Bernoulli and Tail-Dependence Compatibility

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Abstract

The tail-dependence compatibility problem is introduced. It raises the question whether a given $d \times d$-matrix of entries in the unit interval is the matrix of pairwise tail-dependence coefficients of a d-dimensional random vector. The problem is studied together with Bernoulli-compatible matrices, i.e., matrices which are expectations of outer products of random vectors with Bernoulli margins. We show that a square matrix with diagonal entries being 1 is a tail-dependence matrix if and only if it is a Bernoulli-compatible matrix multiplied by a constant. We introduce new copula models to construct tail-dependence matrices, including commonly used matrices in statistics.

Key-words: Tail dependence, Bernoulli random vectors, compatibility, matrices, copulas, insurance application.

MSC2010: 60E05, 62H99, 62H20, 62E15, 62H86.

1 Introduction

The problem of how to construct a bivariate random vector (X_1, X_2) with log-normal marginals $X_1 \sim \text{LN}(0,1)$, $X_2 \sim \text{LN}(0,16)$ and correlation coefficient $\text{Cor}(X_1, X_2) = 0.5$ is well known in the history of dependence modeling, partially because of its relevance to risk management practice. The short answer is: There is no such model; see Embrechts et al. (2002) who studied these kinds of problems in terms of copulas. Problems of this kind were brought to RiskLab at ETH Zurich by the insurance industry in the mid 1990s when dependence was

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thought of in terms of correlation (matrices). For further background to Quantitative Risk Management, see McNeil et al. (2015). Now, almost 20 years later, copulas are a well established tool to quantify dependence in multivariate data and to construct new multivariate distributions. Their use has become standard within industry and regulation. Nevertheless, dependence is still summarized in terms of numbers (as opposed to (copula) functions), so-called *measures of association*. Although there are various ways to compute such numbers in dimension $d > 2$, measures of association are still most widely used in the bivariate case $d = 2$. A popular measure of association is tail dependence. It is important for applications in Quantitative Risk Management as it measures the strength of dependence in either the lower-left or upper-right tail of the bivariate distribution, the regions Quantitative Risk Management is mainly concerned with.

We were recently asked[1] the following question which is in the same spirit as the log-normal correlation problem if one replaces "correlation" by "tail dependence"; see Section 3.1 for a definition.

For which $\alpha \in [0, 1]$ is the matrix

$$\Gamma_d(\alpha) = \begin{pmatrix} 1 & 0 & \cdots & 0 & \alpha \\ 0 & 1 & \cdots & 0 & \alpha \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha \\ \alpha & \alpha & \cdots & \alpha & 1 \end{pmatrix} \tag{1.1}$$

a matrix of pairwise (either lower or upper) tail-dependence coefficients?

Intrigued by this question, we more generally consider the following *tail-dependence compatibility problem* in this paper:

When is a given matrix in $[0, 1]^{d\times d}$ the matrix of pairwise (either lower or upper) tail-dependence coefficients?

In what follows, we call a matrix of pairwise tail-dependence coefficients a *tail-dependence matrix*. The compatibility problems of tail-dependence coefficients were studied in Joe (1997). In particular, when $d = 3$, inequalities for the bivariate tail-dependence coefficients have been established; see Joe (1997, Theorem 3.14) as well as Joe (2014, Theorem 8.20). The sharpness of these inequalities is obtained in Nikoloulopoulos et al. (2009). It is generally open to characterize the tail-dependence matrix compatibility for $d > 3$.

Our aim in this paper is to give a full answer to the tail-dependence compatibility problem; see Section 3. To this end, we introduce and study *Bernoulli-compatible matrices* in Section 2.

[1] by Federico Degen (Head Risk Modeling and Quantification, Zurich Insurance Group) and Janusz Milek (Zurich Insurance Group)

As a main result, we show that a matrix with diagonal entries being 1 is a compatible tail-dependence matrix if and only if it is a Bernoulli-compatible matrix multiplied by a constant. In Section 4 we provide probabilistic models for a large class of tail-dependence matrices, including commonly used matrices in statistics. Section 5 concludes.

Throughout this paper, d and m are positive integers, and we consider an atomless probability space $(\Omega, \mathcal{A}, \mathbb{P})$ on which all random variables and random vectors are defined. Vectors are considered as column vectors. For two matrices A, B, $B \geqslant A$ and $B \leqslant A$ are understood as component-wise inequalities. We let $A \circ B$ denote the Hadamard product, i.e., the element-wise product of two matrices A and B of the same dimension. The $d \times d$ identity matrix is denoted by I_d. For a square matrix A, $\mathrm{diag}(A)$ represents a diagonal matrix with diagonal entries equal to those of A, and $A^\top$ is the transpose of A. We denote I_E the indicator function of an event (random or deterministic) $E \in \mathcal{A}$. $\mathbf{0}$ and $\mathbf{1}$ are vectors with all components being 0 and 1 respectively, as long as the dimension of the vectors is clear from the context.

2 Bernoulli compatibility

In this section we introduce and study the *Bernoulli-compatibility problem*. The results obtained in this section are the basis for the *tail-dependence compatibility problem* treated in Section 3; many of them are of independent interest, e.g., for the simulation of sequences of Bernoulli random variables.

2.1 Bernoulli-compatible matrices

Definition 2.1 (Bernoulli vector, $\mathcal{V}_d$)**.** A *Bernoulli vector* is a random vector $\boldsymbol{X}$ supported by $\{0,1\}^d$ for some $d \in \mathbb{N}$. The set of all d-Bernoulli vectors is denoted by $\mathcal{V}_d$.

Equivalently, $\boldsymbol{X} = (X_1, \ldots, X_d)$ is a Bernoulli vector if and only if $X_i \sim \mathrm{B}(1, p_i)$ for some $p_i \in [0,1]$, $i = 1, \ldots, d$. Note that here we do not make any assumption about the dependence structure among the components of $\boldsymbol{X}$. Bernoulli vectors play an important role in Credit Risk Analysis; see, e.g., Bluhm and Overbeck (2006) and Bluhm et al. (2002, Section 2.1).

In this section, we investigate the following question which we refer to as the *Bernoulli-compatibility problem*:

Question 1. *Given a matrix $B \in [0,1]^{d \times d}$, can we find a Bernoulli vector $\boldsymbol{X}$ such that $B = \mathbb{E}[\boldsymbol{X}\boldsymbol{X}^\top]$?*

For studying the Bernoulli-compatibility problem, we introduce the notion of Bernoulli-compatible matrices.

Definition 2.2 (Bernoulli-compatible matrix, $\mathcal{B}_d$)**.** A $d \times d$ matrix B is a *Bernoulli-compatible matrix*, if $B = \mathbb{E}[\boldsymbol{X}\boldsymbol{X}^\top]$ for some $\boldsymbol{X} \in \mathcal{V}_d$. The set of all $d \times d$ Bernoulli-compatible matrices is denoted by $\mathcal{B}_d$.

Concerning covariance matrices, there is extensive research on the compatibility of covariance matrices of Bernoulli vectors in the realm of statistical simulation and time series analysis; see, e.g., Chaganty and Joe (2006). It is known that, when $d \geqslant 3$, the set of all compatible d-Bernoulli correlation matrices is strictly contained in the set of all correlation matrices. Note that $\mathbb{E}[\boldsymbol{X}\boldsymbol{X}^\top] = \mathrm{Cov}(\boldsymbol{X}) + \mathbb{E}[\boldsymbol{X}]\mathbb{E}[\boldsymbol{X}]^\top$. Hence, Question 1 is closely related to the characterization of compatible Bernoulli covariance matrices.

Before we characterize the set $\mathcal{B}_d$ in Section 2.2 and thus address Question 1, we first collect some facts about elements of $\mathcal{B}_d$.

Proposition 2.1. *Let $B, B_1, B_2 \in \mathcal{B}_d$. Then*

i) $B \in [0,1]^{d\times d}$.

ii) $\max\{b_{ii} + b_{jj} - 1, 0\} \leqslant b_{ij} \leqslant \min\{b_{ii}, b_{jj}\}$ *for* $i, j = 1, \dots, d$ *and* $B = (b_{ij})_{d\times d}$.

iii) $tB_1 + (1-t)B_2 \in \mathcal{B}_d$ *for* $t \in [0,1]$, *i.e.,* $\mathcal{B}_d$ *is a convex set.*

iv) $B_1 \circ B_2 \in \mathcal{B}_d$, *i.e.,* $\mathcal{B}_d$ *is closed under the Hadamard product.*

v) $(0)_{d\times d} \in \mathcal{B}_d$ *and* $(1)_{d\times d} \in \mathcal{B}_d$.

vi) For any $\boldsymbol{p} = (p_1, \dots, p_d) \in [0,1]^d$, *the matrix* $B = (b_{ij})_{d\times d} \in \mathcal{B}_d$ *where* $b_{ij} = p_i p_j$ *for* $i \neq j$ *and* $b_{ii} = p_i$, $i, j = 1, \dots, d$.

