A Class of Multivariate Copulas with Bivariate Fréchet Marginal Copulas

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Abstract

In this paper, we present a class of multivariate copulas whose two-dimensional marginals belong to the family of bivariate Fréchet copulas. The coordinates of a random vector distributed as one of these copulas are conditionally independent. We prove that these multivariate copulas are uniquely determined by their two-dimensional marginal copulas. Some other properties for these multivariate copulas are discussed as well. Two applications of these copulas in actuarial science are given.

Key-words: Multivariate copulas; Bivariate Fréchet copulas; Conditional independence; Marginal copulas

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1 Introduction

Copulas are multivariate distributions with uniform [0,1] marginal distributions. In n-dimensional case, the Fréchet upper bound $C_n^+(u_1, \cdots, u_n) = \min\{u_i, i \leq n\}, u_i \in [0,1], i \leq n$, the Fréchet lower bound $C_n^-(u_1, \cdots, u_n) = \max\{u_1 + u_2 + \cdots + u_n - (n-1), 0\}, u_i \in [0,1], i \leq n$ and the product copula $C_n^\perp(u_1, \cdots, u_n) = \prod_{i=1}^n u_i, u_i \in [0,1], i \leq n$ play significant roles. It is known that every n-dimensional copula C is bounded by the Fréchet upper bound C_n^+ and the Fréchet lower bound C_n^-, i.e.,

$$C_n^-(u_1, \cdots, u_n) \leq C(u_1, \cdots, u_n) \leq C_n^+(u_1, \cdots, u_n) \quad \text{for } u_i \in [0,1],\ i \leq n.$$

See Joe (1997), Mari and Kotz (2001), Nelsen (2006) and Salvadori *et al.* (2007) for details. Note that the Fréchet upper bound C_n^+ is a copula for all $n \geq 2$, and that the Fréchet lower bound C_n^- is a copula only if $n = 2$.

Consider the two-dimensional case. Denote

$$M(u,v) = C_2^+(u,v),\ \Pi(u,v) = C_2^\perp(u,v),\ W(u,v) = C_2^-(u,v).$$

A bivariate Fréchet copula is defined as

$$\alpha M + \beta \Pi + \gamma W,$$

where α, β and γ are non-negative constants with $\alpha + \beta + \gamma = 1$. When modeling risks' dependency, the bivariate Fréchet copula shows its advantages from the following aspects:

- Each term in the bivariate Fréchet copula has its practical implications. In actuarial sciences, two risks X and Y are said to be comonotonic if there exist two non-decreasing functions f and g and a random variable Z such that $X = f(Z), Y = g(Z)$ (Denneberg (1994, pp54-55)). Two risks X and Y are said to be countermonotonic if X and $-Y$ are comonotonic (Dhaene *et al.* (2002b), Embrechets *et al.* (2001)). It is known that X and Y are comonotonic (countermonotonic) if and only if their copula equals M (W) (Nelsen (2006)), and X and Y are independent if and only if their copula equals Π. The bivariate Fréchet copulas model two risks' dependency via weighting the comonotonicity, countermonotonicity and independency respectively. The weights α, β and γ give the percentage of each part. We refer to Nelsen (2006), Kaas *et al.* (2001), Salvadori *et al.* (2007), and the references therein.

- The bivariate Fréchet copulas can be used to approximate bivariate copulas. Each bivariate copula can be approximated by a member of bivariate Fréchet copulas, and the approximation errors can be estimated (Yang, Cheng and Zhang (2006)).

- For two risks with a bivariate Fréchet copula, the stop-loss premium or variance of their sum can be written as a linear sum of three parts with coefficients α, β and γ, and the coefficients are invariant with marginal distributions (Yang, Cheng and Zhang (2006), Mikusinski, Sherwood and Taylor (1991)).

Bivariate Fréchet copulas can not be extended directly to multivariate case, due to the fact that the Fréchet lower bound C_n^- is not a copula when $n \geq 3$. In this paper we shall present a family of multivariate copulas with all two-dimensional marginals belonging to the family of bivariate Fréchet copulas.

We first give the framework of our discussion. Throughout this paper we assume that U_i, $i \leq n$ are uniform [0,1] random variables satisfying the following two assumptions:

- **Assumption A:** There exists a uniform $[0, 1]$ random variable U such that the random variables U_i, $i \leq n$ are conditionally independent on the common factor U.

- **Assumption B:** For each $i \leq n$, the joint distribution of U_i and U is a bivariate Fréchet copula

$$C_i(u, v) = a_{i,1}M(u, v) + a_{i,2}\Pi(u, v) + a_{i,3}W(u, v), \tag{1.1}$$

where $a_{i,j} \geq 0, j = 1, 2, 3$ and $a_{i,1} + a_{i,2} + a_{i,3} = 1$.

Assumption A has its practical implications when modeling risks in insurance and finance. Consider n credit obligors with the loss amount expressed as $H_i(U_i)$, $i \leq n$, where H_i is the inverse of the distribution function of the i-th loss amount. The random variables $U_i, i \leq n$ are correlated through the common factor U. Given the common factor U, the variables $U_i, i \leq n$ are independent. The latent variable U may not be observable. The applications of **Assumption A** can be found in the discussion on collateralized debt obligation (Hull and White (2004)), portfolio loss in credit risk (Credit Suisse First Boston (1997)) and credibility premium (Klugman, Panjer and Willmot (2004)).

Assumption B gives the dependency between the individual variables $U_i, i \leq n$ and the common factor U. The constant $a_{i,1}$ is the percentage of the positive deterministic dependency between U and U_i, $a_{i,3}$ is the percentage of the negative deterministic dependency between U and U_i, and $a_{i,2}$ is the percentage of their independence. The assumption (1.1) is based on the joint distribution of (U_i, U), rather than the two-dimensional marginal distributions of $(U_1, \cdots, U_n)$.

The joint distribution of $U_1, U_2, \cdots, U_n$ defines an n-dimensional copula, denoted as $C^{\mathcal{A},\mathcal{B}}$; that is,

$$C^{\mathcal{A},\mathcal{B}}(u_1, u_2, \cdots, u_n) = P(U_1 \leq u_1, U_2 \leq u_2, \cdots, U_n \leq u_n).$$

In this paper, the multivariate copula $C^{\mathcal{A},\mathcal{B}}$ will be investigated under **Assumption A** and **Assumption B**.

The rest of the paper is organized as follows. In Section 2, we give a mathematical expression of the copula $C^{\mathcal{A},\mathcal{B}}$ and prove that all its two-dimensional marginal copulas belong to the family of bivariate Fréchet copulas. In Section 3 we demonstrate some properties of the copula $C^{\mathcal{A},\mathcal{B}}$. In Section 4 we prove that $C^{\mathcal{A},\mathcal{B}}$ is uniquely determined by all its two-dimensional marginal copulas. In Section 5 we apply our theorems to joint-life status in life insurance and individual risk models. In Section 6 we present some concluding remarks. Some proofs will be given in the appendix.

2 Mathematical expression of the copula $C^{\mathcal{A},\mathcal{B}}$

For the indices $(j_1, j_2, \cdots, j_n)$, where $j_i \in \{1, 2, 3\}$, write

$$C^{(j_1,j_2,\cdots,j_n)}(u_1, u_2, \cdots, u_n) = W\big(\min_{i\leq n, j_i=1}\{u_i\}, \min_{i\leq n, j_i=3}\{u_i\}\big) \prod_{i\leq n, j_i=2} u_i,$$

with convention that for an empty set $\varnothing$ the corresponding minimum and product are defined to be 1. We also write

$$\mathcal{S}_n = \{C^{(j_1,j_2,\cdots,j_n)}|j_i = 1,2,3, i \le n\}.$$

The dependency modeled by $C^{(j_1,j_2,\cdots,j_n)}$ will be given in the following proposition. For convenience, for any given indices $j_1, j_2, \cdots, j_n \in \{1,2,3\}$, we denote $J_k = \{i : j_i = k,\ i \le n\}$ for $k = 1,2,3$ if there is no confusion.

Proposition 2.1. *Fix $j_1, j_2, \cdots, j_n$. Let $(V_1, V_2, \cdots, V_n)$ be a random vector with distribution function $C^{(j_1,j_2,\cdots,j_n)}$. Then the following properties hold:*

(1) For $l, m \in J_1$, $V_l = V_m$, a.e.;

(2) For $l, m \in J_3$, $V_l = V_m$, a.e.;

(3) For each $l \in J_2$, V_l and $\{V_i,\ i \ne l, i \le n\}$ are independent;

(4) For $l \in J_1$ and $m \in J_3$, $V_l = 1 - V_m$, a.e..

