CreditRisk$^+$ Model with Dependent Risk Factors

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Abstract

The CreditRisk$^+$ model is widely used in industry for computing the loss of a credit portfolio. The standard CreditRisk$^+$ model assumes independence among a set of common risk factors, a simplified assumption which leads to computational ease. In this paper, we propose to model the common risk factors by a class of multivariate extreme copulas as a generalization of bivariate Fréchet copulas. Further we present a conditional Compound Poisson model to approximate the credit portfolio, and provide a cost-efficient recursive algorithm to calculate the loss distribution. The new model is more flexible than the standard model, with computational advantages compared to other dependence models of risk factors.

Key-words: CreditRisk$^+$ model; conditional independence; dependent risk factors; Panjer's recursion; multivariate copulas.

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1 Introduction

A key step in modeling credit risks is the interdependency among obligors' default events. As a widely employed model in industry, CreditRisk $^+$ (CR^+) model (Credit Suisse First Boston (1997)) assumes that there exist some systematic risk factors to influence all obligors' default, where the risk factors can be industrial fields, geographical countries, etc; for details we refer to Gupton, Finger and Bhatia (1997). In the standard CR^+ model, by assuming independence among the common risk factors, one is able to compute the loss distribution of the credit portfolio recursively without using an extensive Monte Carlo simulation. For example, when the risk factors are assumed to have gamma distributions, Panjer's recursion (Panjer, 1981) can be employed for the calculation. Gordy (2002) proposed to use the saddlepoint approximation in CR^+ model and Vandendorpea et al. (2008) discussed the issue of parameterization. We refer to Gundlach and Lehrbass (2003) for an overview of the importance, applications and research on the CR^+ model.

In practice, it is arguably questionable to assume independence among risk factors. Extending CR^+ model to cover dependent risk factors has been a practically important topic for research in quantitative methods for credit risk. Bürgisser et al. (1999) introduced the correlation among the risk factors and derived some moments of the credit portfolio. Giese (2004) considered the link between the CR^+ loss distribution and the moment-generating function of the risk factors by incorporating sector correlations. Reiβ (2004) modeled the risk factors by incorporating market risk through geometric Brownian motions. Kostadinov (2006) used elliptical copulas to model the dependence among defaults. Deshpande and Iyer (2009) improved CR^+ model by modeling the sector default rates as linear combinations of a common set of independent variables representing risk factors. When the risk factors are modeled by a dependence model, calculating the loss distribution becomes complicated in general, and Monte Carlo methods are often implemented. In this paper, we propose to model the dependence among the risk factors by a class of extreme copulas introduced in Yang, Qi and Wang (2009), and provide a simple algorithm to calculate the distribution of the total loss. The algorithm involves a numerical integration of at most one dimension and so is almost as simple as the one in the standard CR^+ model.

A copula is a multivariate distribution function with uniform marginal distributions. Sklar's Theorem states that for an n-dimensional distribution function H with marginal distributions $F_1, \ldots, F_n$, there exists a n-copula C such that

$$H(x_1, x_2, \ldots, x_n) = C\Big(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)\Big).$$

And C is unique when the marginal distributions are continuous. In the two-dimensional case, three special copulas, the Fréchet upper copula $M(u,v) = \min\{u,v\}$, the Fréchet lower copula

$W(u,v) = \max\{u+v-1, 0\}$, and the product copula $\Pi(u,v) = uv$, correspond to the three special dependence structures: comonotonicity, countermonotonicity, and independence, respectively (see for example, Dhaene et al. (2002a, 2002b)). The *bivariate Fréchet (BF) copula* is defined as a convex combination of the above three copulas; that is,

$$C(u,v) = \alpha M(u,v) + \beta \Pi(u,v) + \gamma W(u,v) \tag{1.1}$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma = 1$. BF copulas can be used to model the dependence of two risks with focus on comonotonic, countermonotonic and independent parts respectively; see Yang, Cheng and Zhang (2006). For high-dimensional settings, Yang, Qi and Wang (2009) presented a class of multivariate copulas with bivariate Fréchet marginal copulas, called the $C^{\mathcal{A},\mathcal{B}}$ copulas, which are determined uniquely by all their bivariate marginal copulas. Since the copula family of $C^{\mathcal{A},\mathcal{B}}$ has a clear financial background and is easy to compute, we propose to model the common risk factors in the CR^+ model by this class of copulas. For the general applications of copula models in finance, we refer to Cherubini, Luciano and Vecchiato (2004).

In this paper, we first present the credit portfolio model with the common risk factors modeled by the $C^{\mathcal{A},\mathcal{B}}$ copulas. In order to compute the loss distribution, we propose a conditional compound Poisson model, where conditional on the risk factors the total loss of the credit portfolio is a compound Poisson distribution. We show that the conditional compound Poisson model is a variable approximation to the original Credit portfolio, similar to the classic compound Poisson approximations for aggregate risk models. Moreover, we provide a cost-efficient algorithm for calculating the loss distribution of the credit portfolio based on the conditional compound Poisson model.

We organize the rest of this paper as follows. Section 2 presents our credit portfolio model with the correlated risk factors modeled by the proposed copulas. Section 3 provides a conditional compound Poisson model as a variable approximation to the basic model. Some algorithms are given for calculating the distribution of the conditional compound Poisson model. Numerical examples are given in Section 4. Conclusions are summarized in Section 5. Some proofs are put in the Appendix.

2 Credit Portfolio with correlated risk factors

2.1 Credit model

For a credit portfolio of N obligors, its total loss due to credit risk is modeled as follows. Denote the obligors in the credit portfolio by $\mathcal{F} = \{A_1, A_2, \ldots, A_N\}$. For each obligor $A \in \mathcal{F}$, put $I_A = 1$ if the obligor A defaults and $I_A = 0$ if no default occurs for the obligor. The face amount of the obligor A is denoted by L_A. For the simplicity of discussion, we assume that the recovery rates are all equal to zero, and this assumption can be removed without much difficulty. The total loss amount of the credit portfolio can be expressed as

$$L = \sum_{A \in \mathcal{F}} L_A I_A. \tag{2.1}$$

Throughout we assume that $L_A, A \in \mathcal{F}$ are integer-valued for computational considerations. Note that the above setting is exactly the same as the one in CR$^+$. The default probability of the obligor A is denoted by p_A, that is, $p_A = \mathrm{E}(I_A)$. Same as in CR$^+$, we assume that there exist $n+1$ risk factors $X_k, k = 0, 1, \ldots, n$ to influence the obligors' default. Here $X_k, k = 0, 1, \ldots, n$ may be economic factors, such as industry indices or economy states in certain geographical regions, hence they are typically observable. We refer to Credit Suisse First Boston (1997) and Wilson (1998) for interpretations of the risk factors. Further we assume that the default probabilities satisfy the following conditions:

(A1) Given the risk factors $X_k, k = 0, 1, \ldots, n$, the default indicators $I_A, A \in \mathcal{F}$ are conditionally independent. Denote the expectation and standard deviation of X_k by μ_k and σ_k respectively, and let $X_0 = \mu_0$ be a constant.

(A2) For each $A \in \mathcal{F}$, there exist non-negative weights $\theta_{A,k},\ k = 0, 1, \ldots, n$, such that $\sum_{k=0}^{n} \theta_{A,k} = 1$ and

$$x_A =: \mathrm{E}[I_A | X_1, X_2, \ldots, X_n] = p_A(\sum_{k=0}^{n} \theta_{A,k} \frac{X_k}{\mu_k}); \tag{2.2}$$

(A3) For $k \geq 0$, the expectation of the risk factor X_k is chosen as

$$\mu_k = \sum_{A \in \mathcal{F}} \theta_{A,k} p_A. \tag{2.3}$$

The above standard settings (A1),(A2) and (A3) are used in the CR$^+$ model. The risk factor X_0 is a constant which shows each obligor's individual contribution on its default probability. For the other risk factors $X_k, 1 \leq k \leq n$, we denote the distribution of X_k by F_k and assume that it is continuous. The inverse function of F_k is denoted by F_k^{-1}.

Remark 2.1. Assumption (A2) clearly indicates that there exist $n+1$ factors to influence the default probability of the portfolio. For obligor A, $\theta_{A,k}$ represents the weight of the influence of the risk factor X_k on its default, and the coefficient $\theta_{A,0}$ is the individual contribution of the obligor A on its default probability. The sum of the total weights equals to one.

Instead of assuming independent risk factors, we propose to use copula methodology to model the correlation among the risk factors $X_k, 1 \leq k \leq n$. More specifically, assume that $U_k = F_k(X_k), 1 \leq k \leq n$ satisfy the following conditional independence framework introduced in Yang, Qi and Wang (2009):

(X1) There exists a uniformly distributed random factor U on $[0,1]$ such that $U_1, \ldots, U_n$ are conditionally independent given U;

(X2) For each $1 \leq i \leq n$, $C_i(u,v) = \mathrm{P}(U_i \leq u, U \leq v)$ satisfies that

$$C_i(u,v) = a_{i,1}M(u,v) + a_{i,2}\Pi(u,v) + a_{i,3}W(u,v), \tag{2.4}$$

where $a_{i,j} \geq 0, j = 1,2,3$ such that $a_{i,1} + a_{i,2} + a_{i,3} = 1$.