Proof. Write $B_1 = \mathbb{E}[\boldsymbol{X}\boldsymbol{X}^\top]$ and $B_2 = \mathbb{E}[\boldsymbol{Y}\boldsymbol{Y}^\top]$ for $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{V}_d$, and $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent.

i) Clear.

ii) This directly follows from the Fréchet–Hoeffding bounds; see McNeil et al. (2015, Remark 7.9).

iii) Let $A \sim \mathrm{B}(1, t)$ be a Bernoulli random variable independent of $\boldsymbol{X}, \boldsymbol{Y}$, and let $\boldsymbol{Z} = A\boldsymbol{X} + (1-A)\boldsymbol{Y}$. Then $\boldsymbol{Z} \in \mathcal{V}_d$, and $\mathbb{E}[\boldsymbol{Z}\boldsymbol{Z}^\top] = t\mathbb{E}[\boldsymbol{X}\boldsymbol{X}^\top] + (1-t)\mathbb{E}[\boldsymbol{Y}\boldsymbol{Y}^\top] = tB_1 + (1-t)B_2$. Hence $tB_1 + (1-t)B_2 \in \mathcal{B}_d$.

iv) Let $\boldsymbol{p} = (p_1, \dots, p_d)$, $\boldsymbol{q} = (q_1, \dots, q_d) \in \mathbb{R}^d$. Then

$$\begin{aligned}(\boldsymbol{p} \circ \boldsymbol{q})(\boldsymbol{p} \circ \boldsymbol{q})^\top = (p_i q_i)_d (p_i q_i)_d^\top = (p_i q_i p_j q_j)_{d\times d} = (p_i p_j)_{d\times d} \circ (q_i q_j)_{d\times d} \\ = (\boldsymbol{p}\boldsymbol{p}^\top) \circ (\boldsymbol{q}\boldsymbol{q}^\top).\end{aligned}$$

Let $\boldsymbol{Z} = \boldsymbol{X} \circ \boldsymbol{Y}$. It follows that $\boldsymbol{Z} \in \mathcal{V}_d$ and $\mathbb{E}[\boldsymbol{Z}\boldsymbol{Z}^\top] = \mathbb{E}[(\boldsymbol{X} \circ \boldsymbol{Y})(\boldsymbol{X} \circ \boldsymbol{Y})^\top] = \mathbb{E}[(\boldsymbol{X}\boldsymbol{X}^\top) \circ (\boldsymbol{Y}\boldsymbol{Y}^\top)] = \mathbb{E}[\boldsymbol{X}\boldsymbol{X}^\top] \circ \mathbb{E}[\boldsymbol{Y}\boldsymbol{Y}^\top] = B_1 \circ B_2$. Hence $B_1 \circ B_2 \in \mathcal{B}_d$.

v) Consider $\boldsymbol{X} = \mathbf{0} \in \mathcal{V}_d$. Then $(0)_{d\times d} = \mathbb{E}[\boldsymbol{X}\boldsymbol{X}^\top] \in \mathcal{B}_d$. Similarly for $(1)_{d\times d}$.

vi) Consider $\boldsymbol{X} \in \mathcal{V}_d$ with independent components and $\mathbb{E}[\boldsymbol{X}] = \boldsymbol{p}$. □

2.2 Characterization of Bernoulli-compatible matrices

We are now able to give a characterization of the set $\mathcal{B}_d$ of Bernoulli-compatible matrices and thus address Question 1.

Theorem 2.2 (Characterization of $\mathcal{B}_d$)**.** *$\mathcal{B}_d$ has the following characterization:*

$$\mathcal{B}_d = \Big\{ \sum_{i=1}^{n} a_i \boldsymbol{p}_i \boldsymbol{p}_i^\top : \boldsymbol{p}_i \in \{0,1\}^d,\ a_i \geqslant 0,\ i = 1, \dots, n,\ \sum_{i=1}^{n} a_i = 1,\ n \in \mathbb{N} \Big\}; \tag{2.1}$$

i.e., $\mathcal{B}_d$ is the convex hull of $\{\boldsymbol{p}\boldsymbol{p}^\top : \boldsymbol{p} \in \{0,1\}^d\}$. In particular, $\mathcal{B}_d$ is closed under convergence in the Euclidean norm.

Proof. Denote the right-hand side of (2.1) by $\mathcal{M}$. For $B \in \mathcal{B}_d$, write $B = \mathbb{E}[\boldsymbol{X}\boldsymbol{X}^\top]$ for some $\boldsymbol{X} \in \mathcal{V}_d$. It follows that

$$B = \sum_{\boldsymbol{p} \in \{0,1\}^d} \boldsymbol{p}\boldsymbol{p}^\top \mathbb{P}(\boldsymbol{X} = \boldsymbol{p}) \in \mathcal{M},$$

hence $\mathcal{B}_d \subseteq \mathcal{M}$. Let $\boldsymbol{X} = \boldsymbol{p} \in \{0,1\}^d$. Then $\boldsymbol{X} \in \mathcal{V}_d$ and $\mathbb{E}[\boldsymbol{X}\boldsymbol{X}^\top] = \boldsymbol{p}\boldsymbol{p}^\top \in \mathcal{B}_d$. By Proposition 2.1, $\mathcal{B}_d$ is a convex set which contains $\{\boldsymbol{p}\boldsymbol{p}^\top : \boldsymbol{p} \in \{0,1\}^d\}$, hence $\mathcal{M} \subseteq \mathcal{B}_d$. In summary, $\mathcal{M} = \mathcal{B}_d$. From (2.1) we can see that $\mathcal{B}_d$ is closed under convergence in the Euclidean norm. □

A matrix B is *completely positive* if $B = AA^\top$ for some (not necessarily square) matrix $A \geqslant 0$. Denote by $\mathcal{C}_d$ the set of completely positive matrices. It is known that $\mathcal{C}_d$ is the convex cone with extreme directions $\{\boldsymbol{p}\boldsymbol{p}^\top : \boldsymbol{p} \in [0,1]^d\}$; see, e.g., Rüschendorf (1981) and Berman and Shaked-Monderer (2003). We thus obtain the following result.

Corollary 2.3. *Any Bernoulli-compatible matrix is completely positive.*

Remark 2.1. One may wonder whether $B = \mathbb{E}[\boldsymbol{X}\boldsymbol{X}^\top]$ is sufficient to determine the distribution of $\boldsymbol{X}$, i.e., whether the decomposition

$$B = \sum_{i=1}^{2^d} a_i \boldsymbol{p}_i \boldsymbol{p}_i^\top \tag{2.2}$$

is unique for distinct vectors $\boldsymbol{p}_i$ in $\{0,1\}^d$. While the decomposition is trivially unique for $d = 2$, this is in general false for $d \geqslant 3$, since there are $2^d - 1$ parameters in (2.2) and only $d(d+1)/2$ parameters in B. The following is an example for $d = 3$. Let

$$\begin{aligned} B &= \frac{1}{4}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \\ &= \frac{1}{4}\big((1,1,1)^\top(1,1,1) + (1,0,0)^\top(1,0,0) + (0,1,0)^\top(0,1,0) \\ &\qquad + (0,0,1)^\top(0,0,1)\big) \\ &= \frac{1}{4}\big((1,1,0)^\top(1,1,0) + (1,0,1)^\top(1,0,1) + (0,1,1)^\top(0,1,1) \\ &\qquad + (0,0,0)^\top(0,0,0)\big). \end{aligned}$$

Thus, by combining the above two decompositions, $B \in \mathcal{B}_3$ has infinitely many different decompositions of the form (2.2). Note that, as in the case of completely positive matrices, it is generally difficult to find decompositions of form (2.2) for a given matrix B.

2.3 Convex cone generated by Bernoulli-compatible matrices

In this section we study the convex cone generated by $\mathcal{B}_d$, denoted by $\mathcal{B}_d^*$:

$$\mathcal{B}_d^* = \{aB : a \geqslant 0,\ B \in \mathcal{B}_d\}. \tag{2.3}$$

The following proposition is implied by Proposition 2.1 and Theorem 2.2.

Proposition 2.4. *$\mathcal{B}_d^*$ is the convex cone with extreme directions $\{\boldsymbol{p}\boldsymbol{p}^\top : \boldsymbol{p} \in \{0,1\}^d\}$. Moreover, $\mathcal{B}_d^*$ is a commutative semiring equipped with addition $(\mathcal{B}_d^*, +)$ and multiplication $(\mathcal{B}_d^*, \circ)$.*

It is obvious that $\mathcal{B}_d^* \subseteq \mathcal{C}_d$. One may wonder whether $\mathcal{B}_d^*$ is identical to $\mathcal{C}_d$, the set of completely positive matrices. As the following example shows, this is false in general for $d \geqslant 2$.

Example 2.1. Note that $B \in \mathcal{B}_d^*$ also satisfies Proposition 2.1, Part ii). Now consider $\boldsymbol{p} = (p_1, \dots, p_d) \in (0,1)^d$ with $p_i > p_j$ for some $i \neq j$. Clearly, $\boldsymbol{p}\boldsymbol{p}^\top \in \mathcal{C}_d$, but $p_i p_j > p_j^2 = \min\{p_i^2, p_j^2\}$ contradicts Proposition 2.1, Part ii), hence $\boldsymbol{p}\boldsymbol{p}^\top \notin \mathcal{B}_d^*$.

For the following result, we need the notion of diagonally dominant matrices. A matrix $A \in \mathbb{R}^{d\times d}$ is called *diagonally dominant* if, for all $i = 1, \dots, d$, $\sum_{j\neq i} |a_{ij}| \leqslant |a_{ii}|$.

Proposition 2.5. *Let $\mathcal{D}_d$ be the set of non-negative, diagonally dominant $d \times d$-matrices. Then $\mathcal{D}_d \subseteq \mathcal{B}_d^*$.*

Proof. For $i, j = 1, \ldots, d$, let $\boldsymbol{p}^{(ij)} = (p_1^{(ij)}, \ldots, p_d^{(ij)})$ where $p_k^{(ij)} = \mathrm{I}_{\{k=i\}\cup\{k=j\}}$. It is straightforward to verify that the (i,i)-, (i,j)-, (j,i)- and (j,j)-entries of the matrix $M^{(ij)} = \boldsymbol{p}^{(ij)}(\boldsymbol{p}^{(ij)})^\top$ are 1, and the other entries are 0. For $D = (d_{ij})_{d\times d} \in \mathcal{D}_d$, let

$$D^* = (d^*_{ij})_{d\times d} = \sum_{i=1}^{d} \sum_{j=1, j\neq i}^{d} d_{ij} M^{(ij)}.$$

By Proposition 2.4, $D^* \in \mathcal{B}^*_d$. It follows that $d^*_{ij} = d_{ij}$ for $i \neq j$ and $d^*_{ii} = \sum_{j=1,j\neq i}^{d} d_{ij} \leqslant d_{ii}$. Therefore, $D = D^* + \sum_{i=1}^{d}(d_{ii} - d^*_{ii})M^{(ii)}$, which, by Proposition 2.4, is in $\mathcal{B}^*_d$. □

For studying the tail-dependence compatibility problem in Section 3, the subset

$$\mathcal{B}^I_d = \{B : B \in \mathcal{B}^*_d,\ \mathrm{diag}(B) = I_d\}$$

of $\mathcal{B}^*_d$ is of interest. It is straightforward to see from Proposition 2.1 and Theorem 2.2 that $\mathcal{B}^I_d$ is a convex set, closed under the Hadamard product and convergence in the Euclidean norm. These properties of $\mathcal{B}^I_d$ will be used later.

