Diversification limit of quantiles under dependence uncertainty

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February 17, 2016[¶]

Abstract

In this paper, we investigate the asymptotic behavior of the portfolio diversification ratio based on Value-at-Risk (quantile) under dependence uncertainty, which we refer to as "worst-case diversification limit". We show that the worst-case diversification limit is equal to the upper limit of the worst-case diversification ratio under mild conditions on the portfolio marginal distributions. In the case of regularly varying margins, we provide explicit values for the worst-case diversification limit. Under the framework of dependence uncertainty the worst-case diversification limit is significantly higher compared to classic results obtained in the literature of multivariate regularly varying distributions. The results carried out in this paper bring together extreme value theory and dependence uncertainty, two popular topics in the recent study of risk aggregation.

Keywords: Value-at-Risk; diversification ratio; extreme value analysis; asymptotics; dependence uncertainty.

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1 Introduction

Diversification issues are one of the main concerns for financial institutions, as pointed out in BCBS (2006) (see also BCBS, 2012, 2013). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be an atomless probability space and L^0 be the set of all random variables in this space. The *diversification ratio* for a portfolio $\mathbf{X} = (X_1, \ldots, X_d) \in (L^0)^d$ based on a Value-at-Risk (VaR) at level $p \in (0,1)$ is defined as

$$\Delta_p^{\mathbf{X}} = \frac{q_p(X_1 + \cdots + X_d)}{\sum_{i=1}^d q_p(X_i)}, \tag{1.1}$$

where the VaR at probability level p, denoted by q_p, is the left-continuous p-quantile of a random variable,

$$q_p(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leqslant x) \geqslant p\}, \quad p \in (0,1].$$

We assume that $q_p(X_i) > 0$, $i = 1, \ldots, d$ for the interpretation of $X_1, \ldots, X_n$ as financial risks. $\Delta_p^{\mathbf{X}}$ represents the ratio between the aggregation risk and the sum of the individual risks, and hence is a measure of the portfolio performance; see for instance Embrechts et al. (2014) and Emmer et al. (2015) for diversification ratios in financial risk management.

Over the past decades, VaR has become the most widely used regulatory risk measure for financial institutions such as banks, insurance companies and investment funds. It is well-known that VaR is not subadditive, that is, for two random variables X and Y, it might be possible that $q_p(X + Y) > q_p(X) + q_p(Y)$ for some $p \in (0,1)$. See Artzner et al. (1999) and McNeil et al. (2015) for the discussion of non-subadditivity of VaR. The lack of subadditivity implies that, portfolio diversification might be penalized when using VaR as the regulatory risk measure, and it is considered one of the main drawbacks of VaR. If VaR is chosen as the regulatory risk measure the financial institutions may split one risk to several dependent risks to avoid or reduce the required risk capital reservation; further, VaR is additive on risks that are comonotonic (i.e. risks that can be represented as increasing functions of a common random variable), together with the non-subadditivity this implies an inconsistent order of risks, where it might be possible to find a dependence structure which is more penalized (higher capital required) than the comonotonic one which is often regarded as the worst-kind of dependence.

In practice, regulators are most concerned with the tails (extreme behaviors) of the risks. The capital requirement principle in Basel II is computed using VaR at a very high probability level; this probability level has been lifted even further in the proposed Basel III*. As a consequence, there is an extensive literature on the asymptotic behavior ($p \to 1^-$) of the VaR diversification ratio for a portfolio risk:

$$\text{the asymptotic behavior of } \ \Delta_p^{\mathbf{X}}, \quad \text{as} \quad p \to 1^-. \tag{1.2}$$

*Quoting BCBS (2013): "*To maintain consistency with the banking book treatment, the Committee has decided to propose an incremental capital charge for default risk based on a VaR calculation using a one-year time horizon and calibrated to a 99.9th percentile confidence level.*"

When a joint model of $(X_1, \ldots, X_n)$ is specified, the study of (1.2) is often carried out via a *Extreme Value Theory* (EVT) and under specific assumptions on the portfolio risks; see Embrechts et al. (2009a) for the first-order asymptotic expansions under a model of multivariate regular variation, and Degen et al. (2010), Mao and Hu (2013) and Mao and Ng (2015) for second-order expansions under the assumption of independent distributed random variables. We refer to McNeil et al. (2015) for a classic treatment of EVT in quantitative risk management.

It is often the case in practice, that financial companies have enough data/models to properly fit the marginal distributions of their portfolio, while the dependence structure across the portfolio risks is more difficult to be statistically estimated. This leads to the notion of *dependence uncertainty* (DU) in risk aggregation. Formally, under dependence uncertainty, the joint distribution $F_{\mathbf{X}}$ of a given portfolio $\mathbf{X} = (X_1, \ldots, X_d),\ X_i \sim F_i,\ i = 1, \ldots, d$ remains unspecified and varies in the *Fréchet class*

$$\mathcal{F}_d(F_1, \ldots, F_d) = \{\text{joint distributions with margins } F_1, \ldots, F_d\}; \tag{1.3}$$

see Joe (1997) for detailed discussion of Fréchet class, and Embrechts et al. (2013, 2015), Bernard et al. (2014) for recent research under DU. Here and throughout, we write $S_d = X_1 + \cdots + X_d$ and $\mathcal{F}_d = \mathcal{F}_d(F_1, \ldots, F_d)$ when the margins $F_1, \ldots, F_d$ are clear in the context. For a random variable X and a distribution function F, we use $X \sim F$ to indicate that X has the distribution function F, and for a random vector $\mathbf{X}$ and a Fréchet class $\mathcal{F}_d$ defined in (1.3), we use $\mathbf{X} \sim \mathcal{F}_d$ to indicate $F_{\mathbf{X}} \in \mathcal{F}_d$, where $F_{\mathbf{X}}$ is the joint distribution function of $\mathbf{X}$.

The goal of this paper is to investigate the *worst* asymptotic behavior of VaR diversification ratio under dependence uncertainty. When the dependence structure is unknown, the existence of a limit for (1.2) cannot be guaranteed; for this reason we seek the upper limit which corresponds to the *worst-case diversification limit*, that is

$$\overline{\Delta}^{\mathcal{F}_d} := \sup_{\mathbf{X} \sim \mathcal{F}_d} \limsup_{p \to 1^-} \Delta_p^{\mathbf{X}}. \tag{1.4}$$

As a main contribution, we show that under mild conditions the two operations lim sup and sup in (1.4) can be exchanged, that is,

$$\overline{\Delta}^{\mathcal{F}_d} = \limsup_{p \to 1^-} \sup_{\mathbf{X} \sim \mathcal{F}_d} \Delta_p^{\mathbf{X}}. \tag{1.5}$$

This allows us to calculate $\overline{\Delta}^{\mathcal{F}_d}$ via the *worst-case diversification ratio* of VaR at probability level p for a Fréchet class $\mathcal{F}_d$ in (1.3), defined as

$$\overline{\Delta}_p^{\mathcal{F}_d} = \sup_{\mathbf{X} \sim \mathcal{F}_d} \Delta_p^{\mathbf{X}}, \quad p \in (0, 1). \tag{1.6}$$

The worst-case diversification ratios $\overline{\Delta}_p^{\mathcal{F}_d}$ are used as a measure of dependence uncertainty; it represents the ratio within the highest capital required by VaR under dependence uncertainty and the sum of the stand alone risk capitals; see Kortschak and Albrecher (2009) for the discussion of non-existence of the limit of the diversification ratio, and Embrechts et al. (2014) for existing results on the calculation $\overline{\Delta}_p^{\mathcal{F}_d}$. The Rearrangement Algorithm introduced in Embrechts et al. (2013) allows to compute (1.6) numerically for any kind of marginal distributions.

Heavy-tailed risks and their tails are the focus of regulators, especially after the recent financial crisis. In EVT, heavy-tailed risks are often modeled by regular variation of their survival functions due to the Fisher-Tippett-Gnedenko theorem (Gnedenko, 1943) and the Pickands-Balkema-de Haan theorem (Balkema and de Haan, 1974). In this paper, we derive the value of $\overline{\Delta}^{\mathcal{F}_d}$ for regularly varying margins. Specifically, it is shown that $\overline{\Delta}^{\mathcal{F}_d}$ only depends on the maximum of the indexes of regular variation. In addition, we investigate the monotonicity and the boundary values of $\overline{\Delta}^{\mathcal{F}_d}$ with respect to the dimension d and the maximum of the indexes.

Our results suggest that $\overline{\Delta}^{\mathcal{F}_d}$ can be much larger than what is expected from existing results based on mutilvariate regularly varying (MRV) distributions and specific families of copulas, such as the ones presented in Barbe et al. (2006) and Embrechts et al. (2009a,b). If $\mathbf{X}$ follows from a d-dimensional MRV distribution with index $\beta > 0$, one has that

$$\lim_{p\to 1^-} \Delta_p^{\mathbf{X}} \leqslant \begin{cases} d^{1/\beta-1}, & \beta \leqslant 1 \\ 1, & \beta > 1. \end{cases}$$

Thus, when dependence is unspecified, the upper limit of $\Delta_p^{\mathbf{X}}$ can be significantly larger than the results based on MRV distributions. For instance, when $d = 2$, our results suggest that $\overline{\Delta}^{\mathcal{F}_2} = 2^{1/\beta}$ if F_1 and F_2 are asymptotically equivalent regularly varying distribution functions with index $\beta > 0$. When d is large and $\beta < 1$, we show that

$$\overline{\Delta}^{\mathcal{F}_d}/(d^{1/\beta-1}) \sim (1-\beta)^{-1/\beta}.$$

This implies that the assumption of MRV used in quantitative risk management could be overly simplified, especially considering that the statistical evidence for multivariate dependence structures including MRV is often limited.

The results in this paper are the first attempt to bring together EVT and DU, two popular topics in the recent study of risk aggregation in quantitative risk management.

We let $q_0(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leqslant x) > 0\}$ for notational simplicity. Sometimes it will be more convenient to express the quantile of a random variable X with distribution F, as the generalized inverse of F, defined as

$$F^{-1}(p) = q_p(X), \quad \text{for } p \in [0, 1];$$

whichever is convenient will be used in the main sections of this paper.

The rest of the paper is organized as follows. In Section 2, some basic properties of $\overline{\Delta}^{\mathcal{F}_d}$ are introduced and our main results on $\overline{\Delta}^{\mathcal{F}_d}$ are established. In Section 3 the case for Fréchet classes with regularly varying margins is studied. Section 4 draws a conclusion and discusses some future work. Some technical proofs are put in the Appendix.

