Pareto-optimal reinsurance arrangements under general model settings

Jun Cai[*], Haiyan Liu[†] and Ruodu Wang[‡]

Abstract

In this paper, we study Pareto optimality of reinsurance arrangements under general model settings. We give the necessary and sufficient conditions for a reinsurance contract to be Pareto-optimal and characterize all Pareto-optimal reinsurance contracts under more general model assumptions. We also obtain the sufficient conditions that guarantee the existence of the Pareto-optimal reinsurance contracts. When the losses of an insurer and a reinsurer are both measured by the Tail-Value-at-Risk (TVaR) risk measures, we obtain the explicit forms of the Pareto-optimal reinsurance contracts under the expected value premium principle. For the purpose of practice, we use numerical examples to show how to determine the mutually acceptable Pareto-optimal reinsurance contracts among the available Pareto-optimal reinsurance contracts such that both the insurer's aim and the reinsurer's goal can be met under the mutually acceptable Pareto-optimal reinsurance contracts.

Key-words: Pareto optimality, optimal reinsurance, comonotonic-semilinearity, comonotonic-convexity, Tail-Value-at-Risk

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[*]Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, ON N2L3G1, Canada (email: jcai@uwaterloo.ca).

[†]Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, ON N2L3G1, Canada (email: h262liu@uwaterloo.ca).

[‡]Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, ON N2L3G1, Canada (email: wang@uwaterloo.ca).

1 Introduction

Reinsurance, as a type of risk sharing, has been extensively studied in actuarial science. Generally, there are two parties in a reinsurance contract, an insurer and a reinsurer. Suppose that the insurer faces a nonnegative ground-up loss $X \in \mathcal{X}$, where $\mathcal{X}$ is a set of random variables containing all random variables involved in the reinsurance contract. The reinsurer agrees to cover part of the loss X, say $I(X)$, and the insurer will pay a reinsurance premium $\pi(I(X))$ to the reinsurer. The function $I \in \mathcal{I}_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is called the ceded loss function, where $\mathbb{R}_+ = [0, \infty)$ and $\mathcal{I}_0$ is a non-empty set of all feasible reinsurance contracts. With the reinsurance contract (function) I and the premium principle $\pi : \mathcal{X} \to \mathbb{R}$, the loss random variables

$$C_I = C_I(X) = X - I(X) + \pi(I(X)) \qquad \text{and} \qquad R_I = R_I(X) = I(X) - \pi(I(X)) \tag{1.1}$$

represent the risk exposures of the insurer and the reinsurer under the reinsurance contract, respectively. Furthermore, let $\rho_1 : \mathcal{X} \to \mathbb{R}$ and $\rho_2 : \mathcal{X} \to \mathbb{R}$ be the objective functionals of the insurer and the reinsurer, respectively. The functionals describe the preferences of the insurer and the reinsurer. Precisely, the insurer prefers X over Y if and only if $\rho_1(X) \leqslant \rho_1(Y)$, and the reinsurer prefers X over Y if and only if $\rho_2(X) \leqslant \rho_2(Y)$. We call the 5-tuple $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ a *reinsurance setting*. In this setting, the general objective functionals ρ_1 and ρ_2 can be risk measures, variances, and disutility functionals. Moreover, up to a sign change, the objective functionals ρ_1 and ρ_2 can also be mean-variance functionals, expected utilities, rank-dependent expected utilities, and so on.