Proof. We only give the proof of part (1). The other proofs are similar and will be omitted. For simplicity we assume that $l < m$. For $l, m \in J_1$ we have

$$\begin{aligned}
&P(V_l \le u, V_m \le v)\\
&= C^{(j_1,\cdots,j_{l-1},j_l,j_{l+1},\cdots,j_{m-1},j_m,j_{m+1},\cdots,j_n)}(1,\cdots,1,u,1,\cdots,1,v,1,\cdots,1)\\
&= M(u,v),\ u,v \in [0,1].
\end{aligned}$$

Thus V_l and V_m are comonotonic and $V_l = V_m$, a.e.. The proposition is proved. $\square$

For each $(j_1, j_2, \cdots, j_n)$, $C^{(j_1,j_2,\cdots,j_n)}$ is an n-dimensional copula that can be written as a composition of a product copula, a two-dimensional Fréchet lower bound and Fréchet upper bounds. Some special copulas in $\mathcal{S}_n$ are listed in the following:

1. $C^{(1,1,\cdots,1)}(u_1, u_2, \cdots, u_n) = C_n^{+}(u_1, u_2, \cdots, u_n)$, the n-dimensional Fréchet upper bound;

2. $C^{(2,2,\cdots,2)}(u_1, u_2, \cdots, u_n) = C_n^{\perp}(u_1, u_2, \cdots, u_n)$, the n-dimensional product copula;

3. $C^{(1,3,2,\cdots,2)}(u_1,u_2,\cdots,u_n) = W(u_1,u_2)C^{\perp}_{n-2}(u_3,\cdots,u_n)$, the product of the bivariate Fréchet lower bound and the $(n-2)$-dimensional product copula;

4. $C^{(1,1,3,\cdots,3)}(u_1,u_2,\cdots,u_n) = W\big(M(u_1,u_2), C^{+}_{n-2}(u_3,\cdots,u_n)\big)$, the composition of the Fréchet upper bounds and the 2-dimensional Fréchet lower bound;

5. $C^{(1,1,3,3,2,\cdots,2)}(u_1,u_2,\cdots,u_n) = W\big(M(u_1,u_2), M(u_3,u_4)\big)C^{\perp}_{n-4}(u_5,\cdots,u_n)$ when $n \geq 5$, the composition of the Fréchet upper bound, the 2-dimensional Fréchet lower bound and the product copula.

The copula $C^{(1,1,3,\cdots,3)}$ is extremal, and the copula $C^{(1,1,3,3,2,\cdots,2)}$ is the product of an extremal copula and a product copula. Recall that a multivariate distribution function F with marginal distributions F_i, $i \leq n$ is said to be extremal if there exists a partition (I, I^c) of the index-set $\{1,2,\cdots,n\}$ such that

$$F(x_1,x_2,\cdots,x_n) = W\big(\min_{i\in I} F_i(x_i), \min_{j\in I^c} F_j(x_j)\big).$$

See Tiit (1998) for discussions on extremal copulas.

For different $(j_1,j_2,\cdots,j_n)$, their corresponding copulas might be the same. For instance, $C^{(1,1,2,\cdots,2,3,3)} = C^{(3,3,2,\cdots,2,1,1)}$, $C^{(2,2,\cdots,2)} = C^{(1,2,\cdots,2)} = C^{(3,2,\cdots,2)} = C^{\perp}_n$. The following proposition gives the number of distinct copulas in the family $\mathcal{S}_n$ and reveals the uniqueness of the convex expression of these copulas. The proof will be given in Appendix.

Proposition 2.2. *(1) The number of the distinct copulas in $\mathcal{S}_n$ is $\frac{1}{2}(3^n - 2n + 1)$.*
(2) If a copula C can be expressed as a linear combination of the $\frac{1}{2}(3^n - 2n + 1)$ distinct copulas in $\mathcal{S}_n$, the expression is unique.

The following theorem states that the copula $C^{\mathcal{A},\mathcal{B}}$ can be expressed as a convex combination of the copulas in $\mathcal{S}_n$.

Theorem 2.1. *Suppose that* **Assumption A** *and* **Assumption B** *hold.*
(a) For $u_i \in [0,1], i \leq n$, we have

$$C^{\mathcal{A},\mathcal{B}}(u_1,u_2,\cdots,u_n) = \sum_{j_1=1}^{3}\cdots\sum_{j_n=1}^{3}\left(\prod_{i=1}^{n} a_{i,j_i}\right) C^{(j_1,j_2,\cdots,j_n)}(u_1,u_2,\cdots,u_n); \qquad (2.1)$$

(b) The two-dimensional marginal copulas of $C^{\mathcal{A},\mathcal{B}}$ belong to the family of bivariate Fréchet copulas. For $i \neq m$ and $u_i, u_m \in [0,1]$,

$$P(U_i \leq u_i, U_m \leq u_m) = \alpha_{i,m} M(u_i, u_m) + \beta_{i,m}\Pi(u_i, u_m) + \gamma_{i,m} W(u_i, u_m) \tag{2.2}$$

with

$$\alpha_{i,m} = a_{i,1}a_{m,1} + a_{i,3}a_{m,3},\ \gamma_{i,m} = a_{i,1}a_{m,3} + a_{i,3}a_{m,1},\ \beta_{i,m} = 1 - \alpha_{i,m} - \gamma_{i,m}; \tag{2.3}$$

(c) For any $C = C^{(j_1,j_2,\cdots,j_n)} \in \mathcal{S}_n$ which is different from the product copula, its coefficient in (2.1) equals

$$\prod_{i=1}^{n} a_{i,j_i} + \prod_{i=1}^{n} a_{i,4-j_i}.$$

Proof. (a) For almost every $v \in [0,1]$, from (1.1) we have that

$$\frac{\partial}{dv} C_i(u_i, v) = a_{i,1} I_{\{u_i > v\}} + a_{i,2} u_i + a_{i,3} I_{\{u_i + v - 1 > 0\}}. \tag{2.4}$$

Under **Assumption A**, the joint distribution of $U_1, U_2, \cdots, U_n$ can be expressed as

$$\begin{aligned}
&P(U_1 \leq u_1, U_2 \leq u_2, \cdots, U_n \leq u_n) \\
&= E\big\{P(U_1 \leq u_1, U_2 \leq u_2, \cdots, U_n \leq u_n | U)\big\} \\
&= E\big\{\prod_{i=1}^{n} P(U_i \leq u_i | U)\big\} = E\big\{\prod_{i=1}^{n} [\frac{\partial}{dv} C_i(u_i, v)|_{v=U}]\big\}.
\end{aligned}$$

Replacing the partial derivatives by (2.4), we have

$$\begin{aligned}
&P(U_1 \leq u_1, U_2 \leq u_2, \cdots, U_n \leq u_n) \\
&= E\big[\prod_{i=1}^{n} [a_{i,1} I_{\{u_i > U\}} + a_{i,2} u_i + a_{i,3} I_{\{u_i + U - 1 > 0\}}]\big] \\
&= \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} E\left(\prod_{j_i=1, i\leq n} (a_{i,j_i} I_{\{u_i > U\}}) \prod_{j_i=3, i\leq n} (a_{i,j_i} I_{\{U > 1-u_i\}}) \prod_{j_i=2, i\leq n} (a_{i,j_i} u_i) \right) \\
&= \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} (\prod_{i=1}^{n} a_{i,j_i}) W\big(\min_{j_i=1, i\leq n}\{u_i\}, \min_{j_i=3, i\leq n}\{u_i\}\big) \prod_{j_i=2, i\leq n} u_i,
\end{aligned}$$

and (2.1) is proved.

(b) Applying the above result to the two-dimensional case, we have

$$
\begin{aligned}
& P(U_i \le u_i, U_m \le u_m) \\
&= \big(a_{i,1}a_{m,1} + a_{i,3}a_{m,3}\big)M(u_i,u_m) + \big(a_{i,1}a_{m,3} + a_{i,3}a_{m,1}\big)W(u_i,u_m) \\
&\quad + \big(a_{i,1}a_{m,2} + a_{i,2}a_{m,1} + a_{i,2}a_{m,2} + a_{i,2}a_{m,3} + a_{i,3}a_{m,2}\big)\Pi(u_i,u_m).
\end{aligned}
$$

Since the sum of the three coefficients equals one, (2.2) and (2.3) are obtained.

(c) Since the copula $C = C^{(j_1,j_2,\cdots,j_n)}$ is different from the product copula, based on the fact $C^{(j_1,j_2,\cdots,j_n)} = C^{(4-j_1,4-j_2,\cdots,4-j_n)}$, we know that the coefficient of the copula C in (2.1) equals

$$\prod_{i=1}^{n} a_{i,j_i} + \prod_{i=1}^{n} a_{i,4-j_i}.$$

As shown in Proposition 2.2, the coefficient is uniquely determined by $C^{\mathcal{A},\mathcal{B}}$. $\square$

The above theorem states that the copula $C^{\mathcal{A},\mathcal{B}}$ can be written as a linear combination of the copulas $C^{(j_1,\cdots,j_n)}$. Note that the number of the summands increases exponentially with n and in practice it is applicable for moderate values of n.