2 Worst-case diversification limit

2.1 Standard bounds

Let $F_1, \dots, F_d$ be d distribution functions and assume that $q_p(X_i) > 0$ for each $i = 1, \dots, d$. Define $\overline{\Delta}^{\mathcal{F}_d}$ and $\overline{\Delta}_p^{\mathcal{F}_d}$, $p \in (0,1)$, as in (1.4) and (1.6), respectively. It is immediate to verify that

$$\overline{\Delta}^{\mathcal{F}_d} \leqslant \limsup_{p \to 1^-} \overline{\Delta}_p^{\mathcal{F}_d}, \tag{2.1}$$

since

$$\limsup_{p \to 1^-} \Delta_p^{\mathbf{X}} \leqslant \limsup_{p \to 1^-} \overline{\Delta}_p^{\mathcal{F}_d} \qquad \text{for every } \mathbf{X} \sim \mathcal{F}_d.$$

A useful observation that will be extensively used in the paper is listed below. To state the proposition, we define the *Expected Shortfall* (ES) of an integrable random variable X at level $p \in [0,1]$ as

$$\psi_p(X) = \frac{1}{1-p} \int_p^1 q_u(X) \mathrm{d}u, \quad p \in [0,1) \quad \text{and} \quad \psi_1(X) = q_1(X).$$

It is well-known that $\psi_p(X) = \mathbb{E}[X | X > q_u(X)]$ if the distribution function of X is continuous at $q_u(X)$. ψ_p subadditive, i.e., for any two integrable random variables X and Y, we have

$$\psi_p(X+Y) \leqslant \psi_p(X) + \psi_p(Y), \quad p \in [0,1],$$

and the equality holds if X and Y are comonotonic (i.e. they can be represented as increasing functions of a common random variable).

Proposition 2.1. *Let $F_1, \dots, F_d$ be d distribution functions with finite first moment. For any $\mathbf{X} \sim \mathcal{F}_d$,*

$$1 \leqslant \overline{\Delta}^{\mathcal{F}_d} \leqslant \limsup_{p \to 1^-} \frac{\sum_{i=1}^d \psi_p(X_i)}{\sum_{i=1}^d q_p(X_i)}. \tag{2.2}$$

Proof. Let $\mathbf{X}^c = (X_1^c, \dots, X_d^c)$ be a vector of comonotonic risks with $\mathbf{X}^c \sim \mathcal{F}_d$ and denote $S_d^c = X_1^c + \ldots + X_d^c$. Since q_p is comonotonic additive, we have that

$$\Delta_p^{\mathbf{X}^c} = \frac{q_p(S_d^c)}{\sum_{i=1}^d q_p(X_i)} = \frac{\sum_{i=1}^d q_p(X_i^c)}{\sum_{i=1}^d q_p(X_i)} = 1, \quad \text{for all } p \in (0,1)$$

and the first inequality follows. To show the second inequality, from (2.1),

$$\begin{aligned}\overline{\Delta}^{\mathcal{F}_d} &\leqslant \limsup_{p \to 1^-} \sup_{\mathbf{X} \sim \mathcal{F}_d} \frac{q_p(S_d)}{\sum_{i=1}^d q_p(X_i)} \leqslant \limsup_{p \to 1^-} \sup_{\mathbf{X} \sim \mathcal{F}_d} \frac{\psi_p(S_d)}{\sum_{i=1}^d q_p(X_i)} \\ &= \limsup_{p \to 1^-} \frac{\sum_{i=1}^d \psi_p(X_i)}{\sum_{i=1}^d q_p(X_i)},\end{aligned}$$

which concludes the proof. $\square$

A result for rapidly varying distributions, an important subclass of light-tailed distributions, follows immediately from Proposition 2.1. Recall that a random variable X has a rapidly varying distribution F on $[0, \infty)$ if $\lim_{t\to\infty} \overline{F}(tx)/\overline{F}(t) = 0$ for all $x > 0$. Two well-known examples of rapidly varying distributions are the class of normal distributions and the class of subexponential distributions (see Teugels (1975)); their tail probabilities decay faster than those of exponential distributions.

Corollary 2.2. *Let $F_1, \dots, F_d$ be d rapidly varying distribution functions. Then for any $\mathbf{X} \sim \mathcal{F}_d$, we have*

$$\overline{\Delta}^{\mathcal{F}_d} = \limsup_{p\to 1^-} \overline{\Delta}_p^{\mathcal{F}_d} = 1. \tag{2.3}$$

Proof. For any rapidly varying distribution F, we have

$$\lim_{p\to 1^-} \frac{\psi_p(X)}{q_p(X)} = 1,$$

where X is a random variable having the distribution function F; see McNeil et al. (2015, Section 5.2). Hence, for any $\varepsilon > 0$, there exists $p_0 > 0$ such that for $p > p_0$,

$$0 < q_p(X_i) \leqslant \psi_p(X_i) < q_p(X_i)(1+\varepsilon), \quad i = 1, \dots, d.$$

Summing up the above inequalities from 1 to d and letting $\varepsilon \to 0^+$ lead to

$$\lim_{p\to 1^-} \frac{\sum_{i=1}^d \psi_p(X_i)}{\sum_{i=1}^d q_p(X_i)} = 1.$$

The proof then follows by Proposition 2.1 and inequality (2.1). □

Remark 2.1. Corollary 2.2 in particular implies that if all the distributions are bounded then the asymptotic behavior of the worst-superadditivity ratio is trivially equal to 1. Hence, in what follows we will always assume that $F_i^{-1}(1) = \infty$ for at least one $i \in \{1, \dots, d\}$.

Before moving to the main results on the worst-case diversification limit, we present two side-results on two related quantities: the *exact diversification limit* (if it exists)

$$\lim_{p\to 1^-} \Delta_p^{\mathbf{X}} \tag{2.4}$$

and the *best-case diversification limit*

$$\liminf_{p\to 1^-} \inf_{\mathbf{X}\sim\mathcal{F}_d} \Delta_p^{\mathbf{X}}. \tag{2.5}$$

The following two propositions show that the two limits in (2.4) and (2.5) are bounded and often easy to calculate.

Proposition 2.3. *Let $F_1, \ldots, F_d$ be d distributions with finite first moment. Then*

$$\lim_{p \to 1^-} \Delta_p^{\mathbf{X}} \leqslant 1, \tag{2.6}$$

for every $\mathbf{X} \sim \mathcal{F}_d(F_1, \ldots, F_d)$ such that $\Delta_p^{\mathbf{X}}$ has a limit as $p \to 1^-$.

Proof. Since $\lim_{p\to 1^-} \int_p^1 q_u(S_d)\mathrm{d}u = \lim_{p\to 1^-} \int_p^1 q_u(X_i)\mathrm{d}u = 0$ for $i = 1, \ldots, d$, we can apply de l'Hospital's Rule. Using the subadditivity of ψ_p, it immediately follows for all $\mathbf{X} \sim \mathcal{F}_d$ such that $\Delta_p^{\mathbf{X}}$ has a limit as $p \to 1^-$:

$$\lim_{p \to 1^-} \Delta_p^{\mathbf{X}} = \lim_{p \to 1^-} \frac{\int_p^1 q_u(S_d)\mathrm{d}u}{\sum_{i=1}^d \int_p^1 q_u(X_i)\mathrm{d}u} = \lim_{p \to 1^-} \frac{\psi_p(S_d)}{\sum_{i=1}^d \psi_p(X_i)} \leqslant 1. \tag{2.7}$$

□

When the limit for the diversification ratio exists (that is, its upper and lower limits coincide) and all the components of $\mathbf{X}$ have finite mean, the asymptotic diversification ratio is trivially equal to one; the more interesting cases are the ones for which an exact limit does not exist.

A wide majority of papers that deal with risk aggregation under uncertainty (e.g. Chen et al. (2012), Embrechts et al. (2013) and Wang et al. (2013)) has mainly focused on the worst-case diversification limit. It might also be relevant to investigate the lower bound of the diversification ratio of a portfolio $\mathbf{X}$ with fixed margins $F_1, \ldots, F_d$:

$$\underline{\Delta}_p^{\mathcal{F}_d} := \inf_{\mathbf{X} \sim \mathcal{F}_d} \Delta_p^{\mathbf{X}}. \tag{2.8}$$

The bounds $\overline{\Delta}_p^{\mathcal{F}_d}$ and $\underline{\Delta}_p^{\mathcal{F}_d}$ together measure the dependence uncertainty of VaR. We refer to Kaas et al. (2009), Embrechts et al. (2003), Bernard et al. (2014), and Bignozzi et al. (2015) for some related discussions. The following proposition gives a straightforward value for $\underline{\Delta}_p^{\mathcal{F}_d}$, from which we can see that the limit of $\overline{\Delta}_p^{\mathcal{F}_d}$ is mathematically different from that of $\underline{\Delta}_p^{\mathcal{F}_d}$. This asymmetry is well noted in the literature; see for instance Embrechts et al. (2015).

Proposition 2.4. *Let $F_1 = \cdots = F_d =: F$ be distribution functions such that $-\infty < F^{-1}(0) < F^{-1}(1) = \infty$. Then*

$$\lim_{p \to 1^-} \inf_{\mathbf{X} \sim \mathcal{F}_d} \Delta_p^{\mathbf{X}} = \frac{1}{d}.$$

Proof. First, it can be verified that for $p \in (0,1)$,

$$F^{-1}(p) + (d-1)F^{-1}(0) \leqslant \inf_{\mathbf{X} \sim \mathcal{F}_d} q_p(S_d) \leqslant F^{-1}(p) + (d-1)F^{-1}\left(\frac{d-1}{d}\right). \tag{2.9}$$

The first inequality follows by noting that $X_1 + \ldots + X_d \geqslant \text{ess-inf} X_1 + \ldots + \text{ess-inf} X_{d-1} + X_d$ a.s. For the second inequality, let $U \sim \mathrm{U}[0,1]$,

$$U_i = U + \frac{i-1}{d} p \mod (1), \quad i = 1, \ldots, d.$$

and $A_i = \{(d-1)p/d \leqslant U_i < p\}$, $i = 1, \ldots, d$. It is obvious that $U_i \sim \mathrm{U}[0,1]$, $i = 1, \ldots, d$, and $A_1, \ldots, A_d$ are disjoint sets. Define, for $j = 1, \ldots, d$,

$$X_j^* = \begin{cases} F^{-1}((d-1)(p-U_i)), & \text{on } A_i, \quad i \neq j, \\ F^{-1}(U_j), & \text{on } A_j, \\ F^{-1}(U), & \text{on } \overline{A}, \end{cases}$$

where $\overline{A} = \Omega \setminus (\cup_{i=1}^d A_i) = \{p \leqslant U \leqslant 1\}$, and write $S_d^* = X_1^* + \cdots + X_d^*$. One can verify by definition that $X_j^* \sim F$, $j = 1, \ldots, d$. Noting that

$$\mathbb{P}\left(S_d^* > F^{-1}(p) + (d-1)F^{-1}\left((d-1)p/d\right)\right) \leqslant \mathbb{P}(\overline{A}) = 1 - p,$$

we have

$$q_p(S_d^*) \leqslant F^{-1}(p) + (d-1)F^{-1}\left((d-1)p/d\right) \leqslant F^{-1}(p) + (d-1)F^{-1}\left((d-1)/d\right).$$