The copula of the above $(U_1, U_2, \ldots, U_n)$ is denoted as $C^{\mathcal{A},\mathcal{B}}$ in Yang, Qi and Wang (2009). In this paper we will call a copula satisfying (X1) and (X2) a $C^{\mathcal{A},\mathcal{B}}$ copula.

Assumptions (X1) and (X2) imply that the risk factors $X_k, 1 \leq k \leq n$ are correlated through the common latent variable U. Assumption (X2) shows the influence of the common factor U on U_i, where $a_{i,1}$ is the weight of the positive influence, $a_{i,3}$ is the weight of the negative influence, and the obligor U_i is independent of the common factor U with portion $a_{i,2}$. For more discussion on the assumptions (X1), (X2) and their practical applications, see Yang, Qi and Wang (2009). Note that CR^+ assumes that the risk factors $X_k, 1 \leq k \leq n$ are independent, i.e., $a_{i,2} = 1, i \leq n$. Hence, the above assumptions are a generalization of the CR^+ model.

The following is a probabilistic explanation of assumptions (X1) and (X2). Let $A_i^+, A_i^\perp, A_i^-, 1 \leq i \leq n$ be random events and $Y_i,\ 1 \leq i \leq n$ be i.i.d. U[0,1] random variables with the following assumptions:

1. For each $1 \leq i \leq n$, $\{A_i^+, A_i^\perp, A_i^-\}$ is a partition of the probability space and

$$P(A_i^+) = a_{i,1}, P(A_i^\perp) = a_{i,2}, P(A_i^-) = a_{i,3}.$$

   The random event-vectors $(A_i^+, A_i^\perp, A_i^-), 1 \leq i \leq n$ are independent, and independent of the common latent variable U in (X1) and (X2) and the random variables $Y_i,\ 1 \leq i \leq n$.

2. The common latent variable U and $Y_i,\ 1 \leq i \leq n$ are independent.

Define

$$U_i = I_{A_i^+}U + I_{A_i^\perp}Y_i + I_{A_i^-}(1-U),\ 1 \leq i \leq n.$$

Then it is easy to verify that $U_i,\ 1 \le i \le n$ satisfy the assumptions (X1) and (X2). Hence the common risk factors can be expressed as $X_k = F_k^{-1}(U_k)$. The above probabilistic expressions give the relationship among the risk factors and the common factor U. Conditional on the event A_i^+ ($A_i^\perp$, or A_i^-), U_i and U are comonotonic (independent, or countermonotonic). If the risk factors are independent, $a_{i,1} = a_{i,3} = 0, a_{i,2} = 1, i \le n$ and $U_i = Y_i, i \le n$. If the risk factors are comonotonic, $a_{i,2} = a_{i,3} = 0, a_{i,1} = 1, i \le n$ and $U_i = U, i \le n$. The advantage of the proposed model is that the coefficients of $a_{i,1}, a_{i,2}$ and $a_{i,3}$ can be adjusted to reflect the influence of the common factor U on default probabilities.

Define

$$X_i^+ = F_i^{-1}(U), X_i^- = F_i^{-1}(1-U), X_i^\perp = F_i^{-1}(Y_i).$$

Then X_i^+, X_i^- and $X_i^\perp$ have the common distribution F_i. Note that X_i^+ and X_i^- are countmonotonic, and they are independent of $X_i^\perp$. The risk factor $X_i = F_i^{-1}(U_i)$ can then be expressed as

$$X_i = I_{A_i^+} X_i^+ + I_{A_i^\perp} X_i^\perp + I_{A_i^-} X_i^-.$$

The above equation decomposes the probabilistic space into three subspaces to show the dependence structure of X_i and the common latent variable U.

2.2 Notation

In this section we introduce some notations for future use. For the indices $(j_1, j_2, \ldots, j_n)$, where $j_i \in \{1, 2, 3\}$, write

$$C^{(j_1, j_2, \ldots, j_n)}(u_1, u_2, \ldots, u_n) = W\big(\min_{i \le n, j_i = 1}\{u_i\}, \min_{i \le n, j_i = 3}\{u_i\}\big) \prod_{i \le n, j_i = 2} u_i, \quad u_i \in [0,1], i \le n,$$

with the convention that for the empty set the corresponding minimum and product are defined to be 1. As shown in Yang, Qi and Wang (2009), the distribution of $(U_1, \ldots, U_n)$ can be expressed as

$$C^{\mathcal{A},\mathcal{B}}(u_1, u_2, \ldots, u_n) = \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} \left(\prod_{i=1}^{n} a_{i,j_i} \right) C^{(j_1, j_2, \ldots, j_n)}(u_1, u_2, \ldots, u_n). \tag{2.5}$$

For $i \ne j$ and $u_i, u_j \in [0,1]$, the bivariate marginal copula of (U_i, U_j) can be expressed as

$$\mathrm{P}(U_i \le u_i, U_j \le u_j) = \alpha_{i,j} M(u_i, u_j) + \beta_{i,j} \Pi(u_i, u_j) + \gamma_{i,j} W(u_i, u_j) \tag{2.6}$$

with

$$\alpha_{i,j} = a_{i,1}a_{j,1} + a_{i,3}a_{j,3},\ \ \gamma_{i,j} = a_{i,1}a_{j,3} + a_{i,3}a_{j,1},\ \ \beta_{i,j} = 1 - \alpha_{i,j} - \gamma_{i,j}. \tag{2.7}$$

That is, the two-dimensional marginal copulas of $C^{\mathcal{A},\mathcal{B}}$ belong to the family of BF copulas. Moreover, $C^{\mathcal{A},\mathcal{B}}$ is uniquely determined by all two-dimensional marginal copulas. Under the above copula structure, the distribution of $(X_1, X_2, \ldots, X_n)$ can be expressed as

$$\mathrm{P}(X_1 \le x_1, \ldots, X_n \le x_n) = C^{\mathcal{A},\mathcal{B}}(F_1(x_1), \ldots, F_n(x_n)).$$

For the given weights $\theta_{A,i}, A \in \mathcal{F},\ 0 \le i \le n$ in (A2), we define

$$D_{k,i} = \frac{1}{\mu_i} \sum_{A:L_A=k} \theta_{A,i} p_A,\ 0 \le i \le n \quad \text{for each} \quad k \ge 1. \tag{2.8}$$

Therefore

$$\sum_{k=1}^{\infty} D_{k,i} = \frac{1}{\mu_i} \sum_{A \in \mathcal{F}} \theta_{A,i} p_A = 1 \quad \text{for each fixed} \quad 0 \le i \le n, \tag{2.9}$$

which implies that $D_{k,i}$ for each fixed i is a probability mass function on $k \in \mathbb{N}$. Thus we can define its probability generating function (pgf) as

$$P_i(z) = \sum_{k=1}^{\infty} D_{k,i} z^k,\ z \in [0, 1]. \tag{2.10}$$

Note that for fixed $0 \le i \le n$, the probability function $D_{k,i}, k = 1, \ldots$ is generated by the risk factor X_i, and $D_{k,i}$ can be regarded as the contribution to the risk factors X_i for those obligors with face amount k. The above notations will be used later.

Remark 2.2. For estimating the parameters of the copula $C^{\mathcal{A},\mathcal{B}}$ satisfying (X1) and (X2), the standard pseudo maximum likelihood estimation procedure is not applicable due to nonexistence of density. Note that Spearman's rho ($\rho^S_{i,j}$) and Kendall's tau ($\tau_{i,j}$) for a bivariate Fréchet copula are linear and quadratic functions of its parameters (e.g. Nelsen, 2006), i.e., for $i, j = 1, \ldots, n,\ i \ne j$:

$$\rho^S_{i,j} = \alpha_{i,j} - \gamma_{i,j},$$

$$\tau_{i,j} = \frac{1}{3}(\alpha_{i,j} - \gamma_{i,j})(\alpha_{i,j} + \beta_{i,j} + 2) = \frac{1}{3}(\alpha_{i,j} - \gamma_{i,j})(3 - \gamma_{i,j}),$$

where $\alpha_{i,j}$ and $\gamma_{i,j}$ are given in (2.6). Hence, one can first estimate Spearman's rho and Kendall's tau, and then esimate $\alpha_{i,j}$ and $\gamma_{i,j}$ via the above equations, say $\hat{\alpha}_{i,j}$ and $\hat{\gamma}_{i,j}$. Since the multivariate Fréchet copula $C^{\mathcal{A},\mathcal{B}}$ is determined by its bivariate marginal copulas (see Yang, Qi and Wang (2009)), estimators for the parameters the parameters $(a_{i,1},\ a_{i,3},\ i = 1, \ldots, n)$ in $C^{\mathcal{A},\mathcal{B}}$ can be obtained by using (2.7), $\hat{\alpha}_{i,j}$ and $\hat{\gamma}_{i,j}$.