An optimal reinsurance design under the setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ can be formulated as an optimization problem that tries to find an optimal contract $I^* \in \mathcal{I}_0$ such that an objective function is minimized at I^*. Optimal reinsurance designs from either the insurer's perspective (e.g. $\min_{I \in \mathcal{I}_0} \rho_1(C_I)$) or the reinsurer's point of view (e.g. $\min_{I \in \mathcal{I}_0} \rho_2(R_I)$) have been well investigated in the literature. However, as pointed out by Borch (1969), "there are two parties to a reinsurance contract, and that an arrangement which is very attractive to one party, may be quite unacceptable to the other." Hence, an interesting question in optimal reinsurance designs is to consider both the insurer's preference and the reinsurer's preference. To address this issue, Borch (1960) derived the optimal retentions of the quota-share and stop-loss reinsurances by maximizing the product of the expected utility functions of the two parties' terminal wealth; Hürlimann (2011) obtained the optimal retentions of the combined quota-share and stop-loss reinsurances by minimizing the sum of the variances of the losses of the insurer and the reinsurer; Cai et al. (2013) proposed the joint survival and profitable probabilities of an insurer and a reinsurer as optimization criteria to determine optimal reinsurances; Cai et al. (2016) developed optimal reinsurances that minimize the convex combination of the VaRs of the losses of an insurer and a reinsurer under certain constraints; and Lo (2017a) discussed the generalized problems of Cai et al. (2016) by using the Neyman-Pearson approach.

Obviously, an insurer and a reinsurer have conflicting interests in a reinsurance contract. A celebrated economic concept used in optimal decision problems with conflicting interests is Pareto optimality, which has been well studied under various settings in insurance and risk management. For instance, Gerber (1978) discussed Pareto-optimal risk exchanges and Golubin (2006) studied Pareto-optimal insurance policies when both the insurer and the reinsurer are risk averse. In addition, Pareto-optimality in risk sharing with different risk measures can be found in Jouini et al. (2008), Filipović and Svindland (2008), Embrechts et al. (2016), and references therein. Most of the existing results in optimal risk sharing/exchange can not be used to determine optimal reinsurance contracts since the model settings for reinsurance designs are usually different from the ones for risk sharing problems. In particular, a reinsurance setting often has practical constraints such as the constraint that the shared risks should be non-negative and comonotonic or the condition that the risk measure of the insurer's loss is not larger than a given value, or the requirement that the expected net profit of an reinsurer is not less than a given amount or the restriction that the reinsurance premium is not bigger than an insurer's budget.

In this paper, we will use the concept of Pareto-optimality to study Pareto-optimal reinsurance contracts under a general reinsurance setting $(X,\rho_1,\rho_2,\pi,\mathcal{I}_0)$. Generally speaking, a Pareto-optimal reinsurance policy is one in which neither of the two parties can be better off without making the other worse off and it can be defined mathematically as follows.

Definition 1.1. Let $(X,\rho_1,\rho_2,\pi,\mathcal{I}_0)$ be a reinsurance setting. A reinsurance contract $I^* \in \mathcal{I}_0$ is called *Pareto-optimal* under $(X,\rho_1,\rho_2,\pi,\mathcal{I}_0)$, if there is no $I \in \mathcal{I}_0$ such that $\rho_1(C_I) \leqslant \rho_1(C_{I^*})$ and $\rho_2(R_I) \leqslant \rho_2(R_{I^*})$, and at least one of the two inequalities is strict, where C_I and R_I are defined in (1.1).

First, similar to Pareto-optimal problems in other fields such as risk exchanges (e.g. Gerber (1978)) and risk allocations (e.g. Barrieu and Scandolo (2008)), it is easy to see that a Pareto-optimal reinsurance contract exists if there is a contract that minimizes the convex combination of the objective functionals of the insurer and the reinsurer. Indeed, the following proposition gives a sufficient condition for a reinsurance contract to be Pareto-optimal in a general reinsurance setting $(X,\rho_1,\rho_2,\pi,\mathcal{I}_0)$.

Proposition 1.1. *In a reinsurance setting $(X,\rho_1,\rho_2,\pi,\mathcal{I}_0)$, if*

$$I^* \in \arg\min_{I\in\mathcal{I}_0}\{\lambda\rho_1(C_I) + (1-\lambda)\rho_2(R_I)\}, \tag{1.2}$$

for some $\lambda \in (0,1)$, then I^* is a Pareto-optimal reinsurance contract under the setting $(X,\rho_1,\rho_2,\pi,\mathcal{I}_0)$.