Hence, (2.9) holds. Note that as $p \to 1^-$, both the ratios of LHS and RHS of (2.9) to $F^{-1}(p)$ converge to 1. The desired result follows. □

Remark 2.2. For the general case when $F_1, \ldots, F_d$ are possibly different, we have that

$$\begin{aligned} 1/d \leqslant \liminf_{p \to 1} \bigwedge_{1 \leqslant i \leqslant d} \frac{F_i^{-1}(p) + \sum_{j \neq i} F_j^{-1}(0)}{\sum_{j=1}^d F_j^{-1}(p)} &\leqslant \liminf_{p \to 1} \inf_{\mathbf{X} \sim \mathcal{F}_d} \Delta_p^{\mathbf{X}} \\ &\leqslant \lim_{p \to 1} \frac{F_i^{-1}(p) + \sum_{j \neq i} F_j^{-1}((d-1)p/d)}{\sum_{j=1}^d F_j^{-1}(p)} \leqslant 1 \end{aligned}$$

and the extreme cases ($1/d$ and 1) are obtained respectively for a homogeneous portfolio and for a portfolio where one risk dominates all the others.

2.2 Main result in the general setting

In this subsection, we show that (1.5) holds under the condition that $F_1, \ldots, F_d$ are strictly increasing in a neighborhood of ∞. This is a very weak condition satisfied by almost all models in quantitative risk management.

Theorem 2.5. *Suppose that for each $i = 1, \ldots, d$, F_i is a strictly increasing distribution function on a left-neighborhood of $F_i^{-1}(1)$. Then (1.5) holds, that is:*

$$\overline{\Delta}^{\mathcal{F}_d} = \sup_{\mathbf{X} \sim \mathcal{F}_d} \limsup_{p \to 1^-} \overline{\Delta}_p^{\mathbf{X}} = \limsup_{p \to 1^-} \sup_{\mathbf{X} \sim \mathcal{F}_d} \overline{\Delta}_p^{\mathbf{X}}. \tag{2.10}$$

First, notice that since we are working on the limit as $p \to 1^-$, we can conveniently assume that $F_1, \ldots, F_d$ are strictly increasing functions in whatever intervals we need.

To show Theorem 2.5 we need the two following lemmas. Lemma 2.6 is essentially the same as Embrechts et al. (2015, Lemma A.4) and we restate it here without a proof.

Lemma 2.7 is mainly a technical result. To state the lemmas, define the right-continuous quantile $q_p^*: L^0 \to \mathbb{R}$ as

$$q_p^*(X) = \inf\{x \in \mathbb{R} : F(x) > p\}, \quad \text{for } p \in [0,1),$$

and $q_1^*(X) = q_1(X)$ where F is the distribution function of X.

Lemma 2.6. *Let $F_1, \ldots, F_d$ be strictly increasing distribution functions such that $F_i^{-1}(p) \geqslant 0$ for $i = 1, \ldots, d$ and $p \in (0,1)$. Then*

$$\sup_{\mathbf{X} \sim \mathcal{F}_d} q_p(S_d) = \sup_{\mathbf{X} \sim \mathcal{F}_d} q_p^*(S_d) = \sup_{\mathbf{X}^{[p]} \sim \mathcal{F}^{[p]}} q_p^*(S_d^{[p]}),$$

where $S_d^{[p]}$ is the sum of components of the random vector $\mathbf{X}^{[p]}$, $\mathcal{F}^{[p]} = \mathcal{F}(F_1^{[p]}, \ldots, F_d^{[p]})$ is the Fréchet class, defined by (1.3), with margins $F_i^{[p]}$, $i = 1, \ldots, d$, and $F_i^{[p]}$ is the distribution function of $F_i^{-1}(U)1_{\{U \geqslant p\}}$, $U \sim \mathrm{U}[0,1]$, $i = 1, \ldots, d$.

For $0 \leqslant p \leqslant r \leqslant 1$, we denote by $\mathcal{F}_{\mathrm{U}}$ the d-dimensional Fréchet class, defined by (1.3), with uniform margins on $[0,1]$ and

$$\mathcal{F}_{\mathrm{U}}^{[p,r]} = \{F_{\mathrm{U}} \in \mathcal{F}_{\mathrm{U}} : \ F_{\mathrm{U}}([p,r]^d) = r - p, \ F_{\mathrm{U}}([r,1]^d) = 1 - r\}, \tag{2.11}$$

i.e., for $\mathbf{U} \sim \mathcal{F}_{\mathrm{U}}^{[p,r]}$, $\{U_i \in [p,r)\} = \{U_j \in [p,r)\}$ a.s. and $\{U_i \in [r,1]\} = \{U_j \in [r,1]\}$ a.s. for all $i, j = 1, \ldots, d$.

Lemma 2.7. *Let $F_1, \ldots, F_d$ be strictly increasing distribution functions and $p \in (0,1)$ such that $F_i^{-1}(1) = \infty$, $F_i^{-1}(p) \geqslant 0$ for $i = 1, \ldots, d$. Define $W_p(\cdot) : [p,1] \to \mathbb{R}$ as*

$$r \mapsto W_p(r) = \sup_{\mathbf{U} \sim \mathcal{F}_{\mathrm{U}}^{[p,r]}} q_p^* \left(\sum_{i=1}^d F_i^{-1}(U_i) \right).$$

Then $W_p(r)$ is continuous at $r = 1^-$.

The proof is similar to that of Bernard et al. (2014, Lemma 4.4) and is postponed to the Appendix.

Proof of Theorem 2.5. Denote $I = \{i = 1, \ldots, d : F_i^{-1}(1) = \infty\}$. If $I = \emptyset$, then it is easy to verify that both sides of (2.10) equal 1. If $I \neq \emptyset$, denote $\mathcal{F}_I$ the $|I|$-dimensional Fréchet class with margins F_i, $i \in I$. Note that for each $\mathbf{X} \sim \mathcal{F}_d$, $\lim_{p \to 1^-} q_p(\sum_{i \in I} X_i) = \lim_{p \to 1^-} \sum_{i \in I} q_p(X_i) = \infty$. It follows that for $\mathbf{X} \sim \mathcal{F}_d$,

$$\limsup_{p \to 1^-} \Delta_p^{\mathbf{X}} = \limsup_{p \to 1^-} \frac{q_p(\sum_{i \in I} X_i)}{\sum_{i \in I} q_p(X_i)} \quad \text{and} \quad \limsup_{p \to 1^-} \overline{\Delta}_p^{\mathcal{F}_d} = \limsup_{p \to 1^-} \overline{\Delta}_p^{\mathcal{F}_I}.$$

Hence, we only need to deal with the case $F_i^{-1}(1) = \infty$ for $i = 1, \dots, d$. Note that for any $p \in (0,1)$ and $\varepsilon > 0$, there exists $0 < \delta(p) < 1 - p$ such that

$$\frac{\sum_{i=1}^d q_{p+\delta(p)}(X_i)}{\sum_{i=1}^d q_p(X_i)} < 1 + \varepsilon.$$

It follows that

$$\limsup_{p \to 1^-} \frac{q_p(S_d)}{\sum_{i=1}^d q_p(X_i)} \geqslant \limsup_{p \to 1^-} \frac{q_p(S_d)}{\sum_{i=1}^d q_{p+\delta(p)}(X_i)} \geqslant \limsup_{p \to 1^-} \frac{q_p^*(S_d)}{\sum_{i=1}^d q_p(X_i)(1+\varepsilon)}$$

holds for any $\varepsilon > 0$. Letting $\varepsilon \to 0^+$ and noting (2.1), we have that, to show (2.10), it suffices to show that

$$\sup_{\mathbf{X} \sim \mathcal{F}_d} \limsup_{p \to 1^-} \frac{q_p^*(S_d)}{\sum_{i=1}^d q_p(X_i)} \geqslant \limsup_{p \to 1^-} \overline{\Delta}_p^{\mathcal{F}_d}. \tag{2.12}$$

By the definition of the upper limit, there exist $p_n \to 1^-$ as $n \to \infty$ such that $F_i^{-1}(p_n) > 0$ for $i = 1, \dots, d$ and

$$\lim_{n \to \infty} \overline{\Delta}_{p_n}^{\mathcal{F}_d} = \limsup_{p \to 1^-} \overline{\Delta}_p^{\mathcal{F}_d}.$$

For each $n \in \mathbb{N}$, by Lemma 2.7, there exists $r_n \in [p_n, 1)$ such that

$$W_{p_n}(r) > W_{p_n}(1) - \frac{1}{n}, \quad r \in [r_n, 1).$$

For each $n \in \mathbb{N}$, choose $\ell(n)$ such that $p_{\ell(n+1)} > r_{\ell(n)}$ for $n \in \mathbb{N}$. By the definition of the upper limit again, there exist $\mathbf{U}_n = (U_{n1}, \dots, U_{nd}) \sim \mathcal{F}_{\mathrm{U}}^{[p_{\ell(n)}, p_{\ell(n+1)}]}$, $n \in \mathbb{N}$, such that

$$q_{p_{\ell(n)}}^* \left(\sum_{i=1}^d F_i^{-1}(U_{ni}) \right) \geqslant W_{p_{\ell(n)}}(p_{\ell(n+1)}) - \frac{1}{n} > W_{p_{\ell(n)}}(1) - \frac{2}{n}. \tag{2.13}$$

Since $\mathbf{U}_n \sim \mathcal{F}_{\mathrm{U}}^{[p_{\ell(n)}, p_{\ell(n+1)}]}$, we can write

$$A_n := \{p_{\ell(n)} < U_{nk} \leqslant p_{\ell(n+1)}, \ k = 1, \dots, d\} = \{p_{\ell(n)} < U_{nj} \leqslant p_{\ell(n+1)}\}, \ j = 1, \dots, d, \text{ a.s.}$$

Further, since (2.13) only concerns the distributions of $\{\mathbf{U}_n, \ n \in \mathbb{N}\}$, and $\sum_{n=1}^\infty \mathbb{P}(A_n) \leqslant 1$, we can take $\{\mathbf{U}_n, \ n \in \mathbb{N}\}$ such that A_n, $n \in \mathbb{N}$ are disjoint sets.