3 Tail-dependence compatibility

3.1 Tail-dependence matrices

The notion of tail dependence captures (extreme) dependence in the lower-left or upper-right tails of a bivariate distribution. In what follows, we focus on lower-left tails; the problem for upper-right tails follows by a reflection around $(1/2, 1/2)$, i.e., studying the survival copula of the underlying copula.

Definition 3.1 (Tail-dependence coefficient)**.** The *(lower) tail-dependence coefficient* of two continuous random variables $X_1 \sim F_1$ and $X_2 \sim F_2$ is defined by

$$\lambda = \lim_{u\downarrow 0} \frac{\mathbb{P}(F_1(X_1) \leqslant u, F_2(X_2) \leqslant u)}{u}, \tag{3.1}$$

given that the limit exists.

If we denote the copula of (X_1, X_2) by C, then

$$\lambda = \lim_{u\downarrow 0} \frac{C(u,u)}{u}.$$

Clearly $\lambda \in [0,1]$, and λ only depends on the copula of (X_1, X_2), not the marginal distributions. For virtually all copula models used in practice, the limit in (3.1) exists; for how to construct an example where λ does not exist, see Kortschak and Albrecher (2009).

Definition 3.2 (Tail-dependence matrix, $\mathcal{T}_d$)**.** Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a random vector with continuous marginal distributions. The *tail-dependence matrix* of $\boldsymbol{X}$ is $\Lambda = (\lambda_{ij})_{d\times d}$, where λ_{ij} is the tail-dependence coefficient of X_i and X_j, $i, j = 1, \ldots, d$. We denote by $\mathcal{T}_d$ the set of all tail-dependence matrices.

The following proposition summarizes basic properties of tail-dependence matrices. Its proof is very similar to that of Proposition 2.1 and is omitted here.

Proposition 3.1. *For any $\Lambda_1, \Lambda_2 \in \mathcal{T}_d$, we have that*

i) $\Lambda_1 = \Lambda_1^\top$.

ii) $t\Lambda_1 + (1-t)\Lambda_2 \in \mathcal{T}_d$ *for* $t \in [0,1]$*, i.e.,* $\mathcal{T}_d$ *is a convex set.*

iii) $I_d \leqslant \Lambda_1 \leqslant (1)_{d\times d}$ *with* $I_d \in \mathcal{T}_d$ *and* $(1)_{d\times d} \in \mathcal{T}_d$.

As we will show next, $\mathcal{T}_d$ is also closed under the Hadamard product.

Proposition 3.2. *Let $k \in \mathbb{N}$ and $\Lambda_1, \ldots, \Lambda_k \in \mathcal{T}_d$. Then $\Lambda_1 \circ \cdots \circ \Lambda_k \in \mathcal{T}_d$.*

Proof. Note that it would be sufficient to show the result for $k = 2$, but we provide a general construction for any k. For each $l = 1, \ldots, k$, let C_l be a d-dimensional copula with tail-dependence matrix Λ_l. Furthermore, let $g(u) = u^{1/k}$, $u \in [0,1]$. It follows from Liebscher (2008) that $C(u_1, \ldots, u_d) = \prod_{l=1}^k C_l(g(u_1), \ldots, g(u_d))$ is a copula; note that

$$\big(g^{-1}(\max_{1\leqslant l\leqslant k}\{U_{l1}\}), \ldots, g^{-1}(\max_{1\leqslant l\leqslant k}\{U_{ld}\})\big) \sim C \tag{3.2}$$

for independent random vectors $(U_{l1}, \ldots, U_{ld}) \sim C_l$, $l = 1, \ldots, k$. The (i,j)-entry λ_{ij} of Λ corresponding to C is thus given by

$$\begin{aligned}\lambda_{ij} &= \lim_{u\downarrow 0} \frac{\prod_{l=1}^k C_{l,ij}(g(u), g(u))}{u} = \lim_{u\downarrow 0} \prod_{l=1}^k \frac{C_{l,ij}(g(u), g(u))}{g(u)} \\ &= \prod_{l=1}^k \lim_{u\downarrow 0} \frac{C_{l,ij}(g(u), g(u))}{g(u)} = \prod_{l=1}^k \lim_{u\downarrow 0} \frac{C_{l,ij}(u, u)}{u} = \prod_{l=1}^k \lambda_{l,ij},\end{aligned}$$

where $C_{l,ij}$ denotes the (i,j)-margin of C_l and $\lambda_{l,ij}$ denotes the (i,j)th entry of Λ_l, $l = 1, \ldots, k$. □

3.2 Characterization of tail-dependence matrices

In this section, we investigate the following question:

Question 2. *Given a $d \times d$ matrix $\Lambda \in [0,1]^{d\times d}$, is it a tail-dependence matrix?*

The following theorem fully characterizes tail-dependence matrices and thus provides a theoretical (but not necessarily practical) answer to Question 2.

Theorem 3.3 (Characterization of $\mathcal{T}_d$)**.** *A square matrix with diagonal entries being 1 is a tail-dependence matrix if and only if it is a Bernoulli-compatible matrix multiplied by a constant. Equivalently,* $\mathcal{T}_d = \mathcal{B}_d^I$.

Proof. We first show that $\mathcal{T}_d \subseteq \mathcal{B}_d^I$. For each $\Lambda = (\lambda_{ij})_{d\times d} \in \mathcal{T}_d$, suppose that C is a copula with tail-dependence matrix Λ and $\boldsymbol{U} = (U_1, \dots, U_n) \sim C$. Let $\boldsymbol{W}_u = (\mathrm{I}_{\{U_1 \leqslant u\}}, \dots, \mathrm{I}_{\{U_d \leqslant u\}})$. By definition,

$$\lambda_{ij} = \lim_{u \downarrow 0} \frac{1}{u} \mathbb{E}[\mathrm{I}_{\{U_i \leqslant u\}} \mathrm{I}_{\{U_j \leqslant u\}}]$$

and

$$\Lambda = \lim_{u \downarrow 0} \frac{1}{u} \mathbb{E}[\boldsymbol{W}_u \boldsymbol{W}_u^\top].$$

Since $\mathcal{B}_d^I$ is closed and $\mathbb{E}[\boldsymbol{W}_u \boldsymbol{W}_u^\top]/u \in \mathcal{B}_d^I$, we have that $\Lambda \in \mathcal{B}_d^I$.

Now consider $\mathcal{B}_d^I \subseteq \mathcal{T}_d$. By definition of $\mathcal{B}_d^I$, each $B \in \mathcal{B}_d^I$ can be written as $B = \mathbb{E}[\boldsymbol{X}\boldsymbol{X}^\top]/p$ for an $\boldsymbol{X} \in \mathcal{V}_d$ and $\mathbb{E}[\boldsymbol{X}] = (p, \dots, p) \in (0, 1]^d$. Let $U, V \sim \mathrm{U}[0, 1]$, $U, V, \boldsymbol{X}$ be independent and

$$\boldsymbol{Y} = \boldsymbol{X} pU + (\mathbf{1} - \boldsymbol{X})(p + (1 - p)V). \tag{3.3}$$

We can verify that for $t \in [0, 1]$ and $i = 1, \dots, d$,

$$\begin{aligned}\mathbb{P}(Y_i \leqslant t) &= \mathbb{P}(X_i = 1)\mathbb{P}(pU \leqslant t) + \mathbb{P}(X_i = 0)\mathbb{P}(p + (1 - p)V \leqslant t) \\ &= p \min\{t/p, 1\} + (1 - p) \max\{(t - p)/(1 - p), 0\} = t,\end{aligned}$$

i.e., $Y_1, \dots, Y_d$ are $\mathrm{U}[0, 1]$-distributed. Let λ_{ij} be the tail-dependence coefficient of Y_i and Y_j, $i, j = 1, \dots, d$. For $i, j = 1, \dots, d$ we obtain that

$$\begin{aligned}\lambda_{ij} &= \lim_{u \downarrow 0} \frac{1}{u} \mathbb{P}(Y_i \leqslant u, Y_j \leqslant u) = \lim_{u \downarrow 0} \frac{1}{u} \mathbb{P}(X_i = 1, X_j = 1)\mathbb{P}(pU \leqslant u) \\ &= \frac{1}{p} \mathbb{E}[X_i X_j].\end{aligned}$$

As a consequence, the tail-dependence matrix of $(Y_1, \dots, Y_d)$ is B and $B \in \mathcal{T}_d$. □

It follows from Theorem 3.3 and Proposition 2.4 that $\mathcal{T}_d$ is the "1-diagonals" cross-section of the convex cone with extreme directions $\{\boldsymbol{p}\boldsymbol{p}^\top : \boldsymbol{p} \in \{0, 1\}^d\}$. Furthermore, the proof of Theorem 3.3 is constructive. As we saw, for any $B \in \mathcal{B}_d^I$, $\boldsymbol{Y}$ defined by (3.3) has tail-dependence matrix B. This interesting construction will be applied in Section 4 where we show that commonly applied matrices in statistics are tail-dependence matrices and where we derive the copula of $\boldsymbol{Y}$.

Remark 3.1. From the fact that $\mathcal{T}_d = \mathcal{B}_d^I$ and $\mathcal{B}_d^I$ is closed under the Hadamard product (see Proposition 2.1, Part iv)), Proposition 3.2 directly follows. Note, however, that our proof of Proposition 3.2 is constructive. Given tail-dependence matrices and corresponding copulas, we can construct a copula C which has the Hadamard product of the tail-dependence matrices as corresponding tail-dependence matrix. If sampling of all involved copulas is feasible, we can sample C; see Figure 1 for examples[2].