2.3 Properties of the proposed model

Put

$$\mathbf{\Theta} = \begin{pmatrix} \theta_{A_1,0} & \theta_{A_1,1} & \dots & \theta_{A_1,n} \\ \theta_{A_2,0} & \theta_{A_2,1} & \dots & \theta_{A_2,n} \\ \dots & \dots & \dots & \dots \\ \theta_{A_N,0} & \theta_{A_N,1} & \dots & \theta_{A_N,n} \end{pmatrix}.$$

Then (2.2) can be written as

$$\begin{pmatrix} \frac{x_{A_1}}{p_{A_1}} \\ \frac{x_{A_2}}{p_{A_2}} \\ \dots \\ \frac{x_{A_N}}{p_{A_N}} \end{pmatrix} = \mathbf{\Theta} \begin{pmatrix} 1 \\ \frac{X_1}{\mu_1} \\ \dots \\ \frac{X_n}{\mu_n} \end{pmatrix}, \tag{2.11}$$

which implies that the covariance matrix of $x_A, A \in \mathcal{F}$ can be expressed as

$$\mathrm{Cov}(\begin{pmatrix} \frac{x_{A_1}}{p_{A_1}} \\ \frac{x_{A_2}}{p_{A_2}} \\ \dots \\ \frac{x_{A_N}}{p_{A_N}} \end{pmatrix}) = \mathbf{\Theta}\mathrm{Cov}(\begin{pmatrix} 1 \\ \frac{X_1}{\mu_1} \\ \dots \\ \frac{X_n}{\mu_n} \end{pmatrix})\mathbf{\Theta}^T.$$

Proposition 2.1. *Under the assumptions (A1)-(A3) and (X1)-(X2), we have*

$$\mathrm{Cov}(I_A, I_B) = \mathrm{Cov}(x_A, x_B), \quad \mathrm{Var}(I_A) = p_A - \mathrm{E}(x_A^2) + \mathrm{Var}(x_A),$$

and

$$\mathrm{Var}(\frac{x_A}{p_A}) = \sum_{k=1}^{n} \theta_{A,k}^2 \frac{\sigma_k^2}{\mu_k^2} + 2\sum_{k>j\geq 1} \theta_{A,k}\theta_{A,j}[\alpha_{k,j}\mathrm{Cov}(\frac{X_k^+}{\mu_k}, \frac{X_j^+}{\mu_j}) + \gamma_{k,j}\mathrm{Cov}(\frac{X_k^+}{\mu_k}, \frac{X_j^-}{\mu_j})]. \tag{2.12}$$

Proof. Under the framework of conditional independence, it is straightforward to verify that

$$\begin{aligned} \mathrm{Cov}(I_A, I_B) &= \mathrm{E}\{\mathrm{E}(I_A|X_1, X_2, \dots, X_n)\mathrm{E}(I_B|X_1, X_2, \dots, X_n)\} - p_A p_B \\ &= \mathrm{Cov}(x_A, x_B) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Var}(I_A) &= \mathrm{E}(\mathrm{Var}(I_A|X_1, X_2, \dots, X_n)) + \mathrm{Var}(\mathrm{E}[I_A|X_1, X_2, \dots, X_n]) \\ &= p_A - \mathrm{E}(x_A^2) + \mathrm{Var}(x_A). \end{aligned}$$

It follows from (2.2) that

$$\mathrm{Var}(\frac{x_A}{p_A}) = \sum_{i=1}^{n} \theta_{A,i}^2 \frac{\sigma_i^2}{\mu_i^2} + 2\sum_{i>j\geq 1} \theta_{A,i}\theta_{A,j}\mathrm{Cov}(\frac{X_i}{\mu_i}, \frac{X_j}{\mu_j}).$$

By (2.6), we can show that

$$\begin{aligned}\mathrm{Cov}(X_i, X_j) \quad &= \alpha_{i,j}\mathrm{Cov}(X_i^+, X_j^+) + \beta_{i,j}\mathrm{Cov}(X_i^\perp, X_j^\perp) + \gamma_{i,j}\mathrm{Cov}(X_i^+, X_j^-) \\ &= \alpha_{i,j}\mathrm{Cov}(X_i^+, X_j^+) + \gamma_{i,j}\mathrm{Cov}(X_i^+, X_j^-),\end{aligned}$$

which implies (2.12). Hence, the proposition holds.

□

Note that for $A \neq B$,

$$\mathrm{Cov}(\frac{x_A}{p_A}, \frac{x_B}{p_B}) = \mathrm{Cov}(\frac{I_A}{p_A}, \frac{I_B}{p_B}).$$

In (2.12), the variance of x_A is expressed into two parts. The first part is the contribution of the variances of individual risk factors, and the second part is the contribution of correlation among the risk factors. When the risk factors are independent as in CR$^+$ model, i.e., $\alpha_{i,j} = 0, \gamma_{i,j} = 0, i \neq j$, we have

$$\mathrm{Var}(x_A) = p_A^2 \sum_{k=1}^{n} w_{A,k}^2 \sigma_k^2$$

and

$$\mathrm{Cov}(x_A, x_B) = p_A p_B \sum_{k=1}^{n} w_{A,k} w_{B,k} \sigma_k^2,$$

where $w_{A,k} = \theta_{A,k}/\mu_k$ is the weight function defined in the CR$^+$ model. When $U_i = U, 1 \leq i \leq n$, that is, the risk factors are comonotonic, we have

$$\mathrm{Var}(\frac{x_A}{p_A}) = \sum_{k=1}^{n} w_{A,k}^2 \sigma_k^2 + 2 \sum_{k>j\geq 1} w_{A,k} w_{A,j} \mathrm{Cov}(X_k^+, X_j^+).$$

Note that in the CR$^+$ model, the risk factors X_i, $1 \leq i \leq n$, are assumed to be independent, and the following equation is employed:

$$\sqrt{\mathrm{Var}(X_k)} = \sum_{i \leq N} \theta_{A_i,k} \sqrt{\mathrm{Var}(x_{A_i})}, \tag{2.13}$$

that is,

$$\frac{\sqrt{\mathrm{Var}(X_k)}}{\mu_k} = \frac{\sum_{i \leq N} \theta_{A_i,k} p_{A_i} \sqrt{\mathrm{Var}(\frac{x_{A_i}}{p_{A_i}})}}{\sum_{i \leq N} \theta_{A_i,k} p_{A_i}}.$$

The above equation is justified for the case that $n = 1$ and $\theta_{A_i,1} = 1$. However, when $n > 1$, the following example shows that (2.13) is not true.

Example 2.1. **(Counter-example of** (2.13)**)**. Suppose that the risk factors $X_i, i \leq 2$ are independent, $\theta_{A_i,1} = \theta_{A_i,2} = \frac{1}{2}$ and $p_{A_i} = p, i \leq N$, for some $p \in (0, 1)$. Then

$$\mu_1 = \mu_2 = \frac{Np}{2}$$

and

$$\mathrm{Var}(\frac{x_{A_i}}{p}) = \frac{1}{4}\mathrm{Var}(\frac{X_1}{\mu_1}) + \frac{1}{4}\mathrm{Var}(\frac{X_2}{\mu_2}),\ i \le N.$$

It follows that

$$\frac{\sum_{i\le N}\theta_{A_i,k}p_{A_i}\sqrt{\mathrm{Var}(\frac{x_{A_i}}{p_{A_i}})}}{\sum_{i\le N}\theta_{A_i,k}p_{A_i}} = \sqrt{\mathrm{Var}(\frac{x_{A_1}}{p})} < \frac{1}{2}\sqrt{\mathrm{Var}(\frac{X_1}{\mu_1})} + \frac{1}{2}\sqrt{\mathrm{Var}(\frac{X_2}{\mu_1})},$$

which implies that (2.13) does not hold for $k=1$ or $k=2$.

Assume that the copula of $X_1,\dots,X_n$ is $C^{(j_1,\dots,j_n)}$. For $j_i = 1,2,3$ and $B \subset \mathcal{F}$, denote the probability of default for $A \in B$ and no default for $A \in \mathcal{F}\setminus B$ by $f_{j_1,\dots,j_n}(B)$. Then we can easily check that

$$\begin{aligned} f_{j_1,\dots,j_n}(B) \quad = \quad & \mathrm{E}\Bigg(\prod_{A\in B}(\sum_{m=0}^{n}\frac{\theta_{A,m}}{\mu_m}F_m^{-1}(UI_{\{j_m=1\}} + V_m I_{\{j_m=2\}} + (1-U)I_{\{j_m=3\}})) \\ & \times \prod_{A\in\mathcal{F}\setminus B}(1-\sum_{m=0}^{n}\frac{\theta_{A,m}}{\mu_m}F_m^{-1}(UI_{\{j_m=1\}} + V_m I_{\{j_m=2\}} + (1-U)I_{\{j_m=3\}}))\Bigg) \end{aligned}$$

(see Yang, Qi and Wang (2009) for properties of $C^{(j_1,\dots,j_n)}$). In other words,

$$\begin{aligned} & f_{j_1,\dots,j_n}(B) \\ = \; & \int_0^1\cdots\int_0^1\Bigg(\prod_{A\in B}(\sum_{m=0}^{n}\frac{\theta_{A,m}}{\mu_m}F_m^{-1}(xI_{\{j_m=1\}} + y_m I_{\{j_m=2\}} + (1-x)I_{\{j_m=3\}})) \\ & \times \prod_{A\in\mathcal{F}\setminus B}(1-\sum_{m=0}^{n}\frac{\theta_{A,m}}{\mu_m}F_m^{-1}(xI_{\{j_m=1\}} + y_m I_{\{j_m=2\}} + (1-x)I_{\{j_m=3\}}))\Bigg)\,dxdy_1\dots dy_n. \end{aligned}$$

Note that the above function does not depend on the coefficients $a_{i,j}, i \le n, j \le 3$.