Define a random vector $\mathbf{X} = (X_1, \dots, X_d)$ as

$$X_i = F_i^{-1}(U) I_{\{U \leqslant p_{\ell(1)}\}} + \sum_{k=1}^\infty F_i^{-1}(U_{ki}) I_{\{p_{\ell(k)} < U_{ki} \leqslant p_{\ell(k+1)}\}}, \quad i = 1, \dots, d \tag{2.14}$$

where $U \sim \mathrm{U}[0,1]$ satisfies $\{U \leqslant p_{\ell(1)}\} = \bigcap_{n \in \mathbb{N}} A_n^c$. Then we have that $X_i \sim F_i$, $i = 1, \dots, d$ and

$$\limsup_{n \to \infty} \frac{q_{p_{\ell(n)}}^*(S_d)}{\sum_{i=1}^d q_{p_{\ell(n)}}(X_i)} \geqslant \limsup_{n \to \infty} \frac{W_{p_{\ell(n)}}(1) - 2/n}{\sum_{i=1}^d q_{p_{\ell(n)}}(X_i)} = \limsup_{p \to 1^-} \overline{\Delta}_p^{\mathcal{F}_d},$$

i.e., (2.12) holds. Thus, this completes the proof. $\square$

Remark 2.3. (i) From the proof of Theorem 2.5, it is easy to see that we actually show that

$$\overline{\Delta}^{\mathcal{F}_d} = \max_{\mathbf{X}\sim\mathcal{F}_d} \limsup_{p\to 1^-} \overline{\Delta}_p^{\mathbf{X}} = \limsup_{p\to 1^-} \sup_{\mathbf{X}\sim\mathcal{F}_d} \overline{\Delta}_p^{\mathbf{X}}. \tag{2.15}$$

There is a random vector $\mathbf{X} \sim \mathcal{F}_d$ in (2.14) such that its upper asymptotic diversification ratio indeed attains the worst-case diversification limit defined in (1.6).

(ii) Due to the relation between a distribution function and its quantile function, we can show a similar results in terms of distribution functions as that in Theorem 2.5. More specifically, assume that $F_1, \ldots, F_d$ are asymptotic equivalent distribution functions such that they are continuous in a neighborhood of ∞. We have

$$\sup_{\mathbf{X}\sim\mathcal{F}_d} \limsup_{x\to\infty} \frac{\mathbb{P}(S_d > x)}{\overline{F}_1(x)} = \limsup_{x\to\infty} \sup_{\mathbf{X}\sim\mathcal{F}_d} \frac{\mathbb{P}(S_d > x)}{\overline{F}_1(x)}. \tag{2.16}$$

Theorem 2.5 states that the supremum and the upper limit in (2.10) can be exchanged under mild conditions. Note that the RHS of (2.10) involves $\overline{\Delta}_p^{\mathcal{F}_d}$, which have been studied extensively by either numerical methods or analytical methods; see for instance Embrechts et al. (2013), Wang et al. (2013) and Jakobsons et al. (2016). Therefore, Theorem 2.5 provides a practical way to calculate $\overline{\Delta}^{\mathcal{F}_d}$.

In the two-dimensional case $d = 2$, an analytical formula follows immediately from results in Makarov (1981) and Rüschendorf (1982) where $\sup\{\mathbb{P}(X_1 + X_2 \leqslant x) : X_1 \sim F_1, X_2 \sim F_2\}$ for $x \in \mathbb{R}$ was established.

Corollary 2.8. *Assume that F_1 and F_2 are two distribution functions strictly increasing at a neighborhood of ∞. We have that*

$$\overline{\Delta}^{\mathcal{F}_2} = \limsup_{p\to 1^-} \inf_{x\in[0,1-p]} \frac{F_1^{-1}(p+x) + F_2^{-1}(1-x)}{F_1^{-1}(p) + F_2^{-1}(p)}. \tag{2.17}$$

3 Regularly varying distributions

In this section, we focus on risks with regularly varying tails. Regular variation is a crucial concept in modeling extreme risks (heavy-tailed risks). We first recall its definition and its basic properties, and then present the main results of this section, i.e., determine the value of $\overline{\Delta}^{\mathcal{F}_d}$ for regularly varying risks and investigate its properties.

3.1 Regular variation

Definition 3.1. An eventually non-negative (that is, $f(x) \geqslant 0$ for x large enough) measurable function f is said to be *regularly varying* with a regularity index $\gamma \in \mathbb{R}$, if

$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^{\gamma}, \quad \text{for all } x > 0. \tag{3.1}$$

Denote this by $f \in \mathrm{RV}_{\gamma}$.

We list some basic properties of regular variation; see Bingham et al. (1989) and de Haan and Ferreira (2006). These properties are essential to the proof of the main results in this section.

Lemma 3.1. *Let F be a distribution function. Then for any $\beta > 0$, $\overline{F}(\cdot) \in \mathrm{RV}_{-\beta}$ is equivalent to $F^{-1}(1-1/\cdot) \in \mathrm{RV}_{1/\beta}$. Moreover,*

(i) the convergence in (3.1) for $f = \overline{F}$ is uniform with respect to x in any compact subset of $\mathbb{R}$;

(ii) the convergence in (3.1) for $f(\cdot) = F^{-1}(1-1/\cdot)$ is uniform with respect to x in any subset of $\mathbb{R} \cup \{\infty\}$ bounded away from zero.

3.2 Two-dimensional analytical result

In this section, we start with the two-dimensional case where a nice analytical result is available. The general result for $d \geqslant 3$, requires a different and more involved proof and is presented in Section 3.3.

Theorem 3.2. *Assume that F_1 and F_2 are asymptotically equivalent regularly varying distribution functions with index $\beta > 0$. We have*

$$\overline{\Delta}^{\mathcal{F}_2} = 2^{1/\beta}. \tag{3.2}$$

Proof. By Corollary 2.8 and taking $x = (1-p)/2 \in [0, 1-p]$ in (2.17) yields that

$$\overline{\Delta}^{\mathcal{F}_2} \leqslant \limsup_{p \to 1-} \frac{F_1^{-1}((1+p)/2) + F_2^{-1}((1+p)/2)}{F_1^{-1}(p) + F_2^{-1}(p)} = 2^{1/\beta}.$$

Thus, it suffices to show that $\overline{\Delta}^{\mathcal{F}_2} \geqslant 2^{1/\beta}$. There exist $p_n \to 1^-$ and $x_n \in [0, 1-p_n]$ as $n \to \infty$ such that

$$\lim_{n \to \infty} \inf_{x \in [0, 1-p_n]} \frac{F_1^{-1}(p_n + x) + F_2^{-1}(1-x)}{F_1^{-1}(p_n) + F_2^{-1}(p_n)} = \overline{\Delta}^{\mathcal{F}_2} \tag{3.3}$$

and for all $n \in \mathbb{N}$

$$\left| \frac{F_1^{-1}(p_n + x_n) + F_2^{-1}(1-x_n)}{F_1^{-1}(p_n) + F_2^{-1}(p_n)} - \inf_{x \in [0, 1-p_n]} \frac{F_1^{-1}(p_n + x) + F_2^{-1}(1-x)}{F_1^{-1}(p_n) + F_2^{-1}(p_n)} \right| \leqslant \frac{1}{n}. \tag{3.4}$$

We assert that there exist n_0 and $\delta > 0$ such that

$$x_n/(1-p_n) \in [\delta, 1-\delta], \quad \text{for all } n \geqslant n_0. \tag{3.5}$$

To see it, assume that there exists a subsequence n' such that $x_{n'}/(1-p_{n'}) \to 0$ as $n' \to \infty$. Then by (3.3) and (3.4),

$$\overline{\Delta}^{\mathcal{F}_2} \geqslant \lim_{n' \to \infty} \frac{F_2^{-1}(1-(1-p_{n'})\varepsilon)}{2F_2^{-1}(p_{n'})} = \frac{1}{2}\varepsilon^{-1/\beta}$$

which goes to ∞ as $\varepsilon \to 0^+$ and this conflicts with the constraint $\overline{\Delta}^{\mathcal{F}_2} \leqslant 2^{1/\beta}$. Similarly, there exists no subsequence n' such that $x_{n'}/(1-p_{n'}) \to 1$ as $n' \to \infty$. Hence, (3.5) holds. Then by the uniform convergence for regular variation on any compact set, we have

$$\overline{\Delta}^{\mathcal{F}_2} = \lim_{n\to\infty} \left(\left(\frac{1-p_n-x_n}{1-p_n} \right)^{-1/\beta} + \left(\frac{x_n}{1-p_n} \right)^{-1/\beta} \right) \geqslant 2^{1/\beta},$$

as desired. □

Remark 3.1. The result in Theorem 3.2 does not distinguish whether $\beta > 1$ or $\beta \leqslant 1$, as is commonly the case in EVT. Classic results in EVT give

$$\lim_{p\to 1^-} \Delta_p^{\mathbf{X}} \leqslant \begin{cases} d^{1/\beta-1}, & \beta \leqslant 1 \\ 1, & \beta > 1, \end{cases}$$

if $\mathbf{X}$ follows from a multivariate regularly varying (MRV) distribution; see for instance Barbe et al. (2006) and Embrechts et al. (2009a). Theorem 3.2 suggests that when dependence is unspecified, the upper limit of $\Delta_p^{\mathbf{X}}$ when $d = 2$ and $\beta \leqslant 1$ is two times as large as the above results based on MRV distributions.