The copula $C^{\mathcal{A},\mathcal{B}}$ may correspond to at least two groups of $\{a_{i,j} : i \le n, j = 1,2,3\}$ satisfying (1.1). This can be explained as follows. Suppose that (2.3) holds for $\{a_{i,j} : i \le n, j = 1,2,3\}$. We define a new sequence $V, V_i,\ i \le n$ by letting $V_i = U_i, i \le n$ and $V = 1 - U$. Note that

$$P(V_i \le u_i, i \le n) = P(U_i \le u_i, i \le n) = C^{\mathcal{A},\mathcal{B}}(u_1, u_2, \cdots, u_n)$$

and $V_i, i \le n$ are conditionally independent on V. The two-dimensional marginal distribution is

$$
\begin{aligned}
& P(V_i \le u_i, V \le u) \\
&= P(U_i \le u_i, U \ge 1-u) \\
&= P(U_i \le u_i) - P(U_i \le u_i, U < 1-u) \\
&= P(U_i \le u_i) - \alpha_{i,1}M(u_i, 1-u) - \alpha_{i,2}\Pi(u_i, 1-u) - \alpha_{i,3}W(u_i, 1-u) \\
&= \alpha_{i,3}M(u_i,u) + \alpha_{i,2}\Pi(u_i,u) + \alpha_{i,1}W(u_i,u).
\end{aligned}
$$

Thus, in some cases the copula $C^{\mathcal{A},\mathcal{B}}$ can not uniquely determine the coefficients in (1.1).

Remark 2.1. *We see from Theorem 2.1 that the two-dimensional marginal copulas of* $(U_1, U_2, \cdots, U_n)$ *belong to the family of bivariate Fréchet copulas. Hürlimann (2002) presented a family of copulas with two-dimensional marginal copulas of linear Spearman copulas, that is, for* $i \neq m$ *the vector* (U_i, U_m) *has one parameter linear Spearman copulas, given by*

$$\begin{aligned} & P(U_i \leq u_i, U_m \leq u_m) \\ & = (1 - |\theta_{im}|)\Pi(u_i, u_m) + |\theta_{im}| M(u_i, u_m) I_{\{\theta_{im} > 0\}} + |\theta_{im}| W(u_i, u_m) I_{\{\theta_{im} < 0\}}, \end{aligned}$$

where the parameter $\theta_{im} \in [-1, 1]$.

Remark 2.2. *When* $n = 2$, $\mathcal{S}_2 = \{M, W, \Pi\}$ *and the family of all convex combinations of* M, W *and* Π *coincides with the family of bivariate Fréchet copulas. For any bivariate Fréchet copula* C*, let* (U_1, U_2) *be a random vector with the joint distribution* C *and* $U = U_1$*, then* **Assumption A** *and* **Assumption B** *are satisfied. This implies that all convex combinations of* M, W *and* Π *belong to the family of* $C^{\mathcal{A},\mathcal{B}}$*. When* $n \geq 3$*, there exists a convex combination of the copulas in* $\mathcal{S}_n$ *that does not belong to the family of* $C^{\mathcal{A},\mathcal{B}}$*. For illustration, consider the case* $n = 3$*. By applying Theorem 2.1 we can show that a copula* C *containing the components* $\min\{u_1, u_2, u_3\}$ *and* $\prod_{i=1}^{3} u_i$ *in* $\mathcal{S}_3$ *should also contain at least one of the three components* $\min\{u_1, u_2\}u_3$, $\min\{u_2, u_3\}u_1$ *and* $\min\{u_1, u_3\}u_2$ *if it belongs to the family of* $C^{\mathcal{A},\mathcal{B}}$*. Therefore, the convex combination*

$$C(u_1, u_2, u_3) = \alpha \min\{u_1, u_2, u_3\} + (1 - \alpha) \prod_{i=1}^{3} u_i, \;\; u_i \in [0, 1], i \leq 3, \alpha \in (0, 1)$$

does not belong to the family of $C^{\mathcal{A},\mathcal{B}}$.

3 Some properties of the copula $C^{\mathcal{A},\mathcal{B}}$

For the joint distribution of U_i and U in (1.1), the independence coefficient $a_{i,2}$ can be obtained via $a_{i,2} = 1 - a_{i,1} - a_{i,3}$. Hence, when $a_{i,1}, a_{i,3},\ i \leq n$ are given, the copula $C^{\mathcal{A},\mathcal{B}}$ can be obtained. Write

$$B = \begin{pmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ a_{1,3} & a_{2,3} & \cdots & a_{n,3} \end{pmatrix}.$$

Since the matrix B determines the copula $C^{\mathcal{A},\mathcal{B}}$ uniquely, we shall investigate some properties of $C^{\mathcal{A},\mathcal{B}}$ based on the matrix B. The rank of matrix B, denoted as $\operatorname{rank}(B)$, is smaller than or equal to 2. The transpose of matrix B is denoted as B^T.

We define

$$\alpha_{i,i} = a_{i,1}^2 + a_{i,3}^2,\ \gamma_{i,i} = 2a_{i,1}a_{i,3}. \tag{3.1}$$

The coefficients $\alpha_{i,m}, \gamma_{i,m}, i \neq m$ have been defined in (2.3). Thus, if we write

$$s_{i,j}^+ = \alpha_{i,j} + \gamma_{i,j},\ s_{i,j}^- = \alpha_{i,j} - \gamma_{i,j},\ i, j \leq n,$$

then $\beta_{i,j} = 1 - s_{i,j}^+, i \neq j$ and

$$\alpha_{i,j} = \frac{s_{i,j}^+ + s_{i,j}^-}{2},\ \gamma_{i,j} = \frac{s_{i,j}^+ - s_{i,j}^-}{2},\ i, j \leq n.$$

Denote

$$A^+ = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n,1} & \alpha_{n,2} & \cdots & \alpha_{n,n} \end{pmatrix},\quad A^- = \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} & \cdots & \gamma_{1,n} \\ \gamma_{2,1} & \gamma_{2,2} & \cdots & \gamma_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ \gamma_{n,1} & \gamma_{n,2} & \cdots & \gamma_{n,n} \end{pmatrix}.$$

The two matrices give all the information on the two-dimensional marginal copulas of $C^{\mathcal{A},\mathcal{B}}$. Note that

$$A^+ + A^- = (s_{i,m}^+)_{n\times n},\ A^+ - A^- = (s_{i,m}^-)_{n\times n}.$$

Proposition 3.1. *(1) We have*

$$A^+ = B^T B,\ A^- = B^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B \tag{3.2}$$

and

$$\operatorname{rank}(A^-) \leq \operatorname{rank}(A^+) = \operatorname{rank}(B); \tag{3.3}$$

(2) Moreover,

$$s_{i,m}^+ = (a_{i,1} + a_{i,3})(a_{m,1} + a_{m,3}),\ i, m \leq n, \tag{3.4}$$

$$s_{i,m}^- = (a_{i,1} - a_{i,3})(a_{m,1} - a_{m,3}),\ i, m \leq n \tag{3.5}$$

and

$$\text{rank}(A^+ + A^-) \leq 1, \text{rank}(A^+ - A^-) \leq 1; \tag{3.6}$$

(3) $U_i,\ i \leq n$ are independent if and only if $s^+_{i,m} = 0$ for all $i \neq m,\ i, m \leq n$;

(4) For three different positive integers $i, l, k \leq n$, if $s^+_{i,l} \neq 0$ and $s^+_{l,k} \neq 0$, then $s^+_{i,k} \neq 0$.

Proof. (1) Equation (3.2) is the matrix expression of (2.3), and (3.3) follows from (3.2).

(2) From (3.2) we get that

$$A^+ + A^- = B^T \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} B,\ A^+ - A^- = B^T \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} B.$$

Thus (3.4) and (3.5) can be obtained. Since the ranks of the two matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

are all equal to one, (3.6) holds.

(3) When $U_i,\ i \leq n$ are independent, from (2.2) we have that for $i \neq m,\ \beta_{i,m} = 1$ holds. Thus $s^+_{i,m} = 1 - \beta_{i,m} = 0$ follows.

Conversely, when $s^+_{i,m} = 0$ for all $i \neq m,\ i, m \leq n$, from (3.4) we know that there is at most one i such that $a_{i,1} + a_{i,3} \neq 0$. Assume that $a_{i,1} + a_{i,3} = 0,\ i \leq n-1$. Then for every $i \leq n-1$ the random variable U_i and U are independent. By the conditional independence of $U_i,\ i \leq n$ on U, we have that for $u_i \in [0, 1],\ i \leq n$,

$$\begin{aligned}
&P(U_1 \leq u_1, \cdots, U_n \leq u_n) \\
&= E\big[\prod_{i=1}^{n} P(U_i \leq u_i | U)\big] = E\big[P(U_n \leq u_n | U) \prod_{i=1}^{n-1} P(U_i \leq u_i)\big] \\
&= \prod_{i=1}^{n} u_i.
\end{aligned}$$

Thus $U_i,\ i \leq n$ are independent.