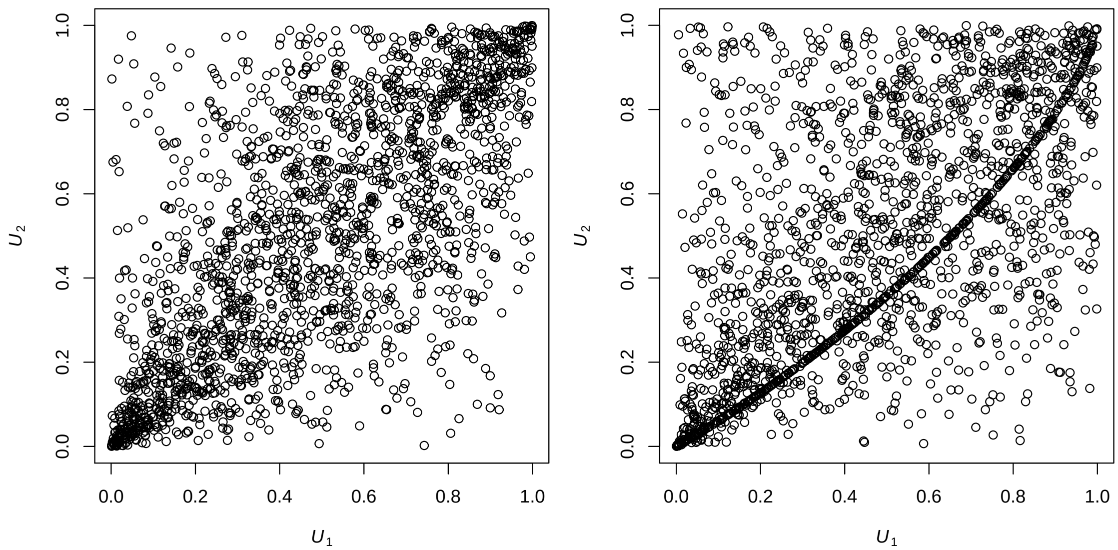

Figure 1: Left-hand side: Scatter plot of 2000 samples from (3.2) for C_1 being a Clayton copula with parameter $\theta = 4$ ($\lambda_1 = 2^{-1/4} \approx 0.8409$) and C_2 being a t_3 copula with parameter $\rho = 0.8$ (tail-dependence coefficient $\lambda_2 = 2t_4(-2/3) \approx 0.5415$). By Proposition 3.2, the tail-dependence coefficient of (3.2) is thus $\lambda = \lambda_1\lambda_2 = 2^{3/4}t_4(-2/3) \approx 0.4553$. Right-hand side: C_1 as before, but C_2 is a survival Marshall–Olkin copula with parameters $\alpha_1 = 2^{-3/4}, \alpha_2 = 0.8$, so that $\lambda = \lambda_1\lambda_2 = 1/2$.

Theorem 3.3 combined with Corollary 2.3 directly leads to the following result.

Corollary 3.4. *Every tail-dependence matrix is completely positive, and hence positive semi-definite.*

Furthermore, Theorem 3.3 and Proposition 2.5 imply the following result.

Corollary 3.5. *Every diagonally dominant matrix with non-negative entries and diagonal entries being* 1 *is a tail-dependence matrix.*

Note that this result already yields the if-part of Proposition 4.7 below.

[2] All plots can be reproduced via the R package `copula` (version $\geqslant$ 0.999-13) by calling `demo(tail_compatibility)`.

4 Compatible models for tail-dependence matrices

4.1 Widely known matrices

We now consider the following three types of matrices $\Lambda = (\lambda_{ij})_{d\times d}$ which are frequently applied in multivariate statistics and time series analysis and show that they are tail-dependence matrices.

a) Equicorrelation matrix with parameter $\alpha \in [0,1]$: $\lambda_{ij} = \mathrm{I}_{\{i=j\}} + \alpha \mathrm{I}_{\{i\neq j\}}$, $i,j = 1,\dots,d$.

b) AR(1) matrix with parameter $\alpha \in [0,1]$: $\lambda_{ij} = \alpha^{|i-j|}$, $i,j = 1,\dots,d$.

c) MA(1) matrix with parameter $\alpha \in [0,1/2]$: $\lambda_{ij} = \mathrm{I}_{\{i=j\}} + \alpha \mathrm{I}_{\{|i-j|=1\}}$, $i,j = 1,\dots,d$.

Chaganty and Joe (2006) considered the compatibility of correlation matrices of Bernoulli vectors for the above three types of matrices and obtained necessary and sufficient conditions for the existence of compatible models for $d = 3$. For the tail-dependence compatibility problem that we consider in this paper, the above three types of matrices are all compatible, and we are able to construct corresponding models for each case.

Proposition 4.1. *Let Λ be the tail-dependence matrix of the d-dimensional random vector*

$$\boldsymbol{Y} = \boldsymbol{X} pU + (\boldsymbol{1} - \boldsymbol{X})(p + (1-p)V), \tag{4.1}$$

where $U, V \sim \mathrm{U}[0,1]$, $\boldsymbol{X} \in \mathcal{V}_d$ and $U, V, \boldsymbol{X}$ are independent.

i) For $\alpha \in [0,1]$, if $\boldsymbol{X}$ has independent components and $\mathbb{E}[X_1] = \cdots = \mathbb{E}[X_d] = \alpha$, then Λ is an equicorrelation matrix with parameter α; i.e., a) is a tail-dependence matrix.

ii) For $\alpha \in [0,1]$, if $X_i = \prod_{j=i}^{i+d-1} Z_j$, $i = 1,\dots,d$, for independent $\mathrm{B}(1,\alpha)$ random variables $Z_1,\dots,Z_{2d-1}$, then Λ is an AR(1) matrix with parameter α; i.e., b) is a tail-dependence matrix.

iii) For $\alpha \in [0,1/2]$, if $X_i = \mathrm{I}_{\{Z\in[(i-1)(1-\alpha),(i-1)(1-\alpha)+1]\}}$, $i = 1,\dots,d$, for $Z \sim \mathrm{U}[0,d]$, then Λ is an MA(1) matrix with parameter α; i.e., c) is a tail-dependence matrix.

Proof. We have seen in the proof of Theorem 3.3 that if $\mathbb{E}[X_1] = \cdots = \mathbb{E}[X_d] = p$, then $\boldsymbol{Y}$ defined through (4.1) has tail-dependence matrix $\mathbb{E}[\boldsymbol{X}\boldsymbol{X}^\top]/p$. Write $\Lambda = (\lambda_{ij})_{d\times d}$ and note that $\lambda_{ii} = 1$, $i = 1,\dots,d$, is always guaranteed.

i) For $i \neq j$, we have that $\mathbb{E}[X_iX_j] = \alpha^2$ and thus $\lambda_{ij} = \alpha^2/\alpha = \alpha$. This shows that Λ is an equicorrelation matrix with parameter α.

ii) For $i < j$, we have that

$$\mathbb{E}[X_i X_j] = \mathbb{E}\left[\prod_{k=i}^{i+d-1} Z_k \prod_{l=j}^{j+d-1} Z_l\right] = \mathbb{E}\left[\prod_{k=i}^{j-1} Z_k\right]\mathbb{E}\left[\prod_{k=j}^{i+d-1} Z_k\right]\mathbb{E}\left[\prod_{k=i+d}^{j+d-1} Z_k\right]$$
$$= \alpha^{j-i}\alpha^{i+d-j}\alpha^{j-i} = \alpha^{j-i+d}$$

and $\mathbb{E}[X_i] = \mathbb{E}[X_i^2] = \alpha^d$. Hence, $\lambda_{ij} = \alpha^{j-i+d}/\alpha^d = \alpha^{j-i}$ for $i < j$. By symmetry, $\lambda_{ij} = \alpha^{|i-j|}$ for $i \neq j$. Thus, Λ is an AR(1) matrix with parameter α.

iii) For $i < j$, note that $2(1-\alpha) \geqslant 1$, so

$$\mathbb{E}[X_i X_j] = \mathbb{P}(Z \in [(j-1)(1-\alpha), (i-1)(1-\alpha)+1])$$
$$= \mathrm{I}_{\{j=i+1\}}\mathbb{P}(Z \in [i(1-\alpha), (i-1)(1-\alpha)+1]) = \mathrm{I}_{\{j=i+1\}}\frac{\alpha}{d}$$

and $\mathbb{E}[X_i] = \mathbb{E}[X_i^2] = \frac{1}{d}$. Hence, $\lambda_{ij} = \alpha \mathrm{I}_{\{j-i=1\}}$ for $i < j$. By symmetry, $\lambda_{ij} = \alpha \mathrm{I}_{\{|i-j|=1\}}$ for $i \neq j$. Thus, Λ is an MA(1) matrix with parameter α. □

4.2 Advanced tail-dependence models

Theorem 3.3 gives a characterization of tail-dependence matrices using Bernoulli-compatible matrices and (3.3) provides a compatible model $\boldsymbol{Y}$ for any tail-dependence matrix $\Lambda(= \mathbb{E}[\boldsymbol{X}\boldsymbol{X}^\top]/p)$.

It is generally not easy to check whether a given matrix is a Bernoulli-compatible matrix or a tail-dependence matrix; see also Remark 2.1. Therefore, we now study the following question.

Question 3. *How can we construct a broader class of models with flexible dependence structures and desired tail-dependence matrices?*

To enrich our models, we bring random matrices with Bernoulli entries into play. For $d, m \in \mathbb{N}$, let

$$\mathcal{V}_{d\times m} = \left\{X = (X_{ij})_{d\times m} : \mathbb{P}(X \in \{0,1\}^{d\times m}) = 1,\ \sum_{j=1}^{m} X_{ij} \leqslant 1,\ i = 1, \ldots, d\right\},$$

i.e., $\mathcal{V}_{d\times m}$ is the set of $d \times m$ random matrices supported in $\{0,1\}^{d\times m}$ with each row being *mutually exclusive*; see Dhaene and Denuit (1999). Furthermore, we introduce a transformation $\mathcal{L}$ on the set of square matrices, such that, for any $i, j = 1, \ldots, d$, the (i,j)th element $\tilde{b}_{ij}$ of $\mathcal{L}(B)$ is given by

$$\tilde{b}_{ij} = \begin{cases} b_{ij}, & \text{if } i \neq j, \\ 1, & \text{if } i = j; \end{cases} \tag{4.2}$$

i.e., $\mathcal{L}$ adjusts the diagonal entries of a matrix to be 1, and preserves all the other entries. For a set S of square matrices, we set $\mathcal{L}(S) = \{\mathcal{L}(B) : B \in S\}$. We can now address Question 3.