Example 3.1. (Pareto model) As an example of Theorems 2.5 and 3.2 consider two Pareto distributions, $F_i(x) = 1 - x^{-\alpha_i}$ with $x_i \geqslant 1$, $\alpha_i > 0$ for $i = 1, 2$ and assume without loss of generality that $\alpha_1 \geqslant \alpha_2$. From Corollary 2.8,

$$\overline{\Delta}^{\mathcal{F}_2} = \limsup_{p\to 1^-} \inf_{x\in[0,1-p]} \frac{(1-p-x)^{-\frac{1}{\alpha_1}} + x^{-\frac{1}{\alpha_2}}}{(1-p)^{\frac{1}{\alpha_1}} + (1-p)^{\frac{1}{\alpha_2}}}. \tag{3.6}$$

For $\alpha_1 = \alpha_2$, the infimum on the right-hand-side of (3.6) is achieved for $x = \frac{1-p}{2}$ and we have

$$\overline{\Delta}^{\mathcal{F}_2} = 2^{\frac{1}{\alpha_1}}, \tag{3.7}$$

as given in Theorem 3.2. When $\alpha_2 < \alpha_1 < 1$, the argument of the infimum in (3.6) is a strictly convex function in x and the minimum can be obtained setting the first derivative (with respect to x) equal to 0, for $\alpha_1 \geqslant 1$ a more detailed analysis is required. In Figures 1-4 we report the graphs of the quantity

$$\inf_{x\in[0,1-p]} \frac{(1-p-x)^{-\frac{1}{\alpha_1}} + x^{-\frac{1}{\alpha_2}}}{(1-p)^{\frac{1}{\alpha_1}} + (1-p)^{\frac{1}{\alpha_2}}},$$

as a function of p for different values of the parameters α_1, α_2. For $\alpha_1 > \alpha_2$, $F_1^{-1}(t)/F_2^{-1}(t) \to 0$ as $t \to 1^-$, and we obtain

$$\sup_{\mathbf{X}\sim\mathcal{F}_2} \limsup_{p\to 1^-} \Delta_p^{\mathbf{X}} = \sup_{Y\sim\mathcal{F}_1} \limsup_{p\to 1^-} \frac{q_p(Y)}{q_p(Y)} = 1.$$

It is interesting to note that the closer α_1, α_2 are, the slower is the convergence to 1.

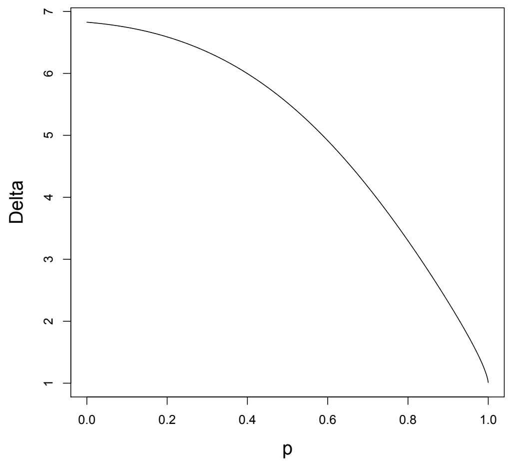

Figure 1: $\alpha_1 = 1/2, \alpha_2 = 1/4$

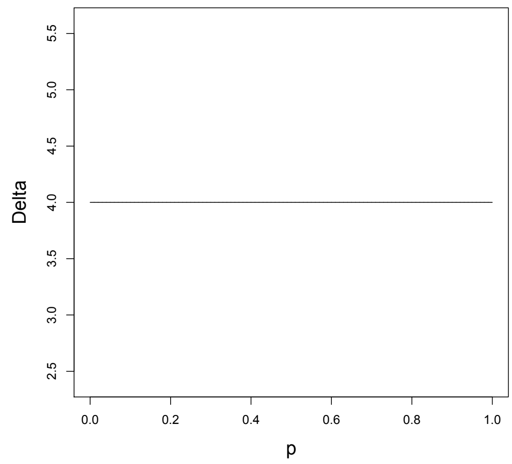

Figure 2: $\alpha_1 = 1/2, \alpha_2 = 1/2$

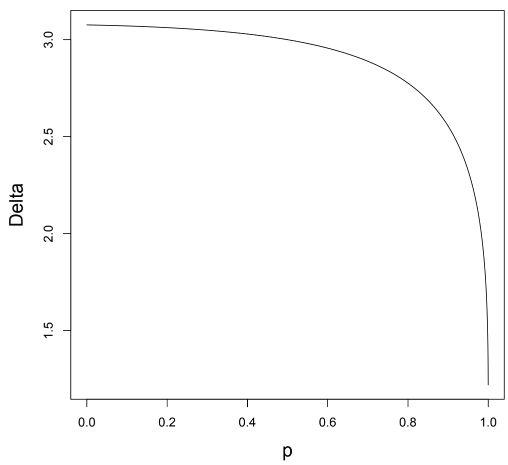

Figure 3: $\alpha_1 = 1/2, \alpha_2 = 3/4$

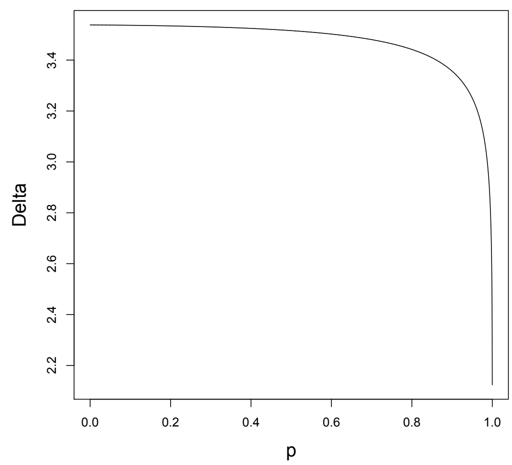

Figure 4: $\alpha_1 = 1/2, \alpha_2 = 3/5$

3.3 Main result in the general setting

We show that $\overline{\Delta}^{\mathcal{F}_d}$ defined by (1.4) only depends on the tail index of regular variation. Specifically, we have the following result.

Theorem 3.3. *Assume that $F_1, \ldots, F_d$, $d \geqslant 3$, are asymptotically equivalent regularly varying distribution functions with index $\beta > 0$ and let $\overline{\Delta}^{\mathcal{F}_d}$ be defined by (1.4). We have that*

$$\overline{\Delta}^{\mathcal{F}_d} = \frac{1}{d} x_d^{-\frac{1}{\beta}} + \frac{d-1}{d}(1 - x_d(d-1))^{-\frac{1}{\beta}}, \tag{3.8}$$

where x_d is the unique solution in $(0, 1/d)$ to the following equation:

$$\frac{1}{1-dx}\int_{(d-1)x}^{1-x}(1-u)^{-\frac{1}{\beta}}\mathrm{d}u = \frac{1}{d}x^{-\frac{1}{\beta}} + \frac{d-1}{d}(1-(d-1)x)^{-\frac{1}{\beta}}. \tag{3.9}$$

When $\beta = 1$, LHS of (3.9) is understood as its limit $\frac{d}{1-dx}\log\frac{1-(d-1)x}{x}$.

Proof. We prove the result in two steps. First we show that it holds in the homogeneous case under the constraint that $F_1 = \cdots = F_d =: F$ has monotone density. Then we extend it to the general case. Before that, we verify a simple fact and give some notation.

(1) The limit of (3.8) is bounded by $d^{1/\beta}$, since

$$\begin{aligned}\overline{\Delta}^{\mathcal{F}_d} &= \limsup_{p\to 1^-}\sup_{\mathbf{X}\sim\mathcal{F}_d}\frac{q_p\left(\sum_{i=1}^d X_i\right)}{\sum_{i=1}^d q_p(X_i)}\\ &\leqslant \limsup_{p\to 1^-}\frac{\sum_{i=1}^d q_{1-(1-p)/d}(X_i)}{\sum_{i=1}^d q_p(X_i)} = d^{1/\beta}.\end{aligned} \tag{3.10}$$

See (A.1) for the details of the proof of the inequality.

(2) For simplicity, denote

$$\Delta_\beta = \frac{1}{d}x_d^{-\frac{1}{\beta}} + \frac{d-1}{d}(1-x_d(d-1))^{-\frac{1}{\beta}}$$

and

$$H(x) := \frac{1}{1-dx}\int_{(d-1)x}^{1-x}(1-u)^{-\frac{1}{\beta}}\mathrm{d}u - \frac{1}{d}x^{-\frac{1}{\beta}} - \frac{d-1}{d}(1-(d-1)x)^{-\frac{1}{\beta}}.$$

With the definition of H, (3.9) reads as $H(x) = 0$.

Step 1. Suppose that $F_1 = \cdots = F_d =: F$ such that F has monotone density. For $p \in (0,1)$, let F_p denote the distribution function of $[F^{-1}(U)|U > p]$ with $U \sim \mathrm{U}[0,1]$ and let $\hat{x}_{d,p}$ be the unique solution to the following equation:

$$\int_{(d-1)x}^{1-x} F_p^{-1}(u)\mathrm{d}u = \frac{d-1}{d}F_p^{-1}((d-1)x) + \frac{1}{d}F_p^{-1}(1-x). \tag{3.11}$$

We first recall some results in Wang et al. (2013). It is shown that there exists $\mathbf{U}^p = (U_1^p, \dots, U_d^p) \sim \mathcal{F}_{\mathrm{U}}^{[p,1]}$, with $\mathcal{F}_{\mathrm{U}}^{[p,1]}$ defined in (2.11), such that

$$\begin{cases} \sum_{i=1}^d F^{-1}(U_i^p) = d\int_{(d-1)\hat{x}_{d,p}}^{1-\hat{x}_{d,p}} F_p^{-1}(u)\mathrm{d}u, & \text{on } A_0, \\ U_j^p = p + (d-1)(1-U_i^p),\ j \neq i, & \text{on } A_i, \quad i = 1, \dots, d, \end{cases} \tag{3.12}$$

where

$$A_0 = \bigcap_{i=1}^d \left\{ p + (1-p)(d-1)\hat{x}_{d,p} \leqslant U_i^p \leqslant 1 - (1-p)\hat{x}_{d,p} \right\},$$

and

$$A_i = \{U_i^p > 1 - (1-p)\hat{x}_{d,p}\}, \quad i = 1, \dots, d.$$

With this construction, we have that

$$A_0 = \bigcup_{i=1}^d \left\{ p + (1-p)(d-1)\hat{x}_{d,p} \leqslant U_i^p \leqslant 1 - (1-p)\hat{x}_{d,p} \right\},$$

and $\mathbf{X}^p$ attains the supremum of $\overline{\Delta}_p^{\mathcal{F}_d}$ with $\mathbf{X}^p = (F^{-1}(U_1^p), \dots, F^{-1}(U_d^p))$, that is,

$$\sup_{\mathbf{X} \sim \mathcal{F}_d} q_p \left(\sum_{i=1}^d X_i \right) = q_p \left(\sum_{i=1}^d X_i^p \right) = d \int_{(d-1)\hat{x}_d}^{1-\hat{x}_d} F_p^{-1}(u)\mathrm{d}u,$$

and it equals $\sum_{i=1}^d F^{-1}(U_i^p)$ on A_0. Details of the above results can be found in Wang et al. (2013, Section 3). It follows that

$$\overline{\Delta}^{\mathcal{F}_d} = \limsup_{p \to 1^-} \frac{\int_{(d-1)\hat{x}_{d,p}}^{1-\hat{x}_{d,p}} F_p^{-1}(u)\mathrm{d}u}{F^{-1}(p)} = \limsup_{p \to 1^-} \frac{(d-1)F_p^{-1}((d-1)\hat{x}_{d,p}) + F_p^{-1}(1-\hat{x}_{d,p})}{dF^{-1}(p)}. \tag{3.13}$$