(4) When $s^+_{i,l} \neq 0$ and $s^+_{l,k} \neq 0$, from (3.4) we have that $a_{i,1} + a_{i,3} \neq 0$ and $a_{k,1} + a_{k,3} \neq 0$. Then

$$s^+_{i,k} = (a_{i,1} + a_{i,3})(a_{k,1} + a_{k,3}) \neq 0.$$

This completes the proof of the proposition. □

Example 3.1. *Let*

$$B = \begin{pmatrix} 1 & 0 & 0.5 & 0 & 0.3 & 0.4 \\ 0 & 1 & 0.5 & 0 & 0.4 & 0.3 \end{pmatrix}.$$

For the random variables $U, U_i, i \le n$ modeled by the above matrix, U_1 and U are comonotonic, U_2 and U are countermonotonic, U_3 and U are uncorrelated but they are dependent, U_4 and U are independent, U_5 and U are negatively correlated, and U_6 and U are positively correlated. According to Proposition 3.1,

$$A^+ = B^T B = \begin{pmatrix} 1 & 0 & 0.5 & 0 & 0.3 & 0.4 \\ 0 & 1 & 0.5 & 0 & 0.4 & 0.3 \\ 0.5 & 0.5 & 0.5 & 0 & 0.35 & 0.35 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0.4 & 0.35 & 0 & 0.25 & 0.24 \\ 0.4 & 0.3 & 0.35 & 0 & 0.24 & 0.25 \end{pmatrix} \tag{3.7}$$

and

$$A^- = B^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B = \begin{pmatrix} 0 & 1 & 0.5 & 0 & 0.4 & 0.3 \\ 1 & 0 & 0.5 & 0 & 0.3 & 0.4 \\ 0.5 & 0.5 & 0.5 & 0 & 0.35 & 0.35 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.3 & 0.35 & 0 & 0.24 & 0.25 \\ 0.3 & 0.4 & 0.35 & 0 & 0.25 & 0.24 \end{pmatrix}. \tag{3.8}$$

The two-dimensional marginal copulas of $(U_1, U_2, \cdots, U_n)$ can be obtained through A^+, A^- and (2.2).

4 The uniqueness of $C^{\mathcal{A},\mathcal{B}}$ with given two-dimensional marginal copulas

It follows from Theorem 2.1 that all two-dimensional marginal copulas of $C^{\mathcal{A},\mathcal{B}}$ belong to the family of the bivariate Fréchet copulas. Conversely, given the two-dimensional marginal copulas of $C^{\mathcal{A},\mathcal{B}}$, it would be interesting to see whether the corresponding $C^{\mathcal{A},\mathcal{B}}$ is unique, and how to get $a_{i,j}$ in (1.1).

From Proposition 2.2, each bivariate Fréchet copula is uniquely determined by any two of the three coefficients of the bivariate Fréchet upper bound, lower bound and the product copula, and thus all two-dimensional marginal copulas of $C^{\mathcal{A},\mathcal{B}}$ correspond uniquely to one group of coefficients $\alpha_{i,m}, \gamma_{i,m},\ i \neq m, i, m \leq n$ via (2.2). In what follows, for the given $\alpha_{i,m}, \gamma_{i,m},\ i \neq m, i, m \leq n$, assume that there exist uniform [0,1] random variables $U_i,\ i \leq n$ with two-dimensional marginal copulas given by (2.2) and **Assumption A** and **Assumption B** are satisfied.

In the case $n = 2$, the copula $C^{\mathcal{A},\mathcal{B}}$ and its two-dimensional marginal copula are the same. Next we let $n \geq 3$. Note that $s^+_{i,j} = \alpha_{i,j} + \gamma_{i,j}$ and $s^-_{i,j} = \alpha_{i,j} - \gamma_{i,j}$.

Proposition 4.1. *Suppose that $s^+_{i,m} > 0$ for all $i \neq m, i, m \leq n$. Then $a_{i,1} + a_{i,3},\ i \leq n$ are uniquely determined by $\alpha_{i,m}, \gamma_{i,m},\ i \neq m, i, m \leq n$ via* (3.4)*. Moreover,*
(1) if $s^-_{i,m} \neq 0$ for all $i \neq m, i, m \leq n$, then $|a_{i,1} - a_{i,3}|,\ i \leq n$ are uniquely determined by $\alpha_{i,m}, \gamma_{i,m},\ i \neq m, i, m \leq n$ via (3.5)*;*
(2) if $s^-_{i,m} = 0$ for all $i \neq m, i, m \leq n$, then there exists at most one i such that $a_{i,1} \neq a_{i,3}$;
(3) if $s^-_{i,m} = 0$ for some $m \neq i$ and there exists an $l \leq n$ such that $s^-_{m,l} \neq 0$, then $a_{i,1} = a_{i,3}$.

Proof. Since $s^+_{i,m} > 0$ for all $i \neq m$, from (3.6) we can see that $\text{rank}(A^+ + A^-) = 1$. Thus for the given $\alpha_{i,m}, \gamma_{i,m},\ i \neq m, i, m \leq n$, the elements $s^+_{i,i},\ i \leq n$ can be determined uniquely. From (3.4) we see that $a_{i,1} + a_{i,3} = \sqrt{s^+_{i,i}},\ i \leq n$.

If $s^-_{i,m} \neq 0$ for all $i \neq m$, from (3.6) we have $\text{rank}(A^+ - A^-) = 1$. Thus the given coefficients $\alpha_{i,m}, \gamma_{i,m},\ i \neq m, i, m \leq n$ determine $s^-_{i,i},\ i \leq n$ uniquely, and from (3.5) we know that $|a_{i,1} - a_{i,3}| = \sqrt{s^-_{i,i}},\ i \leq n$.

In the case $s^-_{i,m} = 0$ for all $i \neq m$, $s^-_{i,m} = (a_{i,1} - a_{i,3})(a_{m,1} - a_{m,3}) = 0$ for all $i \neq m$. Thus there exists at most one i such that $a_{i,1} \neq a_{i,3}$. The last part of the proposition can be proved similarly. The proposition is proved. □

We define

$$I_0 = \{i \leq n : s^+_{i,m} = 0 \text{ for all } \ m \neq i,\ m \leq n\}$$

and

$$I_1 = \{i \le n : i \notin I_0\}.$$

Let

$$I_1^0 = \{i \in I_1 : s^-_{i,m} = 0 \text{ for all } m \ne i, m \in I_1\}.$$

Note that $I_0 \bigcup I_1 = \{1, 2, \cdots, n\}$ and $I_1^0 \subseteq I_1$. Through above definitions, the index set $\{1, 2, \cdots, n\}$ has been divided into several groups by the values of $\alpha_{i,m}, \gamma_{i,m},\ i \ne m, i, m \le n$. For a set Ω, its cardinality is denoted as $\#(\Omega)$ or $\#\Omega$.

Proposition 4.2. *(1) For any $i \in I_0$ and $m \in I_1$, we have $s^+_{i,m} = 0$;*
(2) The index set I_1 is empty or $\#(I_1) \ge 2$. When $\#(I_1) \ge 2$, we have $s^+_{i,m} > 0$ for any $i, m \in I_1$;
(3) $I_1 = I_1^0$ or $\#\{i \in I_1 \setminus I_1^0\} \ge 2$. When $I_1 \ne I_1^0$, we have $a_{i,1} = a_{i,3}$ for all $i \in I_1^0$ and $s^-_{k,m} \ne 0$ for all $k \ne m, k, m \in I_1 \setminus I_1^0$;
(4) The copula $C^{\mathcal{A},\mathcal{B}}$ has the following decomposition

$$C^{\mathcal{A},\mathcal{B}}(u_1, u_2, \cdots, u_n) = P(U_m \le u_m, m \in I_1) \times \prod_{i \in I_0} u_i, \quad u_i \in [0, 1], i \le n. \tag{4.1}$$

Proof. (1) For $i \in I_0$ and $m \in I_1$, by the definition of I_0 we have $s^+_{i,m} = 0$.

(2) If $\#(I_1) = 1$, then for $i \in I_1$, by using the result of part (1) we know that $s^+_{i,m} = 0,\ m \ne i, m \le n$, which leads to that $i \in I_0$. Thus I_1 is an empty set or $\#(I_1) \ge 2$.

When $\#(I_1) \ge 2$, fix $i, m \in I_1$. If $s^+_{i,m} = 0$, from (3.4) we know that $a_{i,1} + a_{i,3} = 0$ or $a_{m,1} + a_{m,3} = 0$, which leads to that $i \in I_0$ or $m \in I_0$, contradicting to the assumption that $i, m \in I_1$. Thus we have $s^+_{i,m} > 0$.