Theorem 4.2 (A class of flexible models)**.** *Let $\boldsymbol{U} \sim C^{\boldsymbol{U}}$ for an m-dimensional copula $C^{\boldsymbol{U}}$ with tail-dependence matrix Λ and let let $\boldsymbol{V} \sim C^{\boldsymbol{V}}$ for a d-dimensional copula $C^{\boldsymbol{V}}$ with tail-dependence matrix I_d. Furthermore, let $X \in \mathcal{V}_{d\times m}$ such that $X, \boldsymbol{U}, \boldsymbol{V}$ are independent and let*

$$\boldsymbol{Y} = X\boldsymbol{U} + \boldsymbol{Z} \circ \boldsymbol{V}, \tag{4.3}$$

where $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ with $Z_i = 1 - \sum_{k=1}^m X_{ik}$, $i = 1, \dots, d$. Then $\boldsymbol{Y}$ has tail-dependence matrix $\Gamma = \mathcal{L}(\mathbb{E}[X\Lambda X^\top])$.

Proof. Write $X = (X_{ij})_{d\times m}$, $\boldsymbol{U} = (U_1, \dots, U_m)$, $\boldsymbol{V} = (V_1, \dots, V_d)$, $\Lambda = (\lambda_{ij})_{d\times d}$ and $\boldsymbol{Y} = (Y_1, \dots, Y_d)$. Then, for all $i = 1, \dots, d$,

$$Y_i = \sum_{k=1}^m X_{ik}U_k + Z_iV_i = \begin{cases} V_i, & \text{if } X_{ik} = 0 \text{ for all } k = 1, \dots, m, \text{so } Z_i = 1, \\ U_k, & \text{if } X_{ik} = 1 \text{ for some } k = 1, \dots, m, \text{so } Z_i = 0. \end{cases}$$

Clearly, $\boldsymbol{Y}$ has $\mathrm{U}[0,1]$ margins. We now calculate the tail-dependence matrix $\Gamma = (\gamma_{ij})_{d\times d}$ of Y for $i \neq j$. By our independence assumptions, we can derive the following results:

i) $\mathbb{P}(Y_i \leqslant u, Y_j \leqslant u, Z_i = 1, Z_j = 1) = \mathbb{P}(V_i \leqslant u, V_j \leqslant u, Z_i = 1, Z_j = 1) = C_{ij}^{\boldsymbol{V}}(u,u)\mathbb{P}(Z_i = 1, Z_j = 1) \leqslant C_{ij}^{\boldsymbol{V}}(u,u)$, where $C_{ij}^{\boldsymbol{V}}$ denotes the (i,j)th margin of $C^{\boldsymbol{V}}$. As $\boldsymbol{V}$ has tail-dependence matrix I_d, we obtain that

$$\lim_{u\downarrow 0} \frac{1}{u}\mathbb{P}(Y_i \leqslant u, Y_j \leqslant u, Z_i = 1, Z_j = 1) = 0.$$

ii) $\mathbb{P}(Y_i \leqslant u, Y_j \leqslant u, Z_i = 0, Z_j = 1) = \sum_{k=1}^m \mathbb{P}(U_k \leqslant u, V_j \leqslant u, X_{ik} = 1, Z_j = 1) = \sum_{k=1}^m \mathbb{P}(U_k \leqslant u)\mathbb{P}(V_j \leqslant u)\mathbb{P}(X_{ik} = 1, Z_j = 1) \leqslant u^2$ and thus

$$\lim_{u\downarrow 0} \frac{1}{u}\mathbb{P}(Y_i \leqslant u, Y_j \leqslant u, Z_i = 0, Z_j = 1) = 0.$$

Similarly, we obtain that

$$\lim_{u\downarrow 0} \frac{1}{u}\mathbb{P}(Y_i \leqslant u, Y_j \leqslant u, Z_i = 1, Z_j = 0) = 0.$$

iii) $\mathbb{P}(Y_i \leqslant u, Y_j \leqslant u, Z_i = 0, Z_j = 0) = \sum_{k=1}^m \sum_{l=1}^m \mathbb{P}(U_k \leqslant u, U_l \leqslant u, X_{ik} = 1, X_{jl} = 1) = \sum_{k=1}^m \sum_{l=1}^m C_{kl}^{\boldsymbol{U}}(u,u)\mathbb{P}(X_{ik} = 1, X_{jl} = 1) = \sum_{k=1}^m \sum_{l=1}^m C_{kl}^{\boldsymbol{U}}(u,u)\mathbb{E}[X_{ik}X_{jl}]$ so that

$$\begin{aligned} &\lim_{u\downarrow 0} \frac{1}{u}\mathbb{P}(Y_i \leqslant u, Y_j \leqslant u, Z_i = 0, Z_j = 0) \\ &= \sum_{k=1}^m \sum_{l=1}^m \lambda_{kl}\mathbb{E}[X_{ik}X_{jl}] = \mathbb{E}\biggl[\sum_{k=1}^m \sum_{l=1}^m X_{ik}\lambda_{kl}X_{jl}\biggr] = \bigl(\mathbb{E}[X\Lambda X^\top]\bigr)_{ij}. \end{aligned}$$

By the Law of Total Probability, we thus obtain that

$$\begin{aligned} \gamma_{ij} &= \lim_{u\downarrow 0} \frac{\mathbb{P}(Y_i \leqslant u, Y_j \leqslant u)}{u} = \lim_{u\downarrow 0} \frac{\mathbb{P}(Y_i \leqslant u, Y_j \leqslant u, Z_i = 0, Z_j = 0)}{u} \\ &= \bigl(\mathbb{E}[X\Lambda X^\top]\bigr)_{ij}. \end{aligned}$$

This shows that $\mathbb{E}[X\Lambda X^\top]$ and Γ agree on the off-diagonal entries. Since $\Gamma \in \mathcal{T}_d$ implies that $\mathrm{diag}(\Gamma) = I_d$, we conclude that $\mathcal{L}(\mathbb{E}[X\Lambda X^\top]) = \Gamma$. $\square$

A special case of Theorem 4.2 reveals an essential difference between the transition rules of a tail-dependence matrix and a covariance matrix. Suppose that for $X \in \mathcal{V}_{d\times m}$, $\mathbb{E}[X]$ is a stochastic matrix (each row sums to 1), and $\boldsymbol{U} \sim C^{\boldsymbol{U}}$ for an m-dimensional copula $C^{\boldsymbol{U}}$ with tail-dependence matrix $\Lambda = (\lambda_{ij})_{d\times d}$. Now we have that $Z_i = 0$, $i = 1,\dots,d$ in (4.3). By Theorem 4.2, the tail dependence matrix of $\boldsymbol{Y} = X\boldsymbol{U}$ is given by $\mathcal{L}(\mathbb{E}[X\Lambda X^\top])$. One can check the diagonal terms of the matrix $\Lambda^* = (\lambda^*_{ij})_{d\times d} = X\Lambda X^\top$ by

$$\lambda^*_{ii} = \sum_{j=1}^m \sum_{k=1}^m X_{ik}\lambda_{kj}X_{ij} = \sum_{k=1}^m X_{ik}\lambda_{kk} = 1, \quad i = 1,\dots,m.$$

Hence, the tail-dependence matrix of $\boldsymbol{Y}$ is indeed $\mathbb{E}[X\Lambda X^\top]$.

Remark 4.1. In summary:

i) If an m-vector $\boldsymbol{U}$ has covariance matrix Σ, then $X\boldsymbol{U}$ has covariance matrix $\mathbb{E}[X\Sigma X^\top]$ for any $d\times m$ random matrix X independent of $\boldsymbol{U}$.

ii) If an m-vector $\boldsymbol{U}$ has uniform [0,1] margins and tail-dependence matrix Λ, then $X\boldsymbol{U}$ has tail-dependence matrix $\mathbb{E}[X\Lambda X^\top]$ for any $X \in \mathcal{V}_{d\times m}$ independent of $\boldsymbol{U}$ such that each row of X sums to 1.

It is noted that the transition property of tail-dependence matrices is more restricted than that of covariance matrices.

The following two propositions consider selected special cases of this construction which are more straightforward to apply.

Proposition 4.3. *For any $B \in \mathcal{B}_d$ and any $\Lambda \in \mathcal{T}_d$ we have that $\mathcal{L}(B\circ\Lambda) \in \mathcal{T}_d$. In particular, $\mathcal{L}(B) \in \mathcal{T}_d$ and hence $\mathcal{L}(\mathcal{B}_d) \subseteq \mathcal{T}_d$.*

Proof. Write $B = (b_{ij})_{d\times d} = \mathbb{E}[\boldsymbol{W}\boldsymbol{W}^\top]$ for some $\boldsymbol{W} = (W_1,\dots,W_d) \in \mathcal{V}_d$ and consider $X = \mathrm{diag}(\boldsymbol{W}) \in \mathcal{V}_{d\times d}$. As in the proof of Theorem 4.2 (and with the same notation), it follows that for $i \neq j$, $\gamma_{ij} = \mathbb{E}[X_{ii}\lambda_{ij}X_{jj}] = \mathbb{E}[W_iW_j\lambda_{ij}]$. This shows that $\mathbb{E}[X\Lambda X^\top] = \mathbb{E}[\boldsymbol{W}\boldsymbol{W}^\top \circ \Lambda]$ and $B\circ\Lambda$ agree on off-diagonal entries. Thus, $\mathcal{L}(B\circ\Lambda) = \Gamma \in \mathcal{T}_d$. By taking $\Lambda = (1)_{d\times d}$, we obtain $\mathcal{L}(B) \in T_d$. $\square$

The following proposition states a relationship between substochastic matrices and tail-dependence matrices. To this end, let

$$\mathcal{Q}_{d\times m} = \left\{ Q = (q_{ij})_{d\times m} : \sum_{j=1}^m q_{ij} \leqslant 1,\ q_{ij} \geqslant 0,\ i = 1,\dots,d,\ j = 1,\dots,m \right\},$$

i.e., $\mathcal{Q}_{d\times m}$ is the set of $d\times m$ *(row) substochastic matrices*; note that the expectation of a random matrix in $\mathcal{V}_{d\times m}$ is a substochastic matrix.