We assert that there exists $\delta > 0$ such that for large enough p, $\hat{x}_{d,p} \in (\delta, 1/d)$. Or else, there exists a sequence $p_n \to 1^-$ such that $\hat{x}_{d,p_n} \to 0^+$ as $n \to \infty$. In this case, the upper limit in RHS of (3.13) is infinity, which conflicts with (3.10). By Potter's inequalities (see de Haan and Ferreira, 2006, Proposition B.1.9 (5)), we have that for any $\varepsilon > 0$, there exits $p_\varepsilon < 1$ such that

$$|H_p(\hat{x}_{d,p}) - H(\hat{x}_{d,p})| = |H(\hat{x}_{d,p})| < \varepsilon \ \text{ for all } p > p_\varepsilon,$$

where

$$H_p(x) := \int_{(d-1)x}^{1-x} \frac{F_p^{-1}(u)}{F^{-1}(p)}\mathrm{d}u - \frac{(d-1)F_p^{-1}((d-1)x) - F_p^{-1}(1-x)}{dF^{-1}(p)}.$$

Then we have that

$$|H(\hat{x}_{d,p}) - H(x_d)| = |H(\hat{x}_{d,p})| < \varepsilon \ \text{ for all } p > p_\varepsilon,$$

where x_d is the unique solution to the equation (3.9). It follows that $\hat{x}_{d,p} \to x_d$ as $p \to 1^-$. Finally by (3.13), we have that

$$\overline{\Delta}^{\mathcal{F}_d} = \Delta_\beta.$$

Step 2. The task is to show that (3.8) holds for general asymptotically equivalent regularly varying functions $F_1, \ldots, F_d$. By de Haan and Ferreira (2006, Proposition B.1.9 (3)), we can find a distribution function F with monotone density such that F is asymptotically equivalent to F_1, and hence also to $F_2, \ldots, F_d$. For $p \in (0,1)$, let $(U_1^p, \ldots, U_d^p) \sim \mathcal{F}_{\mathrm{U}}^{[p,1]}$ satisfy (3.12) and let $X_i^p = F_i^{-1}(U_i^p)$, $i = 1, \ldots, d$. Note that $\{U_i^p \geqslant p\} = \cup_{j=0}^d A_j$ for $i = 1, \ldots, d$. It follows that

$$q_p\left(\sum_{i=1}^d X_i^p\right) = \inf_{0 \leqslant j \leqslant d} \text{ess-inf} \sum_{i=1}^d [X_i^p | A_j]. \tag{3.14}$$

First, it is easy to see that there exists $(u_1^p, \ldots, u_d^p)$ in the support of the distribution function of $[(U_1^p, \ldots, U_d^p)|A_0]$ such that

$$\text{ess-inf} \sum_{i=1}^d [X_i^p | A_0] = \sum_{i=1}^d F_i^{-1}(u_i^p).$$

Note that $p + (d-1)x_d \leqslant U_i^p \leqslant 1 - x_d(1-p)$ almost surely on A_0 for $i = 1, \ldots, d$. Then by the uniform convergence of regular variation on any compact set bounded away from zero, it holds that

$$\begin{aligned}
\limsup_{p \to 1^-} \frac{\text{ess-inf} \sum_{i=1}^d [X_i^p | A_0]}{\sum_{i=1}^d F_i^{-1}(p)} &= \limsup_{p \to 1^-} \frac{\sum_{i=1}^d F_i^{-1}(u_i^p)}{\sum_{i=1}^d F_i^{-1}(p)} \\
&= \limsup_{p \to 1^-} \frac{1}{d} \sum_{i=1}^d \left(\frac{1 - u_i^p}{1-p}\right)^{-1/\beta} \\
&= \limsup_{p \to 1^-} \frac{\text{ess-inf} \sum_{i=1}^d [F^{-1}(U_i^p) | A_0]}{dF^{-1}(p)} \qquad (3.15) \\
&= \limsup_{p \to 1^-} \frac{q_p\left(\sum_{i=1}^d F^{-1}(U_i^p)\right)}{dF^{-1}(p)}. \qquad (3.16)
\end{aligned}$$

The equality (3.15) follows from that $\sum_{i=1}^d F^{-1}(U_i^p)$ is a constant almost surely on A_0, and hence its essential infimum is equal to its value at any point in its support. The equality (3.16) follows from the construction in (3.12).

Second, on A_j for $j = 1, \ldots, d$, from the structure of $(U_1^p, \ldots, U_d^p)$, we have that

$$\text{ess-inf} \sum_{i=1}^d [X_i^p | A_j] = \inf_{0 < y \leqslant \hat{x}_{d,p}} \left\{ F_j^{-1}(1 - y(1-p)) + \sum_{i \neq j} F_i^{-1}(p + (d-1)(1-p)y) \right\}.$$

Since $F^{-1}(1)=\infty$, for each $p\in(0,1)$, the infimum is attained at $y_p\in(0,\hat{x}_{d,p}]$. Then we have that

$$\text{ess-inf}\sum_{i=1}^{d}[X_i^p|A_j]=F_j^{-1}(1-y_p(1-p))+\sum_{i\neq j}F_i^{-1}(p+(d-1)y_p(1-p)).$$

By definition of regularly varying functions, we have that

$$\begin{aligned}\limsup_{p\to1^-}\frac{\text{ess-inf}\sum_{i=1}^{d}[X_i^p|A_j]}{\sum_{i=1}^{d}F_i^{-1}(p)}&=\limsup_{p\to1^-}\frac{1}{d}\left(y_p^{-\frac{1}{\beta}}+\sum_{i\neq j}(1-(d-1)y_p)^{-\frac{1}{\beta}}\right)\\&\geqslant\limsup_{p\to1^-}\frac{1}{d}\left(x_d^{-\frac{1}{\beta}}+(d-1)\left(1-(d-1)x_d\right)^{-\frac{1}{\beta}}\right)\\&=\limsup_{p\to1^-}\frac{q_p\left(\sum_{i=1}^{d}F^{-1}(U_i^p)\right)}{dF^{-1}(p)}.\end{aligned}\tag{3.17}$$

The inequality follows from the fact that $y^{-\frac{1}{\beta}}+(d-1)\left(1-(d-1)y\right)^{-\frac{1}{\beta}}$ is decreasing in $y\in(0,x_d)$. Substituting (3.16) and (3.17) into (3.14) yields that

$$\limsup_{p\to1^-}\frac{q_p\left(\sum_{i=1}^{d}X_i^p\right)}{\sum_{i=1}^{d}q_p\left(X_i\right)}=\Delta_\beta.$$

This means that

$$\overline{\Delta}^{\mathcal{F}_d}\geqslant\Delta_\beta.\tag{3.18}$$

On the other hand, from Remark 2.3 (i), there exists a sequence $p_n\to1^-$ and $\mathbf{U}=(U_1,\dots,U_d)\in\bigcap_{n\in\mathbb{N}}\mathcal{F}_{\mathrm{U}}^{[p_n,1]}$, such that

$$\limsup_{p\to1^-}\Delta_p^{\mathcal{F}_d}=\lim_{n\to\infty}\frac{q_{p_n}(\sum_{i=1}^{d}F_i^{-1}(U_i))}{\sum_{i=1}^{d}F_i^{-1}(p_n)}=\lim_{n\to\infty}\frac{\text{ess-inf}\{\sum_{i=1}^{d}F_i^{-1}(U_i)|U_i\geqslant1-p_n\}}{\sum_{i=1}^{d}F_i^{-1}(p_n)}.$$

Then there exists $(p_n+\lambda_{n1}^0,\dots,p_n+\lambda_{nd}^0)$ in the support of the joint distribution of $[(U_1,\dots,U_d)|p_n\leqslant U_i<p_{n+1},i=1,\dots,d]$ such that

$$\text{ess-inf}\left\{\sum_{i=1}^{d}F_i^{-1}(U_i)|U_i\geqslant1-p_n\right\}=\sum_{i=1}^{d}F_i^{-1}(p_n+\lambda_{ni}^0),\tag{3.19}$$

and hence

$$\limsup_{p\to1^-}\Delta_p^{\mathcal{F}_d}=\lim_{n\to\infty}\frac{\sum_{i=1}^{d}F_i^{-1}(p_n+\lambda_{ni}^0)}{\sum_{i=1}^{d}F_i^{-1}(p_n)}=\lim_{n\to\infty}\left(\frac{1-p_n-\lambda_{ni}^0}{1-p_n}\right)^{-1/\beta}.$$

For any sequence of points $(p_n+\lambda_{n1},\dots,p_n+\lambda_{nd})$ in the support of the joint distribution function of $[(U_1,\dots,U_d)|p_n\leqslant U_i<p_{n+1},i=1,\dots,d]$, we have, from (3.19), that

$$\liminf_{n\to\infty}\frac{\sum_{i=1}^{d}F_i^{-1}(p_n+\lambda_{ni})}{\sum_{i=1}^{d}F_i^{-1}(p_n)}\geqslant\lim_{n\to\infty}\left(\frac{1-p_n-\lambda_{ni}^0}{1-p_n}\right)^{-1/\beta}.\tag{3.20}$$

We assert that

$$\lim_{n\to\infty} \frac{q_{p_n}(\sum_{i=1}^d F^{-1}(U_i))}{\sum_{i=1}^d F^{-1}(p_n)} \geqslant \lim_{n\to\infty} \left(\frac{1-p_n-\lambda_{ni}^0}{1-p_n}\right)^{-1/\beta}. \tag{3.21}$$

If (3.21) does not hold, from Remark 2.3 (i) again, there exists a sequence of points $(p_n + \lambda_{n1}^*, \ldots, p_n + \lambda_{nd}^*)$ in the support of the joint distribution function of $[(U_1, \ldots, U_d)|p_n \leqslant U_i < p_{n+1}, i = 1, \ldots, d]$ such that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^d F^{-1}(p_n+\lambda_{ni}^*)}{\sum_{i=1}^d F^{-1}(p_n)} < \lim_{n\to\infty} \left(\frac{1-p_n-\lambda_{ni}^0}{1-p_n}\right)^{-1/\beta}. \tag{3.22}$$

Then we have that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^d F^{-1}(p_n+\lambda_{ni}^*)}{\sum_{i=1}^d F^{-1}(p_n)} = \lim_{n\to\infty} \sum_{i=1}^d \left(\frac{1-p_n-\lambda_{ni}^*}{1-p_n}\right)^{-1/\beta} = \lim_{n\to\infty} \frac{\sum_{i=1}^d F_i^{-1}(p_n+\lambda_{ni}^*)}{\sum_{i=1}^d F_i^{-1}(p_n)},$$

which conflicts with (3.20) and (3.22). Hence, (3.21) holds, that is,

$$\limsup_{p\to 1^-} \Delta_p^{\mathcal{F}_d} \leqslant \limsup_{p\to 1^-} \Delta_p^{\mathcal{F}_d(F,\ldots,F)} = \Delta_\beta. \tag{3.23}$$

Combining (3.18) and (3.23) completes the proof.

