \{1, ..., n\}$ denotes the vertex set and $E \subseteq V \times V$ satisfies, $(i, j) \in E \implies (j, i) \in E$ and $(i, i) \notin E, \forall i$. We say that i and j are adjacent and write $i \sim j \iff (i, j) \in E$. A clique in G is a subset $S \subset V$ such that for all $i, j \in S, i \neq j \implies (i, j) \in E$.

For graph theorists, note that the cardinality of $E, |E|$ is equal to twice the number of edges.

The adjacency matrix of a graph is the matrix $A_G \in M_n$, is the self-adjoint matrix given by $A_G = \sum_{(i,j) \in E} E_{i,j}$.

We set $\lambda_i = \lambda_i(A_G)$.

1. Prove that $\sum_{i=1}^n \lambda_i^2 = |E|$.
2. Prove that $\sum_{k=1}^i \lambda_k \leq 0$ for $1 \leq i \leq n$.
3. Prove that $\lambda_n \geq \frac{|E|}{n}$.
4. Prove that if G has a clique S with $|S| = k$, then $\lambda_i \leq -1$, for $1 \leq i \leq k - 1$ and $\lambda_n \geq k - 1$.

7. Due 11/16

1. Let $V : \mathbb{C}^d \to \mathbb{C}^n$ be an isometry, which we identify with its matrix $V \in M_{n,d}$. Prove that $VV^* \in M_n$ is the orthogonal projection onto $\mathcal{R}(V)$.

2. Let $\{v_1, ..., v_n\} \subseteq \mathbb{C}^d$. Prove that $\{v_1, ..., v_n\}$ is a Parseval frame for $\mathbb{C}^d$ iff the Grammian $G = (\langle v_j, v_i \rangle)$ satisfies $G = G^2$ and $Tr(G) = d$.

3. Let $A, B \in M_n$ and let $k \in \mathbb{N}$. Prove that if $A \geq 0$, $B \geq 0$ and $A^k = B^k$ then $A = B$.(This shows the uniqueness of the k-th root of a positive semidefinite matrix.)

8. Due 11/23

1. Let $A \in M_n$. $A = USV$ be a singular valued decomposition. For a real number set $t^\dagger = \begin{cases} t^{-1} & t \neq 0 \\ 0 & t = 0 \end{cases}$. If $S = diag(s_1, ..., s_n)$ then set $S^\dagger = diag(s_1^\dagger, ..., s_n^\dagger)$. Define $A^\dagger = V^* S^\dagger U^*$. Prove:

(1) if $A = U_1 S V_1$ with possibly different unitaries then $V_1^* S^\dagger U_1^* = V^* S^\dagger U^*$.
(2) when A is invertible, $A^\dagger = A^{-1}$.
(3) $AA^\dagger A = A$ and $A^\dagger A A^\dagger = A^\dagger$

2. Let $A \in M_n$. A least squares solution to the equation $Ax = b$ is a vector x such that: 1) $\|Ax - b\|_2 \leq \|Ay - b\|_2, \forall y$ i.e., the error is minimized, 2) among all x satisfying 1), $\|x\|_2$ is minimized. Let P denote the orthogonal projection onto the range of A and let Q denote the orthogonal projection onto $\mathcal{N}(A)^\perp$. Prove that x is the least squares solution to $Ax = b$ iff $Ax = P(b)$ and $Q(x) = x$. Deduce that the least squares solution is unique.

3. Prove that $A^\dagger b$ is the least squares solution to $Ax = b$.

4. Prove that if $A, B \in M_n$ are positive definite, then $A \circ B$ is positive definite.

9. Due 12/8

1. Let $A \in M_n, 0 \preceq A$. let $\gamma = sup\{a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$ where the sup is taken over all permutations σ of $1, ..., n$. Prove that $\gamma^{1/n} \leq \rho(A)$.

2. Let $A \in M_n, 0 \prec A$ and let $x = (x_1, ..., x_n)$ be the Perron vector for A. If $min_i \sum_{j=1}^n a_{i,j} = \rho(A)$ then prove that $x_1 = ... = x_n$. Prove that if $max_i \sum_{j=1}^n a_{i,j} = \rho(A)$ then $x_1 = ... = x_n$.

3. Let $A \in M_n, 0 \preceq A$ and let $H = Re(A)$. Prove that $\rho(A) \leq \lambda_n(H)$.

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