The following proposition shows the advantage of the proposed model.

Proposition 2.2. *For positive function g, we have*

$$\mathrm{E}(g(L)) \quad = \sum_{j_1=1}^{3}\cdots\sum_{j_n=1}^{3}(\prod_{r=1}^{n}a_{r,j_r})\sum_{B\subset\mathcal{F}}\big(g(\sum_{A\in B}L_A)f_{j_1,\dots,j_n}(B)\big).$$

The proof can be found in Proposition 5.1 of Yang, Qi and Wang (2009).

Remark 2.3. The above proposition shows that the expectation can be expressed as a linear combination of $f_{j_1,\dots,j_n}(B)$ with the coefficients of $\prod_{r=1}^{n}a_{r,j_r}$, and $f_{j_1,\dots,j_n}(B)$ does not depend on the coefficients a_{i,j_i}. If the values of $f_{j_1,\dots,j_n}(B), i \le N, j_i = 1,2,3$ are obtained, one can compute $\mathrm{E}(g(L))$ for different coefficients of a_{i,j_i} although the calculation of the probability $f_{j_1,\dots,j_n}(B)$ may be complicated.

In the next section, we will propose a compound Poisson model to approximate the credit portfolio, whose distribution function is much easier to calculate.

3 Conditional compound Poisson model

In order to calculate the loss distribution of the credit portfolio efficiently, one standard way is to use a compound Poisson distribution to approximate it. When the risk factors are independent, it is well known that as N goes to infinity and $\sum_A p_A$ stays a constant, the two models become very close in distribution. See Chapter 2 of Gundlach and Lehrbass (2004) for details.

In this section, we will construct a conditional compound Poisson model based on assumptions (X1) and (X2) and provide a simple algorithm for calculating the loss distribution.

3.1 Conditional compound Poisson model

To mathematically set up a compound Poisson approximation for the credit model (2.1), we replace the indicator random variable I_A by a random variable N_A, which is Poisson distributed with the same mean as I_A conditional on the risk factors $X_i,\ 1 \le i \le n$. Such N_A can be chosen as a closest Poisson random variable approximation of I_A; see Section 3.3. We use the following assumptions for this Poisson approximation:

(N1) Given the risk factors $X_k, k = 1, 2, \ldots, n$, the default indicators $N_A, A \in \mathcal{F}$ are Poisson distributed and conditionally independent;

(N2) For each $A \in \mathcal{F}$, the Poisson parameter

$$\mathrm{E}[N_A|X_1, X_2, \ldots, X_n] = x_A = p_A(\sum_{k=0}^{n} \theta_{A,k} \frac{X_k}{\mu_k}). \tag{3.1}$$

Write $\tilde{L} = \sum_A L_A N_A$. Then $\tilde{L}$ is an approximation of the credit portfolio loss L. Note that the two models $\tilde{L}$ and L have the same common risk factors $X_i, i \le n$, and under the common risk factors the obligors are conditionally independent in each model. Moreover, for each $A \in \mathcal{F}$,

$$\mathrm{E}(I_A|X_1, \ldots, X_n) = \mathrm{E}(N_A|X_1, \ldots, X_n) = x_A.$$

The distribution of $\tilde{L}$ is given in the following theorem.

Theorem 3.1. *Conditional on $X_i, 1 \le i \le n$, the approximation $\tilde{L}$ is compound-Poisson distributed with Poisson parameter $\tilde{\lambda} = \sum_{k=0}^{n} X_k$ and severity probability function $\sum_{i=0}^{n} \frac{D_{m,i} X_i}{\tilde{\lambda}}, m = 1, 2, \ldots$.*

Proof. Applying Theorem 6.3.1 in Panjer and Willmot (1992), we know that conditional on the $X_i, 1 \le i \le n$, $\tilde{L}$ is compound-Poisson distributed with Poisson parameter

$$\tilde{\lambda} = \sum_A p_A \sum_{k=0}^{n} \frac{\theta_{A,k}}{\mu_k} X_k = \sum_{k=0}^{n} (\sum_A p_A \frac{\theta_{A,k}}{\mu_k}) X_k = \sum_{k=0}^{n} X_k \tag{3.2}$$

and severity probability functions

$$\sum_{i=0}^{n} \frac{\sum_A p_A \frac{\theta_{A,i}}{\mu_i} I_{\{L_A=m\}}}{\tilde{\lambda}} X_i = \sum_{i=0}^{n} \frac{D_{m,i} X_i}{\tilde{\lambda}}, m = 1, 2, \ldots.$$

The theorem is proved. □

Theorem 3.1 says that $\tilde{L}$ can be written as

$$\tilde{L} = S_0 + S_1 + S_2 + \cdots + S_n,$$

where conditional on the risk factors $X_i, 1 \leq i \leq n$, the variables S_i, $0 \leq i \leq n$ are independent compound Poisson random variables with Poisson parameters X_i and severity distributions $D_{m,i}, m = 1, 2, \ldots$, respectively.

3.2 Algorithms for computing the distribution function of $\tilde{L}$

The advantage of the proposed model is that the distribution of $\tilde{L}$ can be easily calculated. Here we propose a method similar to the original CR$^+$ model. The only difference is that a dependent part of the pgf $G_1^{(j_1,j_2,\ldots,j_n)}(z)$ is involved.

Let $G(z) = \mathrm{E}(z^{\tilde{L}})$ be the pgf of $\tilde{L}$. Since $\tilde{L}$ is an integer-valued random variable, we have $G(z) = \sum_{i=0}^{\infty} \mathrm{P}(\tilde{L} = i) z^i$. Therefore the polynomial expansion of $G(z)$ gives the probabilities of $\tilde{L}$.

By the classical results in compound Poisson distribution and Theorem 3.1, we can write $G(z)$ as

$$G(z) = \mathrm{E}[\exp\{\sum_{i=0}^{n} X_i(P_i(z) - 1)\}] = \exp\{\mu_0(P_0(z) - 1)\} \times \mathrm{E}[\exp\{\sum_{i=1}^{n} X_i(P_i(z) - 1)\}], \quad (3.3)$$

where $P_i(z)$'s are defined in (2.10). Note that the pgf $G_0(z) := \exp\{\mu_0(P_0(z) - 1)\}$ corresponds to a compound Poisson distribution with Poisson parameter μ_0 and a severity with pgf $P_0(z)$.

By (3.3), (X1) and (X2), the distribution of $\tilde{L}$ can be obtained when the copula coefficients $a_{i,j}$, the individual pgf P_i and the distribution of risk factors X_i for $i = 1, \ldots, n$ are known. Therefore, we do not need to know any detailed information of each individual obligor A, such as $\theta_{A,i}$, P_A and L_A.

Since the copula function of $X_i, 1 \leq i \leq n$ can be expressed by (2.5), for fixed index $(j_1, j_2, \ldots, j_n)$ with $j_i = 1, 2, 3$ for $i \leq n$, we can write

$$G_0(z) =: \sum_{l=0}^{\infty} g_{l,0} z^l, \quad (3.4)$$

$$G_1^{(j_1,j_2,\ldots,j_n)}(z) = \mathrm{E}\exp\left\{\left(\sum_{1\le i\le n: j_i=1}(P_i(z)-1)F_i^{-1}(U)+\sum_{i\le n: j_i=3}(P_i(z)-1)F_i^{-1}(1-U)\right)\right\}$$
$$=: \sum_{l=0}^{\infty} g_{l,1}^{(j_1,j_2,\ldots,j_n)} z^l, \tag{3.5}$$

$$G_2^{(j_1,j_2,\ldots,j_n)}(z) = \prod_{1\le i\le n: j_i=2} \mathrm{E}\exp\left\{(P_i(z)-1)F_i^{-1}(V_i)\right\} =: \sum_{l=0}^{\infty} g_{l,2}^{(j_1,j_2,\ldots,j_n)} z^l \tag{3.6}$$

and

$$G^{(j_1,\ldots,j_n)}(z) = G_0(z)\times G_1^{(j_1,\ldots,j_n)}(z)\times G_2^{(j_1,\ldots,j_n)}(z).$$

Actually, $G^{(j_1,\ldots,j_n)}(z)$ is the probability generating function when the copula of $(X_1,\ldots,X_n)$ equals $C^{(j_1,\ldots,j_n)}$. Note that $G_1^{(j_1,j_2,\ldots,j_n)}(z)$ represents the dependent part of the pgf $G^{(j_1,\ldots,j_n)}(z)$, and $G_2^{(j_1,j_2,\ldots,j_n)}(z)$ represents the independent part of the pgf $G^{(j_1,\ldots,j_n)}(z)$.