$\square$

Remark 3.2. (i) Equations of the type (3.9) are used in Wang and Wang (2011) and Wang et al. (2013) to calculate the value of $\sup\{q_p(S_d) : \mathbf{X} \in \mathcal{F}_d\}$ for a fixed p.

(ii) If $\beta = 1/2$, (3.9) has an explicit solution $x_d = (d-1)^{-1}/2$ for $d \geqslant 3$. In this case, Theorem 3.3 gives the explicit value $\overline{\Delta}^{\mathcal{F}_d} = 4(d-1)$. Note that this is also true for $d = 2$ by Theorem 3.2.

From the proof of Theorem 3.3 and the upper bound established by Barbe et al. (2006) and Embrechts et al. (2009a), we have the following corollary immediately.

Corollary 3.4. *Under the assumptions of Theorem 3.3, we have that*

$$d^{1/\beta-1} < \overline{\Delta}^{\mathcal{F}_d} \leqslant d^{1/\beta}, \quad d \in \mathbb{N}. \tag{3.24}$$

Note that although (3.8) in Theorem 3.3 is not in an explicit form, it can be calculated easily via standard numerical methods. Below we compare the worst-case diversification limits with the upper bound of the diversification ratio for MRV in Embrechts et al. (2009a) for several choices of parameters d and β. The numerical values are obtained via the command `uniroot` in `R`. Note that in the case $\beta = 1/2$ the numerical values are slightly different from the explicit values in Remark 3.2 (ii) due to computational errors in high dimensions ($d = 50$ and $d = 100$). The results are reported in Table 1.

	$\beta = 0.95$		$\beta = 0.8$		$\beta = 0.5$		$\beta = 0.3$	
	$\overline{\Delta}^{\mathcal{F}_d}$	$\lambda_{d,\beta}$	$\overline{\Delta}^{\mathcal{F}_d}$	$\lambda_{d,\beta}$	$\overline{\Delta}^{\mathcal{F}_d}$	$\lambda_{d,\beta}$	$\overline{\Delta}^{\mathcal{F}_d}$	$\lambda_{d,\beta}$
$d = 2$	2.07	2.00	2.38	2.00	4.00	2.00	10.09	2.00
$d = 3$	2.90	2.74	3.58	2.72	8.00	2.67	34.01	2.62
$d = 5$	3.85	3.53	5.14	3.43	16.00	3.20	128.84	3.01
$d = 10$	5.05	4.47	7.45	4.19	36.00	3.60	692.06	3.21
$d = 50$	7.74	6.30	14.30	5.38	195.93	3.92	30282.94	3.29
$d = 100$	9.00	7.06	18.31	5.79	399.01	3.99	151694.9	3.27

Table 1: Numerical results for $\overline{\Delta}^{\mathcal{F}_d}$ and the ratio $\lambda_{d,\beta} = \overline{\Delta}^{\mathcal{F}_d}/(d^{1/\beta-1})$

From Table 1, we can see that as β becomes smaller, i.e., the tail of the marginal distributions becomes heavier, the worst-case diversification ratio $\overline{\Delta}^{\mathcal{F}_d}$, i.e., the impact of dependence uncertainty, becomes larger. However, the difference between the worst-case diversification ratio $\overline{\Delta}^{\mathcal{F}_d}$ and the upper bound of the diversification ratio for MRV ($d^{1/\beta-1}$) reduces. In the meanwhile, as d increases, both the impact of dependence uncertainty and the ratio between the worst-case diversification ratio $\overline{\Delta}^{\mathcal{F}_d}$ and the upper bound of the diversification ratio for MRV ($d^{1/\beta-1}$) become more significant.

3.4 Relevance of $\overline{\Delta}^{\mathcal{F}_d}$ on d and β

In this section, we study the relevance of $\overline{\Delta}^{\mathcal{F}_d}$ defined by (1.4) with respect to d and β when F_i, $i = 1, \dots, d$, are asymptotically equivalent and regularly varying with index $\beta > 0$. For a study of the relevance of $\Delta_p^{\mathbf{X}}$ with respect to β when the dependence structure of $\mathbf{X}$ is modelled by Archimedean copulas, see Embrechts et al. (2009b).

Proposition 3.5. *Assume that $F_1, \dots, F_d$, $d \geqslant 3$, are asymptotically equivalent regularly varying distribution functions with index $\beta > 0$ and let $\overline{\Delta}^{\mathcal{F}_d}$ be defined by (1.4). We have*

(i) For a fixed β, $\overline{\Delta}^{\mathcal{F}_d}$ is increasing in $d \in \mathbb{N}$ and satisfies

$$\lim_{d\to\infty} \overline{\Delta}^{\mathcal{F}_d} = \begin{cases} \frac{\beta}{\beta-1}, & \beta > 1, \\ \infty, & 0 < \beta \leqslant 1. \end{cases}$$

Moreover, for $\beta < 1$, we have that

$$\lim_{d\to\infty} \frac{\overline{\Delta}^{\mathcal{F}_d}}{d^{1/\beta-1}} = \left(\frac{1}{1-\beta}\right)^{\frac{1}{\beta}}. \tag{3.25}$$

(ii) For a fixed d, $\overline{\Delta}^{\mathcal{F}_d}$ is decreasing in $\beta \in (0,\infty)$, and satisfies $\lim_{\beta\to 0^+} \overline{\Delta}^{\mathcal{F}_d} = \infty$ and $\lim_{\beta\to\infty} \overline{\Delta}^{\mathcal{F}_d} = 1$.

Proof. (i) Let x_d be the solution to (3.9) and let $y_d = (1-(d-1)x_d)/x_d > 1$ for $d \in \mathbb{N}$. We assert that y_d is decreasing in $d \in \mathbb{N}$. To see it, for $d \in \mathbb{N}$, define $h_d : (1,\infty) \to \mathbb{R}$ as

$$h_d(y) = \frac{\beta}{\beta-1}\frac{y^{-1/\beta+1}-1}{y-1} - \frac{1}{d} - \frac{d-1}{d}y^{-1/\beta}.$$

It can be verified by taking derivatives of h with respect to d and y that $h_d(y)$ is increasing in $d \in \mathbb{N}$ and decreasing in $y \in (1,\infty)$. Also, note that (3.9) is in fact $h_d(y_d) = 0$. It follows that y_d is increasing in $d \in \mathbb{N}$, and hence x_d is decreasing in $d \in \mathbb{N}$. Next, we show that $\overline{\Delta}^{\mathcal{F}_d}$ is increasing in $d \in \mathbb{N}$ in this case. For some $d \in \mathbb{N}$, and for $x \in (0,1/d)$, define

$$\mathrm{H}_x(d) = \frac{1}{d}x^{-\frac{1}{\beta}} + \frac{d-1}{d}(1-x(d-1))^{-\frac{1}{\beta}}, \quad d \in \mathbb{R}_+.$$

It can be easily verified that

$$\frac{\partial \mathrm{H}_x(d)}{\partial d} = \frac{1}{d^2}\left((1-x(d-1))^{-\frac{1}{\beta}} - x^{-\frac{1}{\beta}} + d(d-1)\frac{x}{\beta}(1-x(d-1))^{-\frac{1}{\beta}-1}\right) \geqslant 0.$$

That is, $\mathrm{H}_x(d) \geqslant \mathrm{H}_x(d-1)$ for $x \in (0,1/d)$. Then we have that

$$\overline{\Delta}^{\mathcal{F}_d} = \mathrm{H}_{x_d}(d) \geqslant \mathrm{H}_{x_d}(d-1) \geqslant \mathrm{H}_{x_{d-1}}(d-1) = \overline{\Delta}^{\mathcal{F}_{d-1}},$$

where the second inequality follows from that $\mathrm{H}_x(d)$ is decreasing in x and $x_d < x_{d-1}$. This completes the proof that $\overline{\Delta}^{\mathcal{F}_d}$ is increasing in d.

Note that $\overline{\Delta}^{\mathcal{F}_d}$ is actually the $d^{-1}\sup_{\mathbf{X}\sim\mathcal{F}_d} q_0(S_d)$ where $\mathcal{F}_d$ is the Fréchet class with margins $F_1 = \cdots = F_d$ being the Pareto distribution and given by $F_1(x) = 1 - x^{-\beta}$, $x \geqslant 1$. Let $X \sim F_1$ be a random variable. For any given $M > 0$, $[X|X \leqslant M]$ is d-completely mixable for large enough $d \in \mathbb{N}$ (Wang and Wang, 2011). Thus, we have dx_d goes to 0 as $d \to \infty$. When $\beta > 1$, it is easy to see that $\overline{\Delta}^{\mathcal{F}_d} = \mathbb{E}[F^{-1}(U)|(d-1)x_d \leqslant U \leqslant 1-x_d]$ converges to the mean of X, i.e., $\beta/(\beta-1)$ as $d \to \infty$. For the case that $\beta \leqslant 1$, it suffices to note that $\overline{\Delta}^{\mathcal{F}_d}$ is decreasing in $\beta \in (0,\infty)$ in (ii). To show (3.25), define y_d as above and $\gamma := 1/\beta - 1$. Then the equation $h_d(y_d) = 0$ reduces to

$$\frac{1-y_d^{-\gamma}}{\gamma(y_d-1)} = \frac{1}{d} + \frac{d-1}{d}y_d^{-\gamma-1} = y_d^{-\gamma-1} + \frac{1}{d}\left(1-y_d^{-\gamma-1}\right),$$

i.e.,

$$d = \gamma y_d \frac{1-y_d^{-1}-y_d^{-\gamma-1}+y_d^{-\gamma-2}}{1-(\gamma+1)y_d^{-\gamma}+\gamma y_d^{-\gamma-1}}.$$

Since $y_d \to \infty$ as $d \to \infty$, we have that $\lim_{d\to\infty} y_d/d = 1/\gamma$. It follows that

$$\overline{\Delta}^{\mathcal{F}_d} = (y_d+d-1)^{\gamma+1}\frac{1-y_d^{-\gamma}}{\gamma(y_d-1)}$$

and hence,

$$\lim_{d\to\infty}\frac{\overline{\Delta}^{\mathcal{F}_d}}{d^\gamma}=\lim_{d\to\infty}\left(1+\frac{y_d-1}{d}\right)^{\gamma+1}\frac{d}{\gamma(y_d-1)}(1-y_d^{-\gamma})=\left(1+\frac{1}{\gamma}\right)^{\gamma+1}=\left(\frac{1}{1-\beta}\right)^{1/\beta}.$$

This competes the proof of (3.25).