Proposition 4.4. *For any $Q\in\mathcal{Q}_{d\times m}$ and any $\Lambda\in\mathcal{T}_m$, we have that $\mathcal{L}(Q\Lambda Q^\top)\in\mathcal{T}_d$. In particular, $\mathcal{L}(QQ^\top)\in\mathcal{T}_d$ for all $Q\in\mathcal{Q}_{d\times m}$ and $\mathcal{L}(\boldsymbol{p}\boldsymbol{p}^\top)\in\mathcal{T}_d$ for all $\boldsymbol{p}\in[0,1]^d$.*

Proof. Write $Q=(q_{ij})_{d\times m}$ and let $X_{ik}=\mathrm{I}_{\{Z_i\in[\sum_{j=1}^{k-1}q_{ij},\ \sum_{j=1}^{k}q_{ij})\}}$ for independent $Z_i\sim\mathrm{U}[0,1]$, $i=1,\dots,d$, $k=1,\dots,m$. It is straightforward to see that $\mathbb{E}[X]=Q$, $X\in\mathcal{V}_{d\times m}$ with independent rows, and $\sum_{k=1}^m X_{ik}\leqslant 1$ for $i=1,\dots,d$, so $X\in\mathcal{V}_{d\times m}$. As in the proof of Theorem 4.2 (and with the same notation), it follows that for $i\neq j$,

$$\gamma_{ij}=\sum_{l=1}^m\sum_{k=1}^m\mathbb{E}[X_{ik}]\mathbb{E}[X_{jl}]\lambda_{kl}=\sum_{l=1}^m\sum_{k=1}^m q_{ik}q_{jl}\lambda_{kl}.$$

This shows that $Q\Lambda Q^\top$ and Γ agree on off-diagonal entries, so $\mathcal{L}(Q\Lambda Q^\top)=\Gamma\in\mathcal{T}_d$. By taking $\Lambda=I_d$, we obtain $\mathcal{L}(QQ^\top)\in T_d$. By taking $m=1$, we obtain $\mathcal{L}(\boldsymbol{p}\boldsymbol{p}^\top)\in\mathcal{T}_d$. □

4.3 Corresponding copula models

In this section, we derive the copulas of (3.3) and (4.3) which are able to produce tail-dependence matrices $\mathbb{E}[\boldsymbol{X}\boldsymbol{X}^\top]/p$ and $\mathcal{L}(\mathbb{E}[X\Lambda X^\top])$ as stated in Theorems 3.3 and 4.2, respectively. We first address the former.

Proposition 4.5 (Copula of (3.3))**.** *Let $\boldsymbol{X}\in\mathcal{V}_d$, $\mathbb{E}[\boldsymbol{X}]=(p,\dots,p)\in(0,1]^d$. Furthermore, let $U,V\sim\mathrm{U}[0,1]$, $U,V,\boldsymbol{X}$ be independent and*

$$\boldsymbol{Y}=\boldsymbol{X}pU+(\boldsymbol{1}-\boldsymbol{X})(p+(1-p)V).$$

Then the copula C of $\boldsymbol{Y}$ at $\boldsymbol{u}=(u_1,\dots,u_d)$ is given by

$$C(\boldsymbol{u})=\sum_{\boldsymbol{i}\in\{0,1\}^d}\min\Big\{\frac{\min_{r:i_r=1}\{u_r\}}{p},1\Big\}\ \max\Big\{\frac{\min_{r:i_r=0}\{u_r\}-p}{1-p},0\Big\}\,\mathbb{P}(\boldsymbol{X}=\boldsymbol{i}),$$

with the convention $\min\emptyset=1$.

Proof. By the Law of Total Probability and our independence assumptions,

$$\begin{aligned}C(\boldsymbol{u})&=\sum_{\boldsymbol{i}\in\{0,1\}^d}\mathbb{P}(\boldsymbol{Y}\leqslant\boldsymbol{u},\boldsymbol{X}=\boldsymbol{i})\\&=\sum_{\boldsymbol{i}\in\{0,1\}^d}\mathbb{P}(pU\leqslant\min_{r:i_r=1}\{u_r\},p+(1-p)V\leqslant\min_{r:i_r=0}\{u_r\},\boldsymbol{X}=\boldsymbol{i})\\&=\sum_{\boldsymbol{i}\in\{0,1\}^d}\mathbb{P}\Big(U\leqslant\frac{\min_{r:i_r=1}\{u_r\}}{p}\Big)\,\mathbb{P}\Big(V\leqslant\frac{\min_{r:i_r=0}\{u_r\}-p}{1-p}\Big)\,\mathbb{P}(\boldsymbol{X}=\boldsymbol{i});\end{aligned}$$

the claim follows from the fact that $U,V\sim\mathrm{U}[0,1]$. □

For deriving the copula of (4.3), we need to introduce some notation; see also Example 4.1 below. In the following theorem, let $\operatorname{supp}(X)$ denote the support of X. For a vector $\boldsymbol{u} = (u_1, \dots, u_d) \in [0,1]^d$ and a matrix $A = (A_{ij})_{d\times m} \in \operatorname{supp}(X)$, denote by A_i the sum of the i-th row of A, $i = 1, \dots, d$, and let $\boldsymbol{u}_A = (u_1 \mathrm{I}_{\{A_1=0\}} + \mathrm{I}_{\{A_1=1\}}, \dots, u_d \mathrm{I}_{\{A_d=0\}} + \mathrm{I}_{\{A_d=1\}})$, and $\boldsymbol{u}_A^* = (\min_{r:A_{r1}=1}\{u_r\}, \dots, \min_{r:A_{rm}=1}\{u_r\})$, where $\min\emptyset = 1$.

Proposition 4.6 (Copula of (4.3))**.** *Suppose that the setup of Theorem 4.2 holds. Then the copula C of $\boldsymbol{Y}$ in (4.3) is given by*

$$C(\boldsymbol{u}) = \sum_{A\in\operatorname{supp}(X)} C^V(\boldsymbol{u}_A)\, C^U(\boldsymbol{u}_A^*)\, \mathbb{P}(X = A). \tag{4.4}$$

Proof. By the Law of Total Probability, it suffices to verify that $\mathbb{P}(\boldsymbol{Y} \leqslant \boldsymbol{u} \,|\, X = A) = C^{\boldsymbol{V}}(\boldsymbol{u}_A)C^{\boldsymbol{U}}(\boldsymbol{u}_A^*)$. This can be seen from

$$\begin{aligned}
&\mathbb{P}(\boldsymbol{Y} \leqslant \boldsymbol{u} \,|\, X = A)\\
&= \mathbb{P}\Big(\sum_{k=1}^{m} A_{jk}U_k + (1 - A_j)V_j \leqslant u_j,\ j = 1, \dots, d\Big)\\
&= \mathbb{P}(U_k \mathrm{I}_{\{A_{jk}=1\}} \leqslant u_j, V_j \mathrm{I}_{\{A_j=0\}} \leqslant u_j,\ j = 1, \dots, d,\ k = 1, \dots, m)\\
&= \mathbb{P}\big(U_k \leqslant \min_{r:A_{rk}=1}\{u_r\}, V_j \leqslant u_j \mathrm{I}_{\{A_j=0\}} + \mathrm{I}_{\{A_j=1\}},\ j = 1, \dots, d,\ k = 1, \dots, m\big)\\
&= \mathbb{P}\big(U_k \leqslant \min_{r:A_{rk}=1}\{u_r\},\ k = 1, \dots, m\big)\mathbb{P}\big(V_j \leqslant u_j \mathrm{I}_{\{A_j=0\}} + \mathrm{I}_{\{A_j=1\}},\ j = 1, \dots, d\big)\\
&= C^{\boldsymbol{U}}(\boldsymbol{u}_A^*)C^{\boldsymbol{V}}(\boldsymbol{u}_A).
\end{aligned}$$

□

As long as $C^{\boldsymbol{V}}$ has tail-dependence matrix I_d, the tail-dependence matrix of $\boldsymbol{Y}$ is not affected by the choice of $C^{\boldsymbol{V}}$. This theoretically provides more flexibility in choosing the body of the distribution of $\boldsymbol{Y}$ while attaining a specific tail-dependence matrix. Note, however, that this also depends on the choice of X; see the following example where we address special cases which allow for more insight into the rather abstract construction (4.4).

Example 4.1. 1. For $m = 1$, the copula C in (4.4) is given by

$$C(\boldsymbol{u}) = \sum_{\boldsymbol{A}\in\{0,1\}^d} C^{\boldsymbol{V}}(\boldsymbol{u}_{\boldsymbol{A}})C^{\boldsymbol{U}}(\boldsymbol{u}_{\boldsymbol{A}}^*)\mathbb{P}(\boldsymbol{X} = \boldsymbol{A}); \tag{4.5}$$

note that X, A in Equation 4.4 are indeed vectors in this case. For $d = 2$, we obtain

$$\begin{aligned}
C(u_1, u_2) &= M(u_1, u_2)\mathbb{P}(\boldsymbol{X} = (\begin{smallmatrix}1\\1\end{smallmatrix})) + C^{\boldsymbol{V}}(u_1, u_2)\mathbb{P}(\boldsymbol{X} = (\begin{smallmatrix}0\\0\end{smallmatrix}))\\
&\quad + \Pi(u_1, u_2)\mathbb{P}(\boldsymbol{X} = (\begin{smallmatrix}1\\0\end{smallmatrix}) \text{ or } \boldsymbol{X} = (\begin{smallmatrix}0\\1\end{smallmatrix})),
\end{aligned}$$

and therefore a mixture of the Fréchet–Hoeffding upper bound $M(u_1, u_2) = \min\{u_1, u_2\}$, the copula $C^{\boldsymbol{V}}$ and the independence copula $\Pi(u_1, u_2) = u_1 u_2$. If $\mathbb{P}(\boldsymbol{X} = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)) = 0$ then C is simply a mixture of M and Π and does not depend on $\boldsymbol{V}$ anymore.