Let $L^{(j_1,j_2,\ldots,j_n)}$ be a random variable satisfying that conditional on U, $L^{(j_1,j_2,\ldots,j_n)}$ has pgf

$$\exp\left\{\left(\sum_{i\le n: j_i=1}(P_i(z)-1)F_i^{-1}(U)+\sum_{i\le n: j_i=3}(P_i(z)-1)F_i^{-1}(1-U)\right)\right\}. \tag{3.7}$$

Put $p_m^{(j_1,j_2,\ldots,j_n)}(U) = \mathrm{P}(L^{(j_1,j_2,\ldots,j_n)} = m|U)$. Then

$$\exp\left\{\left(\sum_{i\le n: j_i=1}(P_i(z)-1)F_i^{-1}(U)+\sum_{i\le n: j_i=3}(P_i(z)-1)F_i^{-1}(1-U)\right)\right\} = \sum_{m=0}^{\infty} p_m^{(j_1,j_2,\ldots,j_n)}(U) z^m.$$

Theorem 3.2. *The pgf of $\tilde{L}$ is given by*

$$G(z) = \sum_{j_1=1}^{3}\cdots\sum_{j_n=1}^{3}\Big(\prod_{i=0}^{n} a_{i,j_i}\Big) G^{(j_1,j_2,\ldots,j_n)}(z), \tag{3.8}$$

and for each $m\ge 0$,

$$\mathrm{P}(\tilde{L}=m) = \sum_{j_1=1}^{3}\cdots\sum_{j_n=1}^{3}\Big(\prod_{i=1}^{n} a_{i,j_i}\Big)\sum_{k+l+h=m,k,l,h\ge 0} g_{k,1}^{(j_1,j_2,\ldots,j_n)} g_{l,2}^{(j_1,j_2,\ldots,j_n)} g_{h,0}. \tag{3.9}$$

Furthermore we have $g_{m,1}^{(j_1,j_2,\ldots,j_n)} = \mathrm{E}(p_m^{(j_1,j_2,\ldots,j_n)}(U))$, and $p_m^{(j_1,j_2,\ldots,j_n)}(U)$ satisfies the following Panjer's recursion

$$p_m^{(j_1,j_2,\ldots,j_n)}(U) = \sum_{1\le i\le n: j_i=1} F_i^{-1}(U)\sum_{j=1}^{m}\frac{j}{m}D_{j,i}p_{m-j}^{(j_1,j_2,\ldots,j_n)}(U)$$
$$+\sum_{1\le i\le n: j_i=3} F_i^{-1}(1-U)\sum_{j=1}^{m}\frac{j}{m}D_{j,i}p_{m-j}^{(j_1,j_2,\ldots,j_n)}(U). \tag{3.10}$$

Proof. It follows from the copula of $X_1, \ldots, X_n$ that

$$\begin{aligned}
& G_0(z) \times \mathrm{E}[\exp\{\sum_{i=1}^n X_i(P_i(z)-1)\}] \\
= \;& G_0(z) \times \sum_{j_1=1}^3 \cdots \sum_{j_n=1}^3 (\prod_{i=1}^n a_{i,j_i}) \mathrm{E}\Bigg(\exp\{ \sum_{1\le i\le n: j_i=1} (P_i(z)-1)F_j^{-1}(U) \\
& + \sum_{1\le i\le n: j_i=2} (P_i(z)-1)F_j^{-1}(V_j) + \sum_{i\le n: j_i=3} (P_i(z)-1)F_j^{-1}(1-U)\} \Bigg) \\
= \;& G_0(z) \times \sum_{j_1=1}^3 \cdots \sum_{j_n=1}^3 (\prod_{i=1}^n a_{i,j_i}) \mathrm{E}\Bigg(\exp\{ \sum_{1\le i\le n: j_i=1} (P_i(z)-1)F_j^{-1}(U) \\
& + \sum_{1\le i\le n: j_i=3} (P_i(z)-1)F_j^{-1}(1-U)\} \Bigg) \times \prod_{1\le i\le n, j_i=2} \mathrm{E}\exp\{(P_i(z)-1)F_j^{-1}(V_j)\} \\
= \;& G_0(z) \times \sum_{j_1=1}^3 \cdots \sum_{j_n=1}^3 (\prod_{i=1}^n a_{i,j_i}) G_1^{(j_1,j_2,\ldots,j_n)}(z) G_2^{(j_1,j_2,\ldots,j_n)}(z) \\
= \;& \sum_{j_1=1}^3 \cdots \sum_{j_n=1}^3 (\prod_{i=1}^n a_{i,j_i}) G^{(j_1,j_2,\ldots,j_n)}(z).
\end{aligned}$$

Therefore equation (3.8) follows from (3.3).

Note that $L^{(j_1,j_2,\ldots,j_n)}$ has pgf $G_1^{(j_1,j_2,\ldots,j_n)}(z)$ and

$$g_{l,1}^{(j_1,j_2,\ldots,j_n)} = \mathrm{P}(L^{(j_1,j_2,\ldots,j_n)} = l).$$

Thus for $p_m^{(j_1,j_2,\ldots,j_n)}(U) = \mathrm{P}(L^{(j_1,j_2,\ldots,j_n)} = m|U)$, we have $g_{m,1}^{(j_1,j_2,\ldots,j_n)} = \mathrm{E}(p_m^{(j_1,j_2,\ldots,j_n)}(U))$.

Conditional on U, the pgf

$$\exp\Bigg\{ \Bigg(\sum_{1\le i\le n: j_i=1} (P_i(z)-1)F_i^{-1}(U) + \sum_{1\le i\le n: j_i=3} (P_i(z)-1)F_i^{-1}(1-U) \Bigg) \Bigg\}$$

corresponds to the compound Poisson distribution with Poisson parameter

$$\lambda^{(j_1,j_2,\ldots,j_n)}(U) = \sum_{1\le i\le n: j_i=1} F_i^{-1}(U) + \sum_{1\le i\le n: j_i=3} F_i^{-1}(1-U)$$

and severity probability function

$$\sum_{1\le i\le n: j_i=1} \frac{F_i^{-1}(U)}{\lambda^{(j_1,j_2,\ldots,j_n)}(U)} D_{m,i} + \sum_{1\le i\le n: j_i=3} \frac{F_i^{-1}(1-U)}{\lambda^{(j_1,j_2,\ldots,j_n)}(U)} D_{m,i}.$$

It follows from Panjer's recursion (Panjer, 1981) that

$$\begin{aligned}
& \mathrm{P}(Y^{(j_1,j_2,\dots,j_n)} = m|U) \\
= \; & \lambda^{(j_1,j_2,\dots,j_n)}(U)\sum_{j=1}^{m}\frac{j}{m}\left(\sum_{i\le n:j_i=1}\frac{F_i^{-1}(U)}{\lambda^{(j_1,j_2,\dots,j_n)}(U)}D_{j,i}+\sum_{i\le n:j_i=3}\frac{F_i^{-1}(1-U)}{\lambda^{(j_1,j_2,\dots,j_n)}(U)}D_{j,i}\right)p_{m-j}^{(j_1,j_2,\dots,j_n)}(U) \\
= \; & \sum_{i\le n:j_i=1}F_i^{-1}(U)\sum_{j=1}^{m}\frac{j}{m}D_{j,i}p_{m-j}^{(j_1,j_2,\dots,j_n)}(U)+\sum_{i\le n:j_i=3}F_i^{-1}(1-U)\sum_{j=1}^{m}\frac{j}{m}D_{j,i}p_{m-j}^{(j_1,j_2,\dots,j_n)}(U).
\end{aligned}$$

Hence the theorem follows. □

Theorem 3.2 shows that for each $(j_1, j_2, \dots, j_n)$, (3.9) can be used to calculate the probability function of $\tilde{L}$ once $g_{l,0}$, $g_{l,1}^{(j_1,j_2,\dots,j_n)}$ and $g_{l,2}^{(j_1,j_2,\dots,j_n)}$, $l \ge 0$ are obtained. The advantage of the algorithm is that $g_{l,0}$, $g_{l,1}^{(j_1,j_2,\dots,j_n)}$, and $g_{l,2}^{(j_1,j_2,\dots,j_n)}$, $l \ge 0$ do not involve the coefficients a_{i,j_i}. Thus we can use equation (3.9) to get the numerical values by multiplying the coefficients of a_{i,j_i}'s.