(3) Consider the case $\#\{i \in I_1 \setminus I_1^0\} = 1$. Note that $\#(I_1) \ge 2$. For $i \in I_1 \setminus I_1^0$, according to the definition of I_1^0 we see that $s^-_{i,m} = 0$ for each $m \in I_1^0$, thus $i \in I_1^0$, contradicting to the assumption that $i \in I_1 \setminus I_1^0$. Thus $\#\{i \in I_1 \setminus I_1^0\} = 0$ or $\#\{i \in I_1 \setminus I_1^0\} \ge 2$.

When $I_1 \ne I_1^0$, applying (3.5) we can easily prove $a_{i,1} = a_{i,3}$ for all $i \in I_1^0$ and $s^-_{k,m} \ne 0$ for all $k \ne m, k, m \in I_1 \setminus I_1^0$.

(4) We first consider the case that $I_0 = \{1, 2, \cdots, n\}$. In this case, $s^+_{i,m} = 0$ for all $i \neq m, i, m \leq n$. By Proposition 3.1, the random variables $U_i, i \leq n$ are independent. Thus (4.1) holds.

Next we consider the case that $I_0 \subset \{1, 2, \cdots, n\}$. Then there exists an $i_0 \in I \setminus I_0$. For each $i \in I_0$, from $s^+_{i,i_0} = 0$ we get that $a_{i,1} + a_{i,3} = 0$, which leads to that U_i and U are independent. Due to the conditional independence of $U_1, \cdots, U_n$ on U, for $u_i \in [0, 1], i \leq n$ we have

$$
\begin{aligned}
& P(U_1 \leq u_1, \cdots, U_n \leq u_n) \\
&= E\big[\prod_{m=1}^{n} P(U_m \leq u_m|U)\big] = E\big[\prod_{m \in I_1} P(U_m \leq u_m|U) \times \prod_{i \in I_0} P(U_i \leq u_i|U)\big] \\
&= E\big[\prod_{m \in I_1} P(U_m \leq u_m|U)\big] \prod_{i \in I_0} u_i \\
&= P(U_m \leq u_m, m \in I_1) \times \prod_{i \in I_0} u_i.
\end{aligned}
$$

Hence we get (4.1). $\square$

By Proposition 4.2, the index sets I_1 and $I_1 \setminus I_1^0$ can be expressed as

$$I_1 = \{i \leq n : \text{there exists } m \neq i \text{ such that } s^+_{i,m} > 0\}$$

and

$$I_1 \setminus I_1^0 = \{i \in I_1 : \text{there exists } m \neq i, m \in I_1, \text{ such that } s^-_{i,m} \neq 0\}.$$

The following theorem asserts the uniqueness of $C^{\mathcal{A},\mathcal{B}}$.

Theorem 4.1. *The copula $C^{\mathcal{A},\mathcal{B}}$ is uniquely determined by all its bivariate marginal copulas.*

Proof. Note that $C^{\mathcal{A},\mathcal{B}}$ can be expressed in (4.1). Therefore, it suffices to prove that $P(U_m \leq u_m,\ m \in I_1)$ is uniquely determined by all its bivariate marginal copulas. This holds trivially if I_1 contains at most two indices. For simplicity we assume that $I_1 = \{1, 2, \cdots, k\}$ for some $k \geq 3$. We should show that the coefficients $a_{i,j},\ i = 1, \cdots, k$, $j = 1, 2, 3$, determined by the bivariate marginal copulas, determine a unique copula $P(U_m \leq u_m, m \leq k)$.

(1) Assume that $s^-_{l,m} \neq 0$ for all $l \neq m, l, m \in I_1$. From Proposition 4.1, $a_{i,2}, a_{i,1} + a_{i,3}$ and $|a_{i,1} - a_{i,3}|,\ i \in I_1$ are uniquely determined. Suppose $a^0_{i,j},\ i \in I_1, j \leq 3$ and $a^1_{i,j},\ i \in I_1, j \leq 3$ are two different solutions of $a_{i,j},\ i \in I_1, j \leq 3$. If for some $i_0 \in I_1$,

$$a^0_{i_0,1} = a^1_{i_0,1}, a^0_{i_0,2} = a^1_{i_0,2}, a^0_{i_0,3} = a^1_{i_0,3},$$

then by (3.5) we assert that $a^0_{i,1} - a^0_{i,3} = a^1_{i,1} - a^1_{i,3},\ i \in I_1$, which leads to that

$$a^0_{i,1} = a^1_{i,1}, a^0_{i,2} = a^1_{i,2}, a^0_{i,3} = a^1_{i,3}, i \in I_1,$$

contradicting to the assumption that the two groups of solutions are different. Thus the two groups satisfy that

$$a^0_{i,1} = a^1_{i,3}, a^0_{i,3} = a^1_{i,1}, a^0_{i,2} = a^1_{i,2},\ i \in I_1.$$

From the discussion following Theorem 2.1, the two groups generate the same distribution $P(U_m \leq u_m, m \in I_1)$.

(2) Assume that $s^-_{l,m} = 0$ for some $l \neq m, l, m \in I_1$. We need to prove that for the decomposition of copula $P(U_m \leq u_m, m \in I_1)$ in (2.1), the coefficients of the copulas in $\{C^{(j_1,\cdots,j_k)} : j_i = 1, 2, 3,\ i \leq k\}$ are unique.

For fixed indices $(j_1, j_2, \cdots, j_k)$, let $C = C^{(j_1,j_2,\cdots,j_k)}$. Since the sum of all coefficients equals one, we only need to consider the case that C is different from the product copula. By Theorem 2.1, the coefficient of C can be expressed as $\prod_{i=1}^k a_{i,j_i} + \prod_{i=1}^k a_{i,4-j_i}$.

When $I^0_1 = I_1$, from Proposition 4.1 we have that

$$\begin{aligned}
&\prod_{i=1}^k a_{i,j_i} + \prod_{i=1}^k a_{i,4-j_i} \\
&= \prod_{i\in I_1, j_i=2} a_{i,2}\Big(\prod_{i\in I_1, j_i=1} a_{i,1} \times \prod_{i\in I_1, j_i=3} a_{i,3} + \prod_{i\in I_1, j_i=1} a_{i,3} \times \prod_{i\in I_1, j_i=3} a_{i,1}\Big) \\
&= \Big(\prod_{i\in I_1, j_i=2} a_{i,2}\Big) \times \Big(\prod_{i\in I_1, j_i\neq 2} (a_{i,1} + a_{i,3})\Big)/2^{\#\{i\in I_1, j_i\neq 2\}-1}
\end{aligned}$$

is unique. Thus $P(U_m \leq u_m, m \in I_1)$ is determined uniquely.

Next we assume that $I^0_1 \subset I_1$. From Proposition 4.2, $\#\{i \in I_1 \setminus I^0_1\} \geq 2$ and for $i \in I^0_1$, $a_{i,1} = a_{i,3}$, and $a_{i,1}$ is uniquely determined. Thus the coefficient of C equals

$$\prod_{i\in I_1} a_{i,j_i} + \prod_{i\in I_1} a_{i,4-j_i} = \prod_{i\in I^0_1} a_{i,j_i} \times (\prod_{i\in I_1\setminus I^0_1} a_{i,j_i} + \prod_{i\in I_1\setminus I^0_1} a_{i,4-j_i}). \tag{4.2}$$

In the case that $\#\{i \in I_1 \setminus I_1^0, j_i \neq 2\} \leq 1$, the above equation can be written as

$$\begin{aligned} & \prod_{i\in I_1} a_{i,j_i} + \prod_{i\in I_1} a_{i,4-j_i} \\ = \quad & \prod_{i\in I_1^0} a_{i,j_i} \times (\prod_{i\in I_1\setminus I_1^0, j_i=2} a_{i,2}) \times (\prod_{i\in I_1\setminus I_1^0, j_i\neq 2} a_{i,j_i} + \prod_{i\in I_1\setminus I_1^0, j_i\neq 2} a_{i,4-j_i}), \end{aligned}$$

thus its value is unique. When $\#\{i \in I_1 \setminus I_1^0, j_i \neq 2\} \geq 2$, the copula $C^{(j_i, i\in I_1\setminus I_1^0)}$ is different from the product copula, and $\prod_{i\in I_1\setminus I_0} a_{i,j_i} + \prod_{i\in I_1\setminus I_1^0} a_{i,4-j_i}$ is the coefficient of $C^{(j_i, i\in I_1\setminus I_1^0)}$ in the decomposition of the copula $P(U_i \leq u_i,\ i \in I_1 \setminus I_1^0)$. Thus we only need to prove the uniqueness of $P(U_i \leq u_i,\ i \in I_1 \setminus I_1^0)$. The case $\#\{i \in I_1 \setminus I_1^0\} = 2$ is trivial, so we will focus on the case $\#\{i \in I_1 \setminus I_1^0\} \geq 3$. From Proposition 4.2 we know that $s^-_{i,m} \neq 0,\ i \neq m, i, m \in I_1 \setminus I_1^0$. Following the same lines in part (1) above, we see that the distribution of $U_i, i \in I_1 \setminus I_1^0$ is unique. Thus the coefficient $\prod_{i\in I_1\setminus I_1^0} a_{i,j_i} + \prod_{i\in I_1\setminus I_1^0} a_{i,4-j_i}$ is unique as well. By (4.2) we get that the coefficient of C is unique. Thus the copula $P(U_m \leq u_m, m \in I_1)$ is uniquely determined.