Now consider the special case of (4.5) where $\boldsymbol{V}$ follows the d-dimensional independence copula $\Pi(\boldsymbol{u}) = \prod_{i=1}^{d} u_i$ and $\boldsymbol{X} = (X_1, \dots, X_{d-1}, 1)$ is such that at most one of $X_1, \dots, X_{d-1}$ is 1 (each randomly with probability $0 \leqslant \alpha \leqslant 1/(d-1)$ and all are simultaneously 0 with probability $1-(d-1)\alpha$). Then, for all $\boldsymbol{u} \in [0,1]^d$, C is given by

$$C(\boldsymbol{u}) = \alpha \sum_{i=1}^{d-1} \Bigg(\min\{u_i, u_d\} \prod_{j=1, j\neq i}^{d-1} u_j \Bigg) + (1-(d-1)\alpha) \prod_{j=1}^{d} u_j. \tag{4.6}$$

This copula is a conditionally independent multivariate Fréchet copula studied in Yang et al. (2009). This example will be revisited in Section 4.4; see also the left-hand side of Figure 3 below.

2. For $m = 2$, $d = 2$, we obtain

$$\begin{aligned} C(u_1, u_2) &= M(u_1, u_2)\mathbb{P}(X = \left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right) \text{ or } X = \left(\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right)) \\ &\quad + C^{\boldsymbol{U}}(u_1, u_2)\mathbb{P}(X = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)) + C^{\boldsymbol{U}}(u_2, u_1)\mathbb{P}(X = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)) \\ &\quad + C^{\boldsymbol{V}}(u_1, u_2)\mathbb{P}(X = \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)) \\ &\quad + \Pi(u_1, u_2)\mathbb{P}(X = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right) \text{ or } \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right) \text{ or } \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \text{ or } \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)). \end{aligned} \tag{4.7}$$

Figure 2 shows samples of size 2000 from (4.7) for $\boldsymbol{V} \sim \Pi$ and two different choices of $\boldsymbol{U}$ (in different rows) and X (in different columns). From Theorem 4.2, we obtain that the off-diagonal entry γ_{12} of the tail-dependence matrix Γ of $\boldsymbol{Y}$ is given by

$$\gamma_{12} = p_{(1,2)(1,1)} + p_{(1,2)(2,2)} + \lambda_{12}(p_{(1,2)(2,1)} + p_{(1,2)(1,2)}),$$

where λ_{12} is the off-diagonal entry of the tail-dependence matrix Λ of $\boldsymbol{U}$.

4.4 An example from risk management practice

Let us now come back to Problem (1.1) which motivated our research on tail-dependence matrices. From a practical point of view, the question is whether it is possible to find one financial position, which has tail-dependence coefficient α with each of $d-1$ tail-independent financial risks (assets). Such a construction can be interesting for risk management purposes, e.g., in the context of hedging.

Recall Problem (1.1):

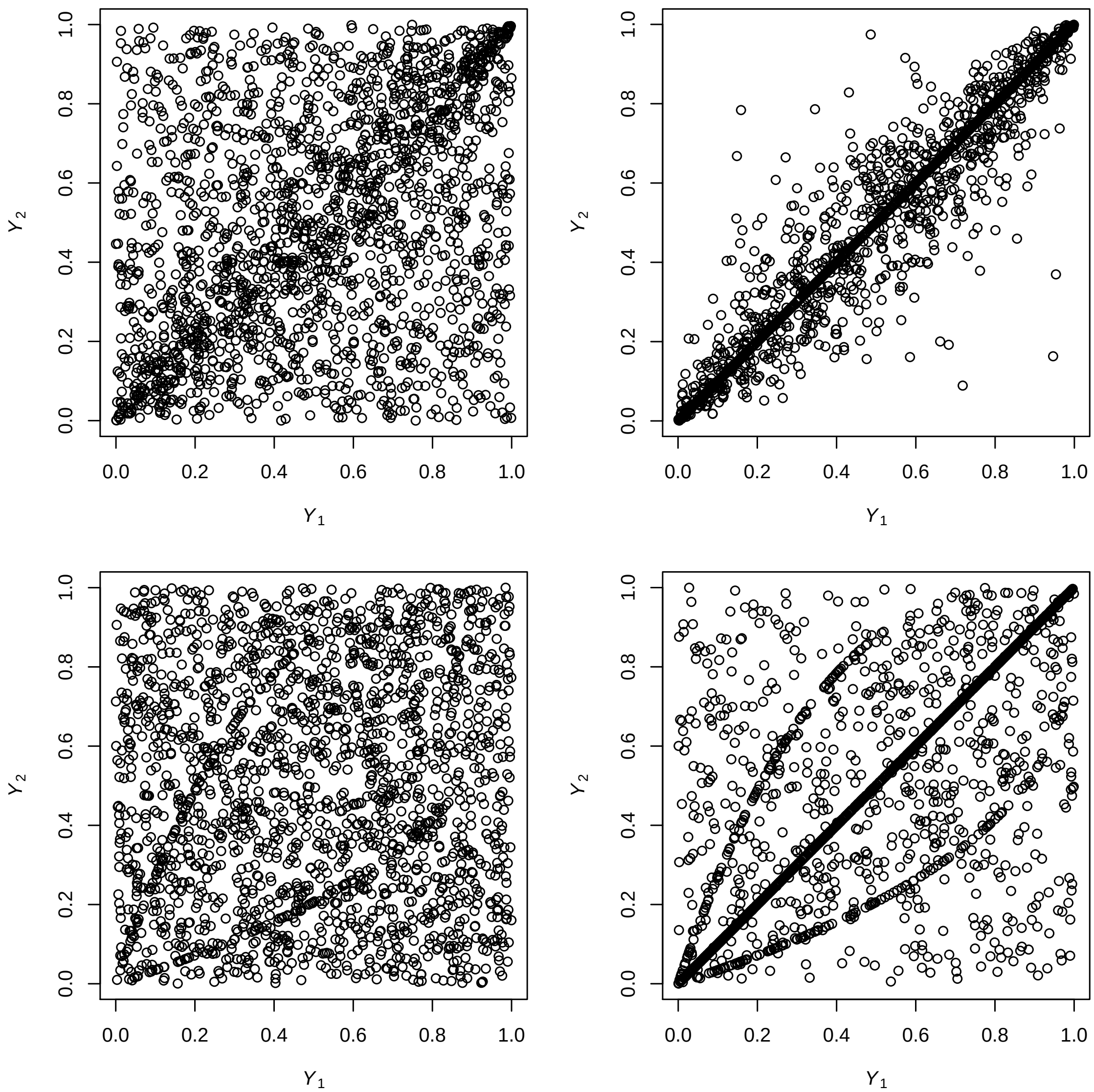

Figure 2: Scatter plots of 2000 samples from $\boldsymbol{Y}$ for $\boldsymbol{V} \sim \Pi$ and $\boldsymbol{U}$ following a bivariate ($m = 2$) t_3 copula with Kendall's tau equal to 0.75 (top row) or a survival Marshall–Olkin copula with parameters $\alpha_1 = 0.25, \alpha_2 = 0.75$ (bottom row). For the plots on the left-hand side, the number of rows of X with one 1 are randomly chosen among $\{0, 1, 2(= d)\}$, the corresponding rows and columns are then randomly selected among $\{1, 2(= d)\}$ and $\{1, 2(= m)\}$, respectively. For the plots on the right-hand side, X is drawn from a multinomial distribution with probabilities 0.5 and 0.5 such that each row contains precisely one 1.

For which $\alpha \in [0,1]$ is the matrix

$$\Gamma_d(\alpha) = \begin{pmatrix} 1 & 0 & \cdots & 0 & \alpha \\ 0 & 1 & \cdots & 0 & \alpha \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha \\ \alpha & \alpha & \cdots & \alpha & 1 \end{pmatrix} \tag{4.8}$$

a matrix of pairwise (either lower or upper) tail-dependence coefficients?

Based on the Fréchet–Hoeffding bounds, it follows from Joe (1997, Theorem 3.14) that for $d = 3$ (and thus also $d > 3$), α has to be in $[0, 1/2]$; however this is not a sufficient condition for $\Gamma_d(\alpha)$ to be a tail-dependence matrix. The following proposition not only gives an answer to (4.8) by providing necessary and sufficient such conditions, but also provides, by its proof, a compatible model for $\Gamma_d(\alpha)$.

Proposition 4.7. $\Gamma_d(\alpha) \in \mathcal{T}_d$ *if and only if* $0 \leqslant \alpha \leqslant 1/(d-1)$.

Proof. The if-part directly follows from Corollary 3.5. We provide a constructive proof based on Theorem 4.2. Suppose that $0 \leqslant \alpha \leqslant 1/(d-1)$. Take a partition $\{\Omega_1, \ldots, \Omega_d\}$ of the sample space Ω with $\mathbb{P}(\Omega_i) = \alpha$, $i = 1, \ldots, d-1$, and let $\boldsymbol{X} = (\mathrm{I}_{\Omega_1}, \ldots, \mathrm{I}_{\Omega_{d-1}}, 1) \in \mathcal{V}_d$. It is straightforward to see that

$$\mathbb{E}[\boldsymbol{X}\boldsymbol{X}^\top] = \begin{pmatrix} \alpha & 0 & \cdots & 0 & \alpha \\ 0 & \alpha & \cdots & 0 & \alpha \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha & \alpha \\ \alpha & \alpha & \cdots & \alpha & 1 \end{pmatrix}.$$

By Proposition 4.3, $\Gamma_d(\alpha) = \mathcal{L}(\mathbb{E}[\boldsymbol{X}\boldsymbol{X}^\top]) \in \mathcal{T}_d$.