The general Panjer's recursion method to get the probabilities $g_{l,0}$, $g_{l,1}^{(j_1,j_2,\dots,j_n)}$, and $g_{l,2}^{(j_1,j_2,\dots,j_n)}$, $l \ge 0$ is summarized as follows.

- Simply applying the Panjer recursion one can compute the probabilities $g_{k,0}, k = 1, 2, \dots$ recursively by

$$g_{k,0} = \mu_0 \sum_{j=1}^{k} \frac{j}{k} g_{k-j,0} D_{j,0} \tag{3.11}$$

  with the initial value $g_{0,0} = e^{-\mu_0}$.

- The probabilities $g_{l,1}^{(j_1,j_2,\dots,j_n)}$ can be calculated by using (3.10). Note that for fixed $U = u$, we can obtain $g_{m,1}^{(j_1,j_2,\dots,j_n)}(u) = \mathrm{P}(L^{(j_1,j_2,\dots,j_n)} = m|U = u)$ for arbitrary m. The probability $g_{m,1}^{(j_1,j_2,\dots,j_n)}$ can be calculated by

$$g_{m,1}^{(j_1,j_2,\dots,j_n)} = \mathrm{E}[\mathrm{P}(L^{(j_1,j_2,\dots,j_n)} = m|U)] = \int_0^1 \mathrm{P}(L^{(j_1,j_2,\dots,j_n)} = m|U = u)du.$$

  Sometimes the above integral does not have a closed form and has to be done numerically.

- For the term $g_{m,2}^{(j_1,j_2,\dots,j_n)}$, we have

$$G_2^{(j_1,j_2,\dots,j_n)}(z) = \prod_{1\le i\le n:j_i=2} \mathrm{E}\exp\left\{(P_i(z)-1)F_i^{-1}(V_i)\right\}. \tag{3.12}$$

  For fixed i and given V_i, the probability at point m corresponding to the pgf $\exp\{(P_i(z) - 1)F_i^{-1}(V_i)\}$ is denoted by $p_m^{(i)}(V_i)$, $m = 0, 1, \dots$. Then $p_m^{(i)}(V_i), m = 0, 1, \dots$ satisfy the following recursive equation

$$p_m^{(i)}(V_i) = F_i^{-1}(V_i)\sum_{j=0}^{m}\frac{j}{m}D_{j,i}p_{m-j}^{(i)}(V_i).$$

Like the calculation of $G_1^{(j_1,j_2,\ldots,j_n)}(z)$, we compute $p_m^{(i)}(V_i)$ for given V_i. Denote

$$p_m^{(i)} = \mathrm{E}[p_m^{(i)}(V_i)] = \int_0^1 p_m^{(i)}(v)dv. \tag{3.13}$$

Then by the convolution of $p_m^{(i)}, i \in \{1 \le k \le n : j_k = 2\}$, we have $g_{l,2}^{(j_1,j_2,\ldots,j_n)}$. This method is the same as in CR^+.

Remark 3.1. The above calculation only involves an integration of at most one dimension, and hence it is very cost-efficient. The traditional methods using copulas, such as the elliptic copulas, involve an n-dimensional integration, and thus are difficult to apply in practice. Also note that if we want the first M terms of the above probabilities, we only need to apply the recursions to the M-th step.

When (3.13) does not have a closed form, some numerical integration is needed. However, when $X_i, 1 \le i \le n$ are Gamma-distributed as in CR^+, $g_{k,2}^{(j_1,j_2,\ldots,j_n)}$ can be calculated recursively without using any approximation.

Proposition 3.1. *When F_i is gamma-distributed with mean $\frac{\alpha_i}{\beta_i}$ and variance $\frac{\alpha_i}{\beta_i^2}$, then $g_{k,2}^{(j_1,j_2,\ldots,j_n)}$ satisfies the following recursive equations:*

$$g_{n+1,2}^{(j_1,j_2,\ldots,j_n)} = \frac{1}{b_0(n+1)}\left(\sum_{i=0}^{\min(r,n)} a_i g_{n-i,2}^{(j_1,j_2,\ldots,j_n)} - \sum_{j=0}^{\min(s-1,n-1)} (n-j)b_{j+1}g_{n-j,2}^{(j_1,j_2,\ldots,j_n)}\right),$$

where r, s, a_i, b_j are numbers such that

$$\frac{a_0 + \cdots + a_r z^r}{b_0 + \cdots + b_s z^s} = \sum_{1 \le k \le n : j_k = 2} \frac{\alpha_k P_k'(z)}{1 + \beta_k - P_k(z)}.$$

The proof is given in the Appendix.

3.3 Variable approximation of L by $\tilde{L}$

Note that the commonly used approximation of L is in terms of distributions. In the following we discuss the approximation $\tilde{L}$ in terms of variable approximation.

Given the risk factors $X_1, \ldots, X_n$, the conditional distribution function of I_A is written as $F_A(x|X_1, X_2, \ldots, X_n)$ and its inverse function is written as $F^{-1}_{A|X_1,\ldots,X_n}(\cdot)$, and let $F_{Poi(x_A)}$ be a Poisson distribution with mean x_A.

Conditional on $X_i, i \le n$, we can construct random variables $Y_A, A \in \mathcal{F}$ satisfying that the sequence is i.i.d. U[0,1] random variables and for each $A \in \mathcal{F}$ the variable Y_A is independent of

$I_B, B \neq A$, such that $F^{-1}_{A|X_1,\ldots,X_n}(Y_A) = I_A$. See Yang, Zhou and Zhang (2005) for details. Then define $N_A = F^{-1}_{Poi(x_A)}(Y_A)$, where $F^{-1}_{Poi(x_A)}$ is the inverse function of $F_{Poi(x_A)}$. We can verify that conditional on the risk factors $X_1, \ldots, X_n$, $N_A, A \in \mathcal{F}$ is an independent Poisson sequence. Note that N_A and I_A are comonotonic when $X_1, \ldots, X_n$ are known, and they are connected by the common variable Y_A.

In order to show the optimality of $N_A, A \in \mathcal{F}$, we introduce a sequence of random variables $M_A, A \in \mathcal{F}$ satisfying the following conditions:

(N3) Given the risk factors $X_k, k = 1, 2, \ldots, n$, the variables $M_A, A \in \mathcal{F}$ are Poisson distributed and conditionally independent, and for each $A \in \mathcal{F}$ the variable M_A is independent of $I_B, B \neq A$;

(N4) For each $A \in \mathcal{F}$,

$$\mathrm{E}[M_A | X_1, X_2, \ldots, X_n] = x_A. \tag{3.14}$$

Hence, conditional on $X_i, i \leq n$, M_A is Poisson-distributed with mean x_A, and $M_A, A \in \mathcal{F}$ are independent. Next theorem shows that $\tilde{L}$ is an approximation of L.

Theorem 3.3. *For each $M_A, A \in \mathcal{F}$ satisfying (N3) and (N4), we have*

$$\mathrm{E}[(L - \tilde{L})^2] \leq \mathrm{E}\left[\left(L - \sum_{A \in \mathcal{F}} L_A M_A\right)^2\right]. \tag{3.15}$$

If a function f has a continuous derivative bounded by M, we have

$$\mathrm{E}|f(L) - f(\tilde{L})| \leq M\sqrt{\mathrm{E}[(L - \tilde{L})^2]}. \tag{3.16}$$

Proof. Since

$$\mathrm{E}[(I_A - N_A)^2 | X_1, X_2, \ldots, X_n] \leq \mathrm{E}[(I_A - M_A)^2 | X_1, X_2, \ldots, X_n],$$

it follows from the conditional independence that

$$\begin{aligned}
&\mathrm{E}[(L - \tilde{L})^2 | X_1, X_2, \ldots, X_n] \\
&= \sum_{A \in \mathcal{F}} L_A^2 \mathrm{E}[(I_A - N_A)^2 | X_1, X_2, \ldots, X_n] \leq \sum_{A \in \mathcal{F}} L_A^2 \mathrm{E}[(I_A - M_A)^2 | X_1, X_2, \ldots, X_n] \\
&= \mathrm{E}[(L - \sum_{A \in \mathcal{F}} L_A M_A)^2 | X_1, X_2, \ldots, X_n].
\end{aligned}$$

Taking expectation results in (3.15). Equation (3.16) follows easily.

□

Theorem 3.3 shows that $\tilde{L}$ is an optimal approximation of L in the family $\{\sum_{A \in \mathcal{F}} L_A M_A : M_A, A \in \mathcal{F}$ satisfy (N3) and (N4)$\}$.

4 Numerical examples

4.1 Model parameters

In the following we apply the proposed algorithm to an example. Let $\tilde{L}$ be the total lose, and the model for $\tilde{L}$ is described in Section 3.1. Assume that for $i = 1, 2, \ldots, n$ the copula of (U_i, U) is a BF copula with

$$C_{U_i,U}(u, v) = a_{i,1}M(u, v) + a_{i,2}\Pi(u, v) + a_{i,3}W(u, v), \tag{4.1}$$

and the pgf $P_i(z)$ in Section 2 is written as

$$P_i(z) = \sum_k D_{k,i} z^k.$$

The risk factor X_i is exponentially distributed with parameter λ_i, for $i = 1, 2, \ldots, n$. Assume $n = 6$ and $\mu_0 = 1$.