Combining the above results with (4.1), the uniqueness of $C^{\mathcal{A},\mathcal{B}}$ is proved. $\square$

Remark 4.1. *Normal copulas are widely used in actuarial sciences and finance to model the correlation between risks (Cherubini, Luciano and Vecchiato(1998)). The copulas $C^{\mathcal{A},\mathcal{B}}$ allow us to model the dependency of risks by setting weights on comonotonicity, countermonotonicity and independency, respectively. The normal copulas and the copulas $C^{\mathcal{A},\mathcal{B}}$ are all uniquely determined by their two-dimensional marginal copulas. The two-dimensional marginal copulas of a normal copula are one-parameter distributions, and those of $C^{\mathcal{A},\mathcal{B}}$ are two-parameter distributions.*

Example 4.1. *(Continuing of Example 3.1) Given the two matrices A^+ and A^- in* (3.7) *and* (3.8)*, where the main diagonal elements are unknown, we can solve for the matrix*

B. *We have*

$$A^+ + A^- = \begin{pmatrix} s^+_{1,1} & 1 & 1 & 0 & 0.7 & 0.7 \\ 1 & s^+_{2,2} & 1 & 0 & 0.7 & 0.7 \\ 1 & 1 & s^+_{3,3} & 0 & 0.7 & 0.7 \\ 0 & 0 & 0 & s^+_{4,4} & 0 & 0 \\ 0.7 & 0.7 & 0.7 & 0 & s^+_{5,5} & 0.49 \\ 0.7 & 0.7 & 0.7 & 0 & 0.49 & s^+_{6,6} \end{pmatrix}$$

and

$$A^+ - A^- = \begin{pmatrix} s^-_{1,1} & -1 & 0 & 0 & -0.1 & 0.1 \\ -1 & s^-_{2,2} & 0 & 0 & 0.1 & -0.1 \\ 0 & 0 & s^-_{3,3} & 0 & 0 & 0 \\ 0 & 0 & 0 & s^-_{4,4} & 0 & 0 \\ -0.1 & 0.1 & 0 & 0 & s^-_{5,5} & -0.01 \\ 0.1 & -0.1 & 0 & 0 & -0.01 & s^-_{6,6} \end{pmatrix}.$$

Note that $I_0 = \{4\}$ *and* $I_1 = \{1, 2, 3, 5, 6\}$. *Since* $\mathrm{rank}(A^+ + A^-) = \mathrm{rank}(A^+ - A^-) = 1$, *then we get that*

$$s^+_{1,1} = s^+_{2,2} = s^+_{3,3} = 1,\ s^+_{4,4} = 0,\ s^+_{5,5} = s^+_{6,6} = 0.49$$

and

$$s^-_{1,1} = s^-_{2,2} = 1,\ s^+_{3,3} = 0,\ s^-_{4,4} = 0,\ s^+_{5,5} = s^+_{6,6} = 0.01.$$

Using $a_{i,1} + a_{i,3} = \sqrt{s^+_{i,i}}$, $|a_{i,1} - a_{i,3}| = \sqrt{s^-_{i,i}}$ *and* (3.5), *finally we obtain*

$$B = \begin{pmatrix} 1 & 0 & 0.5 & 0 & 0.3 & 0.4 \\ 0 & 1 & 0.5 & 0 & 0.4 & 0.3 \end{pmatrix} \text{ or } B = \begin{pmatrix} 0 & 1 & 0.5 & 0 & 0.4 & 0.3 \\ 1 & 0 & 0.5 & 0 & 0.3 & 0.4 \end{pmatrix}.$$

Given one family of bivariate Fréchet copulas, one natural problem is that whether there exists a copula $C^{\mathcal{A},\mathcal{B}}$ having the given family as its two-dimensional marginal copulas. Our next theorem gives a necessary and sufficient condition.

Theorem 4.2. *Give two-dimensional Fréchet copulas* $C_{i,m}, 1 \le i < m \le n$ *with*

$$C_{i,m}(u, v) = d^+_{i,m} M(u, v) + d^\perp_{i,m} \Pi(u, v) + d^-_{i,m} W(u, v),$$

where constants $d^+_{i,m}, d^\perp_{i,m}, d^-_{i,m} \geq 0$, *and* $d^+_{i,m} + d^\perp_{i,m} + d^-_{i,m} = 1$. *There exist uniform [0,1] random variables* $W_i, i \leq n$ *with a copula* $C^{\mathcal{A},\mathcal{B}}$ *such that for each* $1 \leq i < m \leq n$,

$$P(W_i \leq u, W_m \leq v) = C_{i,m}(u, v)$$

if and only if there exist non-negative constants $a_{i,j}, i \leq n, j = 1, 2, 3$ *satisfying* $\sum_{j=1}^3 a_{i,j} = 1, i \leq n$, *such that for each* $1 \leq i < m \leq n$ *the following equations hold:*

$$d^+_{i,m} = a_{i,1}a_{m,1} + a_{i,3}a_{m,3},\ d^-_{i,m} = a_{i,1}a_{m,3} + a_{i,3}a_{m,1},\ d^\perp_{i,m} = 1 - d^+_{i,m} - d^-_{i,m}. \tag{4.3}$$

Proof. We first prove the sufficiency. Suppose that there exist $a_{i,j} \geq 0, i \leq n, j = 1, 2, 3$ such that (4.3) holds for all $1 \leq i < m \leq n$. Let $W, V_i, i \leq n$ be independent uniform [0,1] random variables, and for each $i \leq n$ the random partition $(A^+_i, A^-_i, A^\perp_i)$ of the probability space satisfies that

$$P(A^+_i) = a_{i,1}, P(A^-_i) = a_{i,3}, P(A^\perp_i) = a_{i,2}.$$

Assume that $W, V_i, (A^+_i, A^-_i, A^\perp_i), i \leq n$ are independent. The random variables $W_i, i \leq n$ are defined as follows:

$$W_i = WI_{A^+_i} + V_iI_{A^\perp_i} + (1 - W)I_{A^-_i}.$$

Then $W_i, i \leq n$ are conditionally independent on W, and

$$P(W_i \leq u, W \leq v) = a_{i,1}M(u, v) + a_{i,2}\Pi(u, v) + a_{i,3}W(u, v).$$

Applying Theorem 2.1, we have

$$P(W_i \leq u, W_m \leq v) = C_{i,m}(u, v).$$

Conversely, suppose that there exist uniform [0,1] random variables $W_i, i \leq n$ with a copula $C^{\mathcal{A},\mathcal{B}}$ such that for each $1 \leq i < m \leq n$,

$$P(W_i \leq u, W_m \leq v) = C_{i,m}(u, v).$$

By the definition of the copula $C^{\mathcal{A},\mathcal{B}}$, there exist uniform [0,1] random variables $U, U_i, i \leq n$ and constants $a_{i,j}, i \leq n, j = 1, 2, 3$ satisfying **Assumption A** and **Assumption B**, such that

$$P(U_i \leq u_i, i \leq n) = C^{\mathcal{A},\mathcal{B}}(u_1, \cdots, u_n).$$

Thus $(W_1, \cdots, W_n)$ and $(U_1, \cdots, U_n)$ have the same distribution, which leads to that

$$P(U_i \le u, U_m \le v) = P(W_i \le u, W_m \le v) = C_{i,m}(u, v).$$

Then (4.3) follows from Theorem 2.1. This proves the necessity part. The proof of the theorem is complete. $\square$

5 The applications of copula $C^{\mathcal{A},\mathcal{B}}$

In this section we focus on the applications of the copula $C^{\mathcal{A},\mathcal{B}}$ in two insurance risk models: the joint-life status where the future lifetimes of the individuals in the group are correlated with the copula $C^{\mathcal{A},\mathcal{B}}$, and the individual risk models with the individual risks' dependency modeled by the copula $C^{\mathcal{A},\mathcal{B}}$.