(ii) We also use the fact that $\overline{\Delta}^{\mathcal{F}_d}$ is actually the $d^{-1}\sup_{\mathbf{X}\sim\mathcal{F}_d}q_0(S_d)$ where $\mathcal{F}_d$ is the Fréchet class with margins $F_1=\cdots=F_d$ being the Pareto distribution and given by $F_1(x)=1-x^{-\beta}$, $x\geqslant 1$. Then the monotonicity in β follows immediately from that $F_1^{-1}(x)$ is decreasing $\beta\in(0,\infty)$ for all x. For $d=2$,

$$\lim_{\beta\to 0^+}\overline{\Delta}^{\mathcal{F}_d}=\lim_{\beta\to 0^+}2^{1/\beta}=\infty.$$

Then by monotonicity in d, we have that $\lim_{\beta\to 0^+}\overline{\Delta}^{\mathcal{F}_d}=\infty$ for all $d\in\mathbb{N}$. By (3.10) in the proof of Theorem 3.3, we have that

$$\lim_{\beta\to\infty}\overline{\Delta}^{\mathcal{F}_d}\leqslant\lim_{\beta\to\infty}d^{1/\beta}=1.$$

This completes the proof.

□

Remark 3.3. (i) It is well-known that for a random variable X having regularly varying tail with index $\beta>0$, then

$$\lim_{p\to 1^-}\frac{\psi_p(X)}{q_p(X)}=\begin{cases}\frac{\beta}{\beta-1}, & \beta>1\\ \infty, & \beta\leqslant 1,\end{cases}$$

which is exactly the limit of Proposition 3.5 (i) as $d\to\infty$. This means that the upper bound of (2.2) is achieved as $d\to\infty$. For fixed $p\in(0,1)$, similar result holds under weak conditions; see Embrechts et al. (2015).

(ii) From Proposition 3.5 (ii), we can find that a more heavy-tailed marginal distribution leads to a higher impact of dependence uncertainty. This is in accordance with the numerical results in Table 1.

We give an example to show that if the margins $F_1,\dots,F_d$ have heavier tailed compared to regularly varying (power-type) distributions, then $\overline{\Delta}^{\mathcal{F}_d}$ defined by (1.4) is possibly infinity.

Example 3.2. Let F be a distribution function given by $F(x)=(1-1/\log x)\mathrm{I}_{\{x\geqslant e\}}$. Then from Corollary 2.8 we have that

$$\overline{\Delta}^{\mathcal{F}_2}=\limsup_{p\to 1^-}\inf_{x\in[0,1-p]}\frac{e^{1/(1-p-x)}+e^{1/x}}{2e^{1/(1-p)}}=\limsup_{p\to 1^-}\frac{2e^{2/(1-p)}}{2e^{1/(1-p)}}=\limsup_{p\to 1^-}e^{1/(1-p)}=\infty.$$

4 Conclusion

In the present paper, we studied the worst-case diversification limit

$$\overline{\Delta}^{\mathcal{F}_d} = \sup_{\mathbf{X}\sim\mathcal{F}_d} \limsup_{p\to 1^-} \frac{q_p(S_d)}{\sum_{i=1}^d q_p(X_i)}$$

under dependence uncertainty. As the main contribution of this paper, we showed that the above two operators can be exchanged, that is

$$\overline{\Delta}^{\mathcal{F}_d} = \limsup_{p\to 1^-} \sup_{\mathbf{X}\sim\mathcal{F}_d} \frac{q_p(S_d)}{\sum_{i=1}^d q_p(X_i)},$$

and the right-hand-side of the above equation can be calculated either numerically or explicitly in most practical cases. Furthermore, the upper bound is sharp, since it can be achieved by a specific random vector. In the case of regularly varying margins, explicit values of $\overline{\Delta}^{\mathcal{F}_d}$ are given.

The main results in the paper tell us that the impact of dependence uncertainty might be substantial, especially when the number of risks is large and the quantile levels are far in right tail, often the case in practice. The literature on EVT already highlighted the relevance of non-subadditivity of VaR when handling heavy-tailed risks, however we showed that under the framework of dependence uncertainty $\overline{\Delta}^{\mathcal{F}_d}$ is significantly larger than the limit of the diversification ratio obtained in models of multivariate regularly varying distributions as shown in Table 1. Beside emphasizing once more the non-subadditivity of VaR, the results in the paper stress that the effort posed in the construction of models in EVT might not be sufficient to characterize the worst non-subadditive behavior, obtained for dependence structures that are generally unknown and difficult to understand. This work hopefully serves as a first brick to connect Extreme Value Theory and Dependence Uncertainty, two popular topics in the recent study of risk aggregation in quantitative risk management.

Acknowledgments Valeria Bignozzi and Bin Wang are grateful to the financial support and kind hospitality provided by the University of Waterloo for their respective visits in 2014 and 2015, when part of this paper was written. Tiantian Mao was supported by the NNSF of China (Nos. 11301500, 11371340) and a postdoctoral fellowship in the Department of Statistics and Actuarial Science at the University of Waterloo. Bin Wang was supported by the NNSF of China (No. 11501017). Ruodu Wang acknowledges support from the Natural Sciences and Engineering Research Council of Canada (NSERC RGPIN-435844-2013).

A Appendix

Proof of Lemma 2.7. First we show that for any $p \in (0,1)$, $W_p(1)$ is finite. To see it, for each $p \in (0,1)$, let $x_i(p) = F_i^{-1}(1-(1-p)/(d+1))$, $i = 1,\dots,d$. Note that for any

$\mathbf{X} \sim \mathcal{F}_d$, we have that

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^d X_i > \sum_{i=1}^d x_i(p)\right) &\leqslant \mathbb{P}\left(\cup_{i=1}^d \{X_i > x_i(p)\}\right) \\
&\leqslant \sum_{i=1}^d \mathbb{P}\left(X_i > x_i(p)\right) \leqslant 1 - \frac{dp}{d+1}.
\end{aligned}$$

It follows that

$$q_p^*\left(\sum_{i=1}^d X_i\right) \leqslant \sum_{i=1}^d x_i(p) \quad \text{for all } \mathbf{X} \sim \mathcal{F}_d. \tag{A.1}$$

On the other hand, note that, by Lemma 2.6, it suffices to show the lemma for the case when $F_i^{-1}(0) \geqslant 0$ for $i = 1, \dots, n$, since $W_p(q)$, $q \in (p, 1]$ do not depend on the values of $F_i^{-1}(u)$, $u \leqslant p$, as long as they are equal to or less than $F_i^{-1}(p)$, $i = 1, \dots, d$. Since $W_p(1)$ is finite and $\lim_{p \to 1^-} q_p(X_i) = \infty$ for $i = 1, \dots, d$, there exists $\delta \in (0, 1-p)$ such that

$$q_{1-2\delta}(X_i) > W_p(1), \quad i = 1, \dots, d. \tag{A.2}$$

Take $\varepsilon \in (0, \delta)$ and $F_i^{(\varepsilon)}$ being the distribution function of $F_i^{-1}(U) 1_{\{U \leqslant 1-\varepsilon\}}$, $i = 1, \dots, d$, $U \sim \mathrm{U}[0, 1]$. Then it holds that

$$q_{u-\varepsilon}^*(X_i) = q_u^*(F_i^{-1}(U) 1_{\{U \leqslant 1-\varepsilon\}}), \quad \forall\, u \in (\varepsilon, 1). \tag{A.3}$$

On the other hand, there exists $\mathbf{U}^* \sim \mathcal{F}_U$ with copula C^* such that

$$W_p(1) \geqslant q_p^*\left(\sum_{i=1}^d F_i^{-1}(U_i^*))\right) > W_p(1) - \varepsilon. \tag{A.4}$$

Note that, by Lemma 2.6,

$$W_p(1-\varepsilon) = \sup_{\mathbf{X} \sim \mathcal{F}^{(\varepsilon)}} q_p^*\left(\sum_{i=1}^d X_i\right) \geqslant q_p^*\left(\sum_{i=1}^d (F_i^{(\varepsilon)})^{-1}(U_i^*)\right).$$

It follows from (A.4) that

$$\begin{aligned}
0 \leqslant W_p(1) - W_p(1-\varepsilon) &\leqslant q_p^*\left(\sum_{i=1}^d F_i^{-1}(U_i^*))\right) - q_p^*\left(\sum_{i=1}^d (F_i^{(\varepsilon)})^{-1}(U_i^*)\right) + \varepsilon \\
&= q_p^*\left(\sum_{i=1}^d F_i^{-1}(U_i^*))\right) - q_p^*\left(\sum_{i=1}^d F_i^{-1}((U_i^* - \varepsilon)_+)\right) + \varepsilon \\
&\leqslant \sup_{u_i \leqslant 1-\delta} \left\{ \left(\sum_{i=1}^d q_{u_i}(X_i)\right) - \left(\sum_{i=1}^d q_{(u_i-\varepsilon)_+}(X_i)\right) \right\} + \varepsilon \\
&\leqslant \sum_{i=1}^d \sup_{0 \leqslant u_i \leqslant 1-\delta} \left(q_{u_i}(X_i) - q_{(u_i-\varepsilon)_+}(X_i)\right) + \varepsilon,
\end{aligned} \tag{A.5}$$

where the second equality and the third inequality follow from (A.3) and (A.2), respectively. The right hand side of (A.5) converges to 0 as $\varepsilon \downarrow 0$ by the monotone convergence theorem and $q_{(\cdot)}(X_i)$, $i = 1, \ldots, d$ are continuous function and hence a uniformly continuous function on $[0, 1-\delta]$. Hence, we have that $W_p(\cdot)$ is left continuous at 1. $\square$

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