We take $a_{i,1}, a_{i,2}, a_{i,3}$ in Table 4.1, $\lambda_i, i = 1, 2, \ldots, 6$ in Table 4.2, and the probability functions $D_{k,i}, k = 0, 1, \ldots$ for $1 \le i \le 6$ in Table 4.3. Note that when the information on individual obligors $A \in \mathcal{F}$, such as $\theta_{A,i}$, P_A and L_A is given, we can get $D_{k,i}, k = 0, 1, \ldots$ for $1 \le i \le 6$ by (2.8).

i	1	2	3	4	5	6
$a_{i,1}$	1	0	0	0.9	0	0.3
$a_{i,2}$	0	1	0	0.1	0.7	0.4
$a_{i,3}$	0	0	1	0	0.3	0.3

Table 4.1: The copula coefficients of X_i

i	1	2	3	4	5	6
λ_i	0.1	0.2	0.5	1	3	5

Table 4.2: λ_i: $X_i \sim \text{Expo}(\lambda_i)$

4.1.1 Results and analysis

Here we calculate the distribution of the total loss for different choices of the dependent structure among risk factors.

- Case 1. $X_1, \ldots, X_6$ have copula (4.1) with coefficients in Table 4.1.
- Case 2. $X_1, \ldots, X_6$ are independent as in the CR^+ Model. Note that in this case, the pgf of the total loss is $G^{(2,2,2,2,2,2)}(z)$.

$i\backslash k$	1	2	3	5	10	20	50	100
0	0.40	0.20	0.15	0.10	0.08	0.04	0.02	0.01
1	0.60	0.25	0.10	0.05	0	0	0	0
2	0.80	0.10	0.05	0.04	0.01	0	0	0
3	0.50	0.25	0.10	0.10	0.03	0.02	0	0
4	0.20	0.25	0.30	0.10	0.06	0.05	0.03	0.01
5	0	0	0	0.60	0.25	0.10	0.04	0.01
6	0	0	0	0	0.64	0.30	0.04	0.02

Table 4.3: Table of $D_{k,i}$, $P_i(z) = \sum_{k=1}^{\infty} D_{k,i} z^k$

- Case 3. $X_1, \ldots, X_6$ are comonotonic. Note that in this case, the pgf of the total loss is $G^{(1,1,1,1,1,1)}(z)$. This case gives the maximum variance of the total loss over all possible dependent structure among factors.

Results are given in Figures 4.1 and 4.2. Some more cases of $G^{(j_1,\ldots,j_6)}$ are provided in Figure 4.3. The variance and skewness for different cases are reported in Table 4.4.

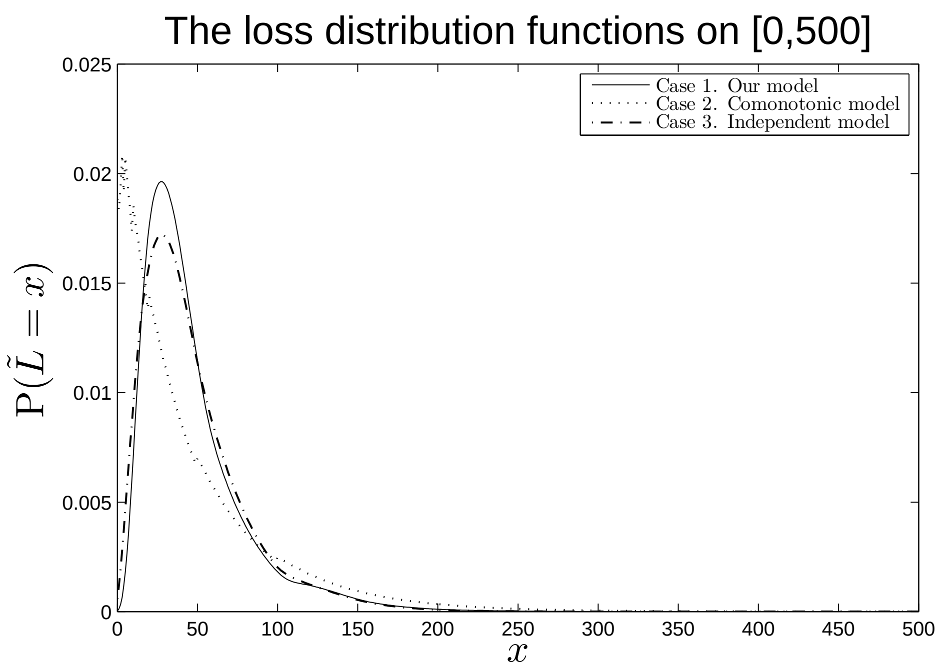

Figure 4.1: The probability functions for three cases on [0,500]

From the figures, we observe that the distribution functions in Case 1 and Case 2 are similar in shape. This is due to the fact that the pgf $G(z)$ is the weighted average of different $G^{(j_1,\ldots,j_6)}(z)$'s

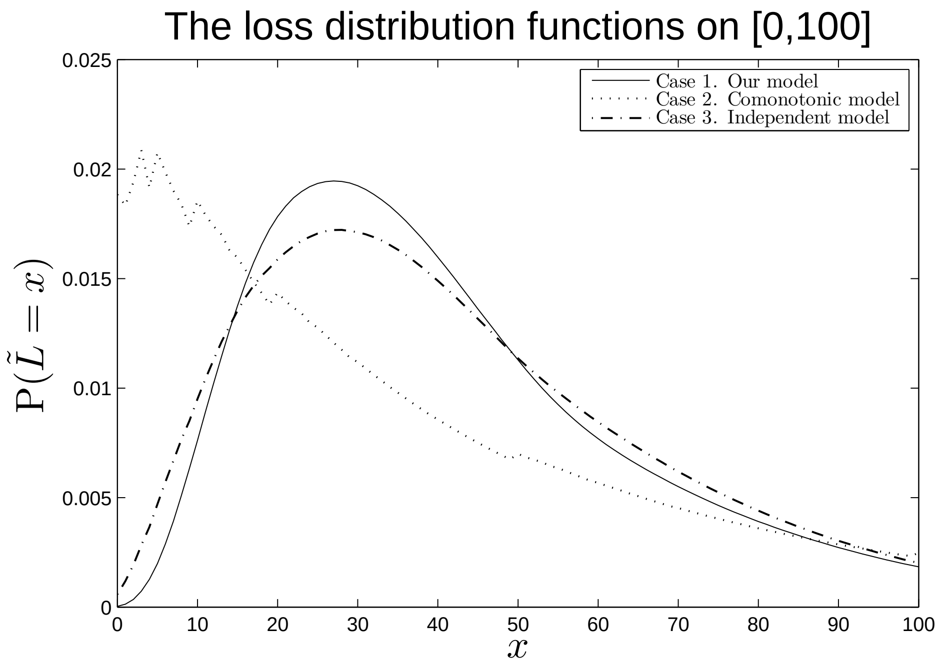

Figure 4.2: The probability functions for three cases on [0,100]

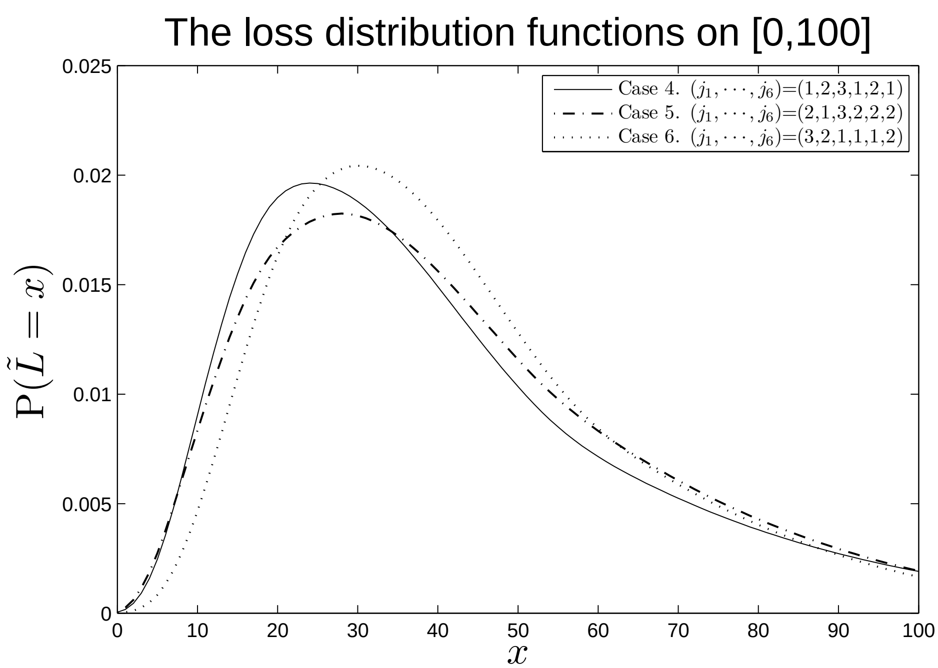

Figure 4.3: The probability functions for other pgf $G^{(j_1,\ldots,j_6)}(z)$

and the distributions with pgf $G^{(j_1,\ldots,j_6)}(z)$ have the same mean and support, while in some of those distributions, risk factors are positively related and in some cases they are negatively related.