5.1 Joint-life status

For n individuals with ages $x_1, x_2, \cdots, x_n$, their future lifetimes are denoted as $T_1(x_1), T_2(x_2), \cdots, T_n(x_n)$. The future lifetime on the joint-life status is defined as

$$T(x_1 : x_2 : \cdots : x_n) = \min\{T_1(x_1), T_2(x_2), \cdots, T_n(x_n)\}.$$

Consider the payment of one unit at time $T(x_1 : x_2 : \cdots : x_n)$ with force of interests r. The actuarial present value of the payment can be expressed as

$$\begin{aligned} APV &= : E(\exp(-rT(x_1 : x_2 : \cdots : x_n))) \\ &= \int_0^\infty e^{-rt} dP(T(x_1 : x_2 : \cdots : x_n) \le t) = -\int_0^\infty e^{-rt} dP(T(x_1 : x_2 : \cdots : x_n) > t). \end{aligned}$$

Integration by parts leads to

$$APV = 1 - r\int_0^\infty e^{-rt} P(T(x_1 : x_2 : \cdots : x_n) > t)dt. \tag{5.1}$$

For simplicity, we assume that $x_1 = x_2 = \cdots = x_n = x$, and that the marginal distributions of $(T_1(x), T_2(x), \cdots, T_n(x))$ are the same, denoted as F_x. Assume that there exist uniform $[0, 1]$ random variables $U, U_i, i \le n$ satisfying **Assumption A** and **Assumption B**, such that

$$(T_1(x), T_2(x), \cdots, T_n(x)) = (F_x^-(U_1), F_x^-(U_2), \cdots, F_x^-(U_n)).$$

Here F_x^- denotes the left-continuous inverse function of F_x. For given $(j_1, \cdots, j_n)$, for simplicity we denote

$$\delta(j_1, \cdots, j_n) = I_{\{\#\{i:j_i=1\}>0, \#\{i:j_i=3\}>0\}}$$

and

$$\eta(j_1, \cdots, j_n) = \#\{i : j_i = 2\} + I_{\{\#\{i:j_i=1\}\times\#\{i:j_i=3\}=0, \#\{i:j_i=1\}+\#\{i:j_i=3\}>0\}}.$$

Detailed calculation shows that

$$\begin{aligned}
&\int_0^\infty e^{-rt} P(T(x : x : \cdots : x) > t) dt \\
&= \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} (\prod_{i=1}^{n} a_{i,j_i}) \int_0^\infty e^{-rt} (1 - 2F_x(t))_+^{\delta(j_1,\cdots,j_n)} (1 - F_x(t))^{\eta(j_1,\cdots,j_n)} dt \\
&=: \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} (\prod_{i=1}^{n} a_{i,j_i}) h_{j_1,j_2,\cdots,j_n}. \qquad (5.2)
\end{aligned}$$

Hence APV can be expressed as a linear combination of $h_{j_1,j_2,\cdots,j_n}, j_i \leq 3, i \leq n$. Note that the coefficients $a_{i,j}$ have no influence on $h_{j_1,j_2,\cdots,j_n}$.

The actuarial notations ${}_tq_x = F_x(t)$, ${}_tp_x = 1 - F_x(t)$ and $q_x = F_x(1)$, $p_x = 1 - F_x(1)$ will be used here. Assume that the mortality of the group follows the uniform distribution of death over each age interval (Bowers *et al.*(1997)), that is, for each non-negative integer y the equation

$${}_tq_y = tq_y, t \in [0, 1]$$

holds. For given $j_1, j_2, \cdots, j_n$,

$$\begin{aligned}
h_{j_1,j_2,\cdots,j_n} &= \sum_{k=0}^{\infty} \int_k^{k+1} e^{-rt} (1 - 2\,{}_tq_x)_+^{\delta(j_1,\cdots,j_n)} ({}_tp_x)^{\eta(j_1,\cdots,j_n)} dt \\
&= \sum_{k=0}^{\infty} \int_k^{k+1} e^{-rt} (2\,{}_tp_x - 1)_+^{\delta(j_1,\cdots,j_n)} ({}_tp_x)^{\eta(j_1,\cdots,j_n)} dt \\
&= \sum_{k=0}^{\infty} \int_0^1 e^{-r(k+t)} (2\,{}_kp_x(1 - tq_{x+k}) - 1)_+^{\delta(j_1,\cdots,j_n)} \\
&\qquad \times ({}_kp_x \times (1 - tq_{x+k}))^{\eta(j_1,\cdots,j_n)} dt. \qquad (5.3)
\end{aligned}$$

The equations (5.1)-(5.3) can be used for calculating APV. Assume that $r = 0.025, n = 4$ and $a_{i,j} = a_{1,j}$ for $i \leq 4, j \leq 3$. We use the mortality for male nonsmokers in *2001 Valuation Basic Table – Ultimate Only* (CSO Task Force Report (2002)). The four cases in Table 5.1 are considered. For Case 1 and Case 2, U_i and U are dependent and uncorrelated for each i. For Case 3, U_i and U are positive correlated for each i. Case 4 describes the situation that U_i and U are independent, thus $U_i, i \leq 4$ are independent in this case.

	Case 1	Case 2	Case 3	Case 4
$a_{i,1}$	0.1	0.2	0.3	0
$a_{i,2}$	0.8	0.6	0.7	1
$a_{i,3}$	0.1	0.2	0	0

Table 5.1: The coefficients $a_{i,j}$

The numerical results for $x = 20, 50$ and 80 are given in Table 5.2. We also calculate the ratios of Cases 1-3 to Case 4 to demonstrate the influence of the dependency assumptions on APV.

	Case 1	Case 2	Case 3	Case 4	$\frac{\text{Case 1}}{\text{Case 4}}$	$\frac{\text{Case 2}}{\text{Case 4}}$	$\frac{\text{Case 3}}{\text{Case 4}}$
$x = 20$	0.3453	0.3387	0.3356	0.3476	0.9933	0.9744	0.9655
$x = 50$	0.6307	0.6228	0.6175	0.6334	0.9957	0.9834	0.9750
$x = 80$	0.9273	0.9240	0.9198	0.9284	0.9988	0.9953	0.9908

Table 5.2: APV under different $a_{i,j}$

5.2 Individual risk models

Individual risk models play an important role in insurance to model the total claims of an insurance portfolio; see, e.g., Kaas *et al.*(2001) for details.

Let $Y_1, Y_2, \cdots, Y_n$ be n individual risks with marginal distributions F_i and $Y_i = F_i^-(U_i)$, where $U_i, i \leq n$ satisfy **Assumption A** and **Assumption B** and F_i^- denotes the left-continuous inverse function of F_i. Then the copula of $(Y_1, Y_2, \cdots, Y_n)$ equals

$C^{\mathcal{A},\mathcal{B}}$. In the following we define $0 \times \infty = 0$.

Proposition 5.1. *Let f be a non-negative n-variable function. Then*

$$Ef(Y_1, \cdots, Y_n) = \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} (\prod_{i=1}^{n} a_{i,j_i}) Ef(Y_1^{(j_1,\cdots,j_n)}, \cdots, Y_n^{(j_1,\cdots,j_n)}).$$

Here for each $(j_1, j_2, \cdots, j_n)$ the random vector $(Y_1^{(j_1,\cdots,j_n)}, \cdots, Y_n^{(j_1,\cdots,j_n)})$ has marginal distributions $F_i, i \le n$ and copula $C^{(j_1,j_2,\cdots,j_n)}$.

Proof. Using Theorem 2.1, we have

$$\begin{aligned}
&Ef(Y_1, \cdots, Y_n) \\
&= \int_0^1 \cdots \int_0^1 f(F_1^-(u_1), \cdots, F_n^-(u_n)) C^{\mathcal{A},\mathcal{B}}(du_1, \cdots, du_n) \\
&= \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} (\prod_{i=1}^{n} a_{i,j_i}) \int_0^1 \cdots \int_0^1 f(F_1^-(u_1), \cdots, F_n^-(u_n)) C^{(j_1,j_2,\cdots,j_n)}(du_1, \cdots, du_n) \\
&= \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} (\prod_{i=1}^{n} a_{i,j_i}) Ef(Y_1^{(j_1,\cdots,j_n)}, \cdots, Y_n^{(j_1,\cdots,j_n)}).
\end{aligned}$$

The proposition is proved. □

Proposition 5.1 shows the advantage of **Assumption A** and **Assumption B** on analyzing the influence of the correlation on risk portfolios. Note that the coefficients $\prod_{i=1}^{n} a_{i,j_i}$ don't depend on the marginal distributions.