For the only if part, suppose that $\Gamma_d(\alpha) \in \mathcal{T}_d$; thus $\alpha \geqslant 0$. By Theorem 3.3, $\Gamma_d(\alpha) \in \mathcal{B}_d^I$. By the definition of $\mathcal{B}_d^I$, $\Gamma_d(\alpha) = B_d/p$ for some $p \in (0,1]$ and a Bernoulli-compatible matrix B_d. Therefore,

$$p\Gamma_d(\alpha) = \begin{pmatrix} p & 0 & \cdots & 0 & p\alpha \\ 0 & p & \cdots & 0 & p\alpha \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & p & p\alpha \\ p\alpha & p\alpha & \cdots & p\alpha & p \end{pmatrix}$$

is a compatible Bernoulli matrix, so $p\Gamma_d(\alpha) \in \mathcal{B}_d$. Write $p\Gamma_d(\alpha) = \mathbb{E}[\boldsymbol{X}\boldsymbol{X}^\top]$ for some $\boldsymbol{X} = (X_1, \dots, X_d) \in \mathcal{V}_d$. It follows that $\mathbb{P}(X_i = 1) = p$ for $i = 1, \dots, d$, $\mathbb{P}(X_i X_j = 1) = 0$ for $i \neq j$, $i, j = 1, \dots, d-1$ and $\mathbb{P}(X_i X_d = 1) = p\alpha$ for $i = 1, \dots, d-1$. Note that $\{X_i X_d = 1\}$, $i = 1, \dots, d-1$, are almost surely disjoint since $\mathbb{P}(X_i X_j = 1) = 0$ for $i \neq j$, $i, j = 1, \dots, d-1$. As a consequence,

$$p = \mathbb{P}(X_d = 1) \geqslant \mathbb{P}\left(\bigcup_{i=1}^{d-1} \{X_i X_d = 1\}\right) = \sum_{i=1}^{d-1} \mathbb{P}(X_i X_d = 1) = (d-1)p\alpha,$$

and thus $(d-1)\alpha \leqslant 1$. □

It follows from the proof of Theorem 4.2 that for $\alpha \in [0, 1/(d-1)]$, a compatible copula model with tail-dependence matrix $\Gamma_d(\alpha)$ can be constructed as follows. Consider a partition $\{\Omega_1, \dots, \Omega_d\}$ of the sample space Ω with $\mathbb{P}(\Omega_i) = \alpha$, $i = 1, \dots, d-1$, and let $\boldsymbol{X} = (X_1, \dots, X_d) = (\mathrm{I}_{\Omega_1}, \dots, \mathrm{I}_{\Omega_{d-1}}, 1) \in \mathcal{V}_d$; note that $m = 1$ here. Furthermore, let $\boldsymbol{V}$ be as in Theorem 4.2, $U \sim \mathrm{U}[0, 1]$ and $U, \boldsymbol{V}, \boldsymbol{X}$ be independent. Then,

$$\boldsymbol{Y} = (UX_1 + (1 - X_1)V_1, \dots, UX_{d-1} + (1 - X_{d-1})V_{d-1}, U)$$

has tail-dependence matrix $\Gamma_d(\alpha)$. Example 4.1, Part 1 provides the copula C of $\boldsymbol{Y}$ in this case. It is also straightforward to verify from this copula that $\boldsymbol{Y}$ has tail-dependence matrix $\Gamma_d(\alpha)$. Figure 3 displays pairs plots of 2000 realizations of $\boldsymbol{Y}$ for $\alpha = 1/3$ and two different copulas for $\boldsymbol{V}$.

Remark 4.2. Note that $\Gamma_d(\alpha)$ is not positive semidefinite if and only if $\alpha > 1/\sqrt{d-1}$. For $d < 5$, element-wise non-negative and positive semidefinite matrices are completely positive; see Berman and Shaked-Monderer (2003, Theorem 2.4). Therefore $\Gamma_3(2/3)$ is completely positive. However, it is not in $\mathcal{T}_3$. It indeed shows that the class of completely positive matrices with diagonal entries being 1 is strictly larger than $\mathcal{T}_d$.

5 Conclusion and discussion

Inspired by the question whether a given matrix in $[0, 1]^{d \times d}$ is the matrix of pairwise tail-dependence coefficients of a d-dimensional random vector, we introduced the tail-dependence compatibility problem. It turns out that this problem is closely related to the Bernoulli-compatibility problem which we also addressed in this paper and which asks when a given matrix in $[0, 1]^{d \times d}$ is a Bernoulli-compatible matrix (see Question 1 and Theorem 2.2). As a main finding, we characterized tail-dependence matrices as precisely those square matrices with diagonal entries being 1 which are Bernoulli-compatible matrices multiplied by a constant (see Question 2

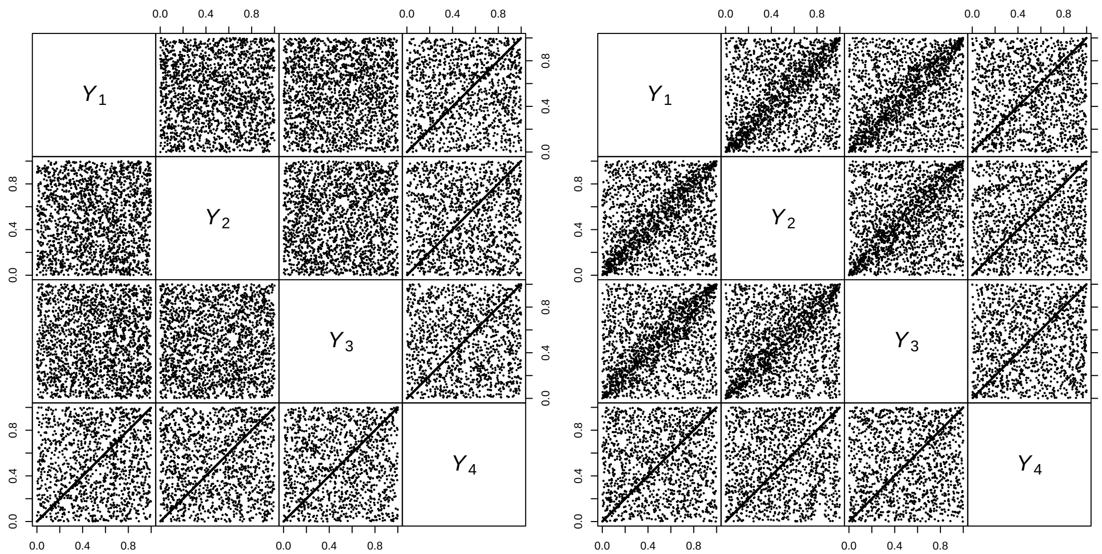

Figure 3: Pairs plot of 2000 samples from $\boldsymbol{Y} \sim C$ which produces the tail dependence matrix $\Gamma_4(1/3)$ as given by (1.1). On the left-hand side, $\boldsymbol{V} \sim \Pi$ (α determines how much weight is on the diagonal for pairs with one component being Y_4; see (4.6)) and on the right-hand-side, $\boldsymbol{V}$ follows a Gauss copula with parameter chosen such that Kendall's tau equals 0.8.

and Theorem 3.3). Furthermore, we presented and studied new models (see, e.g., Question 3 and Theorem 4.2) which provide answers to several questions related to the tail-dependence compatibility problem.

The study of compatibility of tail-dependence matrices is mathematically different from that of covariances matrices. Through many technical arguments in this paper, the reader may have already realized that the tail-dependence matrix lacks a linear structure which is essential to covariance matrices based on tools from Linear Algebra. For instance, let $\boldsymbol{X}$ be a d-random vector with covariance matrix Σ and tail-dependence matrix Λ, and A be an $m \times d$ matrix. The covariance matrix of $A\boldsymbol{X}$ is simply given by $A\Sigma A^\top$, however, the tail-dependence matrix of $A\boldsymbol{X}$ is generally not explicit (see Remark 4.1 for special cases). This lack of linearity can also help to understand why tail-dependence matrices are realized by models based on Bernoulli vectors as we have seen in this paper, in contrast to covariance matrices which are naturally realized by Gaussian (or generally, elliptical) random vectors. The latter have a linear structure, whereas Bernoulli vectors do not. It is not surprising that most classical techniques in Linear Algebra such as matrix decomposition, diagonalization, ranks, inverses and determinants are not very helpful for studying the compatibility problems we address in this paper.

Concerning future research, an interesting open question is how one can (theoretically or numerically) determine whether a given arbitrary non-negative, square matrix is a tail-dependence or Bernoulli-compatible matrix. To the best of our knowledge there are no corresponding algorithms available. Another open question concerns the compatibility of other matrices of pairwise measures of association such as rank-correlation measures (e.g., Spearman's rho or Kendall's tau); see (Embrechts et al., 2002, Section 6.2). Recently, Fiebig et al. (2014) and Strokorb et al. (2015) studied the concept of tail-dependence functions of stochastic processes. Similar results to some of our findings were found in the context of max-stable processes.

From a practitioner's point-of-view, it is important to point out limitations of using tail-dependence matrices in quantitative risk management and other applications. One possible such limitation is the statistical estimation of tail-dependence matrices since, as limits, estimating tail dependence coefficients from data is non-trivial (and typically more complicated than estimation in the body of a bivariate distribution).

After presenting the results of our paper at the conferences "Recent Developments in Dependence Modelling with Applications in Finance and Insurance – 2nd Edition, Brussels, May 29, 2015" and "The 9th International Conference on Extreme Value Analysis, Ann Arbor, June 15–19, 2015", the references Fiebig et al. (2014) and Strokorb et al. (2015) were brought to our attention (see also Acknowledgments below). In these papers, a very related problem is treated, be it from a different, more theoretical angle, mainly based on the theory of max-stable and Tawn-Molchanov processes as well as results for convex-polytopes. For instance, our Theorem 3.3 is similar to Theorem 6 c) in Fiebig et al. (2014).

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