Generating function	Mean	Variance	Skewness
$G(z)$	47.08	1.1143×10^3	1.9866
$G^{(2,2,2,2,2,2)}(z)$	47.08	2.3925×10^3	2.0344
$G^{(1,1,1,1,1,1)}(z)$	47.08	1.0733×10^3	1.6367
$G^{(1,2,3,1,2,1)}(z)$	47.08	1.2577×10^3	2.0727
$G^{(2,1,3,2,2,2)}(z)$	47.08	1.0266×10^3	1.7146
$G^{(3,2,1,1,1,2)}(z)$	47.08	0.9005×10^3	1.9055

Table 4.4: The mean, variance and skewness of different dependence structures

Remark 4.1. Recall that in our algorithm of getting $G(z)$, we only calculate those $G^{(j_1,\dots,j_6)}(z)$'s with positive weight $\prod_{i=1}^{6} a_{i,j_i}$. This significantly reduces the computation when some of the coefficients a_{i,j_i} are zero. In the above example, we do not need to calculate $G^{(1,\dots,1)}(z)$ or $G^{(2,\dots,2)}(z)$ since $\prod_{i=0}^{6} a_{i,1} = \prod_{i=1}^{6} a_{i,2} = 0$.

5 Final remarks

In this paper, we generalize CR^+ model to the case that the common risk factors are dependent via a class of extreme copulas presented in Yang, Qi and Wang (2009). Further we propose a conditional compound Poisson model to approximate the original credit portfolio, and set up a variable connection between the original credit portfolio and the conditional compound Poisson model. A recursive algorithm for computing the loss distribution based on the conditional Compound Poisson model is provided too. The computational advantage of this new model is shown by some numerical examples.

Like all other models, there are some limitations of the $C^{\mathcal{A},\mathcal{B}}$ copulas in practical use. The first limitation is that there are no established goodness-of-fit tests for the $C^{\mathcal{A},\mathcal{B}}$ copulas. The $C^{\mathcal{A},\mathcal{B}}$ copulas are shown to be useful for approximating the overall dependence, but may not be accurate for capturing local dependence properties. Another issue is that the $C^{\mathcal{A},\mathcal{B}}$ copula, similar to the Gaussian copula, only relies on the bivariate structures. This is convenient for estimation and modeling, but may result in oversimplified models.

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6 Appendix

Proposition 3.1 follows directly from the following lemmas.

Lemma 6.1. *(Credit Suisse First Boston (1997)) A power series expansion* $H(z) = \sum_{n=0}^{\infty} A_n z^n$ *has a recurrence relation*

$$A_{n+1} = \frac{1}{b_0(n+1)} \left(\sum_{i=0}^{\min(r,n)} a_i A_{n-i} - \sum_{j=0}^{\min(s-1,n-1)} (n-j) b_{j+1} A_{n-j} \right)$$

if

$$\frac{d}{dz}(\log H(z))) = \frac{1}{H(z)} \frac{dH(z)}{dz} = \frac{A(z)}{B(z)},$$

where

$$A(z) = a_0 + \cdots + a_r z^r,$$

$$B(z) = b_0 + \cdots + b_s z^s.$$

In other words, the logarithmic derivative of $H(z)$ *is a rational function.*

Lemma 6.2. *The logarithmic derivative of* $G_2^{(j_1,j_2,\ldots,j_n)}(z)$ *is a rational function, and*

$$\frac{1}{G_2^{(j_1,j_2,\ldots,j_n)}(z)} \frac{dG_2^{(j_1,j_2,\ldots,j_n)}(z)}{dz} = \frac{A(z)}{B(z)} = \sum_{k \le n: j_k = 2} \frac{\alpha_k P_k'(z)}{1 + \beta_k - P_k(z)}.$$

Proof. From the risk theory, the corresponding random variable of $G_2^{(j_1,j_2,\ldots,j_n)}(z)$ is an independent sum of compound negative binomial risks, i.e.

$$G_2^{(j_1,j_2,\ldots,j_n)}(z) = \prod_{k \le n: j_k = 2} \left(\frac{\beta_k}{1 + \beta_k - P_k(z)} \right)^{\alpha_k}.$$

Put

$$H_k(z) = \left(\frac{\beta_k}{1 + \beta_k - P_k(z)} \right)^{\alpha_k}.$$

Then

$$\frac{\mathrm{d}(\log G_2^{(j_1,j_2,\ldots,j_n)}(z))}{\mathrm{d}z} = \sum_{k \le n: j_k = 2} \frac{H_k'(z)}{H_k(z)} = \sum_{k \le n: j_k = 2} \frac{\alpha_k P_k'(z)}{1 + \beta_k - P_k(z)}.$$

Since $P_k(z)$'s are polynomials with finite terms, the logarithmic derivative of $G_2^{(j_1,j_2,\ldots,j_n)}(z)$ is a rational function. □

References

[1] Bürgisser, P., Kurth, A., Wagner, A., and Wolf, M. (1999). Integrating correlations. *Risk Magazine 12(7), 1-7.*

[2] Cherubini, U., Luciano, E. and Vecchiato, W. (2004). *Copula Methods in Finance.* Wiley.

[3] Credit Suisse First Boston (1997). CreditRisk$^+$–A credit Risk Management Framework. Available at http://www.defaultrisk.com/pp_model_21.htm.

[4] Deshpande, A. and Iyer, S. K. (2009). The credit risk$^+$ model with general sector correlations. *Central European Journal of Operations Research 17(2), 219–228.*

[5] Dhaene, J., Denuit, M., Goovaerts, M. J., Kaas, R., and Vyncke, D. (2002a). The concept of comonotonicity in actuarial science and insurance: theory. *Insurance: Mathematics and Economics 31,3-33.*

[6] Dhaene, J., Denuit, M., Goovaerts, M. J.,Kaas, R., and Vyncke, D. (2002b). The concept of comonotonicity in actuarial science and insurance: applications. *Insurance: Mathematics and Economics 31, 133-161.*

[7] Giese, G. (2004). Enhanced CreditRisk$^+$. In *CreditRisk$^+$ in the Banking Industry* (eds. Gundlach, M. and Lehrbass, F.), *79-90.* Springer-Verlag.

[8] Gordy, M. B. (2002). Saddlepoint approximation of CreditRisk$^+$. *Journal of Banking and Finance 26(7), 1335–1353.*

[9] Gundlach, M. and Lehrbass, F. (Eds.) (2004). *CreditRisk$^+$ in the Banking Industry.* Springer-Verlag.

[10] Gupton, G. M., Finger, C. C., and Bhatia, M. (1997). CreditMetrics – Technical Document. Available at http://www.defaultrisk.com/pp_model_20.htm.

[11] Kostadinov, K. (2006). Tail Approximation for Credit Risk Portfolios with Heavy-tailed Risk Factors. *Journal of Risk 8(2), 81-107.*

[12] Nelsen, R. B. (2006). *An Introduction to Copulas.* New York: Springer. Second edition.

[13] Panjer, H. (1981). Recursive evaluation of a family of compound distributions. *Astin Bulletin 12, 22-26.*

[14] Panjer, H. and Willmot, E. (1992). *Insurance Risk Models.* Society of Actuaries, Schaumberg.

[15] Reiβ, O. (2004). Dependent sectors and an Extension to Incorporate Market Risk. In *CreditRisk$^+$ in the Banking Industry* (eds. Gundlach, M. and Lehrbass, F.), *215-230.* Springer-Verlag.

[16] Vandendorpea, A., Hob, N.-D., Vanduffelc, S. and Doorenb, P.V. (2008). On the parameterization of the CreditRisk$^+$ model for estimating credit portfolio risk. *Insurance: Mathematics and Economics 42(2), 736–745.*

[17] Wilson, T. C. (1998). Portfolio Credit Risk. *FRBNY Economic Policy Review, October, 71–82.*

[18] Yang, J., Qi, Y., and Wang, R. (2009). A class of multivariate copulas with bivariate Frechet marginal copulas. *Insurance: Mathematics and Economics 45(1), 139–147.*

[19] Yang, J., Cheng, S,. and Zhang, L. (2006). Bivariate copula decomposition in terms of comonotonicity, countermonotonicity and independence. *Insurance: Mathematics and Economics 39(2), 267–284.*

[20] Yang, J., Zhou, S., and Zhang, Z. (2005). The compound Poisson random variable approximation to the individual risk model. *Insurance: Mathematics and Economics 36, 57–77.*