Next we focus on the stop-loss premium. It is easily obtained that

$$E(Y_1 + \cdots + Y_n - t)_+ = \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} (\prod_{i=1}^{n} a_{i,j_i}) E(Y_1^{(j_1,\cdots,j_n)} + \cdots + Y_n^{(j_1,\cdots,j_n)} - t)_+.$$

It follows from Dhaene *et al.*(2002a) that

$$E(Y_1^{(1,\cdots,1)} + \cdots + Y_n^{(1,\cdots,1)} - t)_+ \ge E(Y_1^{(j_1,\cdots,j_n)} + \cdots + Y_n^{(j_1,\cdots,j_n)} - t)_+.$$

By comparing terms $E(Y_1^{(j_1,\cdots,j_n)} + \cdots + Y_n^{(j_1,\cdots,j_n)} - t)_+$, we can see the influence of the different correlations on the stop-loss premiums. Let $n = 3$ and the marginal distribution be $F_i(x) = 1 - x^{-\alpha}$, $x \ge 1$ with parameter $\alpha > 1$. Denote

$$g_{j_1,j_2,j_3}(t) = E(Y_1^{(j_1,j_2,j_3)} + Y_2^{(j_1,j_2,j_3)} + Y_3^{(j_1,j_2,j_3)} - t)_+.$$

Note that $g_{j_1,j_2,j_3}(t)$ must equal one of $g_{1,1,1}(t)$, $g_{1,1,2}(t)$, $g_{1,1,3}(t)$, $g_{2,2,2}(t)$ and $g_{1,2,3}(t)$, and that $g_{j_1,j_2,j_3}(t) = \frac{3\alpha}{\alpha-1} - t, t \leq 3$ for all possible (j_1, j_2, j_3). For $\alpha = 2$ and $\alpha = 3$, we plot $g_{1,1,1}(t)$, $g_{1,1,2}(t)$, $g_{1,1,3}(t)$, $g_{2,2,2}(t)$ and $g_{1,2,3}(t)$ in Figure 5.1 and Figure 5.2. Based on the above numerical results, we can calculate the stop-loss premium $E(Y_1+Y_2+Y_3-t)_+$. For comparison we consider the two cases given in Table 5.3 and Table 5.4. For Case 1, U_1, U_2 and U_3 are pairwise positively correlated. For Case 2, U_1 is negatively correlated with U_2 and U_3. The numerical results are given in Table 5.5.

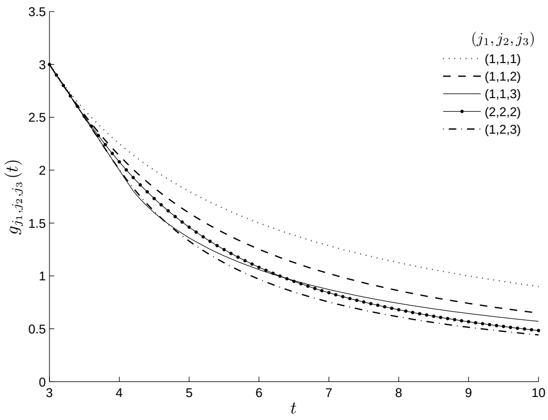

Figure 5.1: $g_{j_1,j_2,j_3}(t)$, $\alpha = 2$

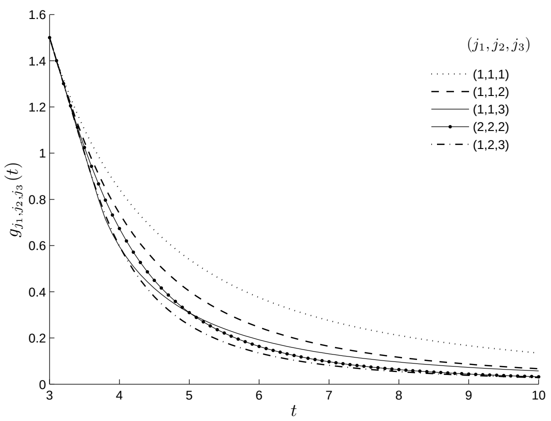

Figure 5.2: $g_{j_1,j_2,j_3}(t)$, $\alpha = 3$

	$i=1$	$i=2$	$i=3$
$a_{i,1}$	0.8	0.7	0.8
$a_{i,2}$	0.2	0	0.1
$a_{i,3}$	0	0.3	0.1

Table 5.3: $a_{i,j}, j \leq 3$, Case 1

	$i=1$	$i=2$	$i=3$
$a_{i,1}$	0	0.5	0.5
$a_{i,2}$	0.1	0.4	0.5
$a_{i,3}$	0.9	0.1	0

Table 5.4: $a_{i,j}, j \leq 3$, Case 2

	$t=3$	$t=4$	$t=5$	$t=7$	$t=10$	$t=20$	$t=50$
$\alpha=2$, Case 1	3.0000	2.1379	1.5974	1.0724	0.7188	0.3429	0.1337
$\alpha=2$, Case 2	3.0000	2.0315	1.3891	0.8257	0.5014	0.2130	0.0778
$\alpha=3$, Case 1	1.5000	0.7336	0.4242	0.1967	0.0908	0.0215	0.0034
$\alpha=3$, Case 2	1.5000	0.6261	0.2939	0.1047	0.0399	0.0078	0.0011

Table 5.5: Numerical results of $E(Y_1+Y_2+Y_3-t)_+$

6 Conclusions

Under the assumption of conditional independence, the multivariate copulas with bivariate Fréchet marginals are obtained. These copulas can be expressed as weighted sums of some special copulas. Some properties of the copulas are investigated. In particular, it is proved that these multivariate copulas are uniquely determined by their two-dimensional marginal copulas. Some applications of the copulas are discussed.

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7 Appendix

Proof of Proposition 2.2: When $l < n-1$, the number of the copulas in the set

$$\big\{C^{(j_1,j_2,\cdots,j_n)} : \#\{i : j_i = 2, 1 \le i \le n\} = l, j_i \in \{1,2,3\},\ 1 \le i \le n\big\}$$

equals

$$\binom{n}{l} \times \frac{1}{2}\left\{\binom{n-1}{0} + \binom{n-l}{1} + \cdots + \binom{n-l}{n-l}\right\} = 2^{n-l-1}\binom{n}{l}.$$

For the case $l \ge n-1$, we have $C^{(2,2,\cdots,2,3)} = C^{(2,2,\cdots,2,1)} = C^{(2,2,\cdots,2,2)}$. Thus, the total number of the distinct copulas in the family $\mathcal{S}_n$ is

$$1 + \sum_{l=0}^{n-2} 2^{n-l-1}\binom{n}{l} = \frac{1}{2}(3^n - 2n + 1).$$

Next we prove the second part. In $[0,1]^n$, we define the set

$$\begin{aligned}
&D^{(j_1,j_2,\cdots,j_n)}\\
&= \big\{u_i = u_m,\ i,m \in J_1\big\} \bigcap \big\{u_l = u_k,\ l,k \in J_3\big\} \bigcap \big\{u_i + u_l = 1,\ i \in J_1, l \in J_3\big\}\\
&\qquad\qquad \bigcap \big\{u_r + u_s \ne 1,\ u_r \ne u_s, r \in J_2, r \ne s, s \le n\big\},
\end{aligned}$$

the support of copula $C^{(j_1,j_2,\cdots,j_n)}$. Note that $D^{(j_1,j_2,\cdots,j_n)} = D^{(4-j_1,4-j_2,\cdots,4-j_n)}$.

The probability measure generated by $C^{(j_1,j_2,\cdots,j_n)}$ is denoted as $P^{(j_1,j_2,\cdots,j_n)}$. Then

$$P^{(j_1,j_2,\cdots,j_n)} = P^{(4-j_1,4-j_2,\cdots,4-j_n)}$$

and

$$P^{(j_1,j_2,\cdots,j_n)}(D^{(j_1,j_2,\cdots,j_n)}) = P^{(4-j_1,4-j_2,\cdots,4-j_n)}(D^{(j_1,j_2,\cdots,j_n)}) = 1.$$

For simplicity, the copulas in $\mathcal{S}_n$ are denoted as C_i, $i = 1,2,\cdots,\frac{1}{2}(3^n - 2n + 1)$. For each i, the probability measure generated by C_i is denoted as P_i and the corresponding support is denoted as D_i. Then $P_i(D_j) = 0$ for $i \ne j$.

Assume that for $f_i, g_i \ge 0$, copula C can be expressed as

$$C = \sum_i f_i C_i = \sum_i g_i C_i.$$

Suppose that there exists i such that $f_i \neq g_i$. We can define a probability measure

$$Q =: \frac{\sum_{g_i - f_i > 0}(g_i - f_i)P_i}{\sum_{g_i - f_i > 0}(g_i - f_i)}.$$

Then we have

$$Q = \frac{\sum_{f_i - g_i > 0}(f_i - g_i)P_i}{\sum_{f_i - g_i > 0}(f_i - g_i)}.$$

Note that for any j with $f_j - g_j > 0$,

$$Q(D_j) = \frac{\sum_{f_i - g_i > 0}(f_i - g_i)P_i(D_j)}{\sum_{f_i - g_i > 0}(f_i - g_i)} > 0.$$

On the other hand,

$$Q(D_j) = \frac{\sum_{f_i - g_i < 0}(g_i - f_i)P_i(D_j)}{\sum_{f_i - g_i < 0}(g_i - f_i)} = 0,$$

contradicting to that $Q(D_j) > 0$. Thus $f_i = g_i$ holds for all i, and the expression of copula C is unique. The proposition is proved.

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