Proof. If I^* is not Pareto-optimal, then there exists an $\hat{I} \in \mathcal{I}_0$ such that $\rho_1(C_{\hat{I}}) \leqslant \rho_1(C_{I^*})$ and $\rho_2(R_{\hat{I}}) \leqslant \rho_2(R_{I^*})$, and at least one of the two inequalities is strict. Then $\lambda\rho_1(C_{\hat{I}}) + (1-\lambda)\rho_2(R_{\hat{I}}) < \lambda\rho_1(C_{I^*}) + (1-\lambda)\rho_2(R_{I^*})$. Thus, $I^* \notin \arg\min_{I\in\mathcal{I}_0}\{\lambda\rho_1(C_I) + (1-\lambda)\rho_2(R_I)\}$, a contradiction. □

Proposition 1.1 holds without any assumptions on $(X,\rho_1,\rho_2,\pi,\mathcal{I}_0)$. Nevertheless, the minimization problem in (1.2) for $\lambda \in (0,1)$ may have no solutions. Furthermore, the conditions in Proposition 1.1 are not necessary for a Pareto-optimal reinsurance contract. Indeed, there are other Pareto-optimal reinsurance contracts that are not the solutions to the minimization problem $\min_{I\in\mathcal{I}_0}\{\lambda\rho_1(C_I) + (1-\lambda)\rho_2(R_I)\}$ for $\lambda \in (0,1)$. In fact, as shown in Theorem 2.1 of this paper, under certain assumptions on $(X,\rho_1,\rho_2,\pi,\mathcal{I}_0)$, Pareto-optimal reinsurance contracts also exist in the solutions to the minimization problems $\min_{I\in\mathcal{I}_0}\{\rho_1(C_I)\}$ and $\min_{I\in\mathcal{I}_0}\{\rho_2(R_I)\}$, and all Pareto-optimal reinsurance contracts are included in the solutions to the minimization problem

$$\min_{I\in\mathcal{I}_0}\{\lambda\rho_1(C_I)+(1-\lambda)\rho_2(R_I)\},\quad \lambda\in[0,1]. \tag{1.3}$$

Therefore, the key to find Pareto-optimal reinsurance contracts is to solve the problem (1.3). Theorem 2.4 of this paper establishes the sufficient conditions that guarantee the existence of the solutions to the problem (1.3) or for the existence of Pareto-optimal reinsurance contracts under the setting $(X,\rho_1,\rho_2,\pi,\mathcal{I}_0)$.

The problem (1.3) itself is also of interest. Mathematically, when $\lambda=1$ and $\lambda=0$, the problem (1.3) is reduced to the problems of finding the optimal reinsurance contracts that minimize an insurer's objective functional and a reinsurance's objective functional, respectively. In addition, from an economical point of view, if the reinsurer is designing a contract based on the solutions to the problem (1.3), such a contract will be more attractive to the insurer than ones designed based on the solutions to the problem $\min_{I\in\mathcal{I}_0}\rho_2(R_I)$. On the other hand, if the insurer is asking the reinsurer to sell a contract based on the solutions to the problem (1.3), the reinsurer is more willing to sell such a contract than ones designed based on the solutions to the problem $\min_{I\in\mathcal{I}_0}\rho_1(C_I)$.

Although Theorem 2.4 of this paper gives the conditions such that the solutions to the problem (1.3) exist, it is not a trivial work to find the solutions to the problem (1.3) even for simple choices of ρ_1, ρ_2, and π. In the literature, many researchers studied the problem (1.3) in the case of $\lambda=0$ or $\lambda=1$ with special choices of ρ_1,ρ_2,π, and $\mathcal{I}_0$. See e.g. Chi and Tan (2011), Bernard and Tian (2009), Cui et al. (2013), Cheung et al. (2014), Cheung and Lo (2015), and Lo (2017b) for minimization of Value-at-Risk (VaR) / Tail-Value-at-Risk (TVaR), tail risk measures, general distortion risk measures, general law-invariant convex risk measures, and insurer's risk–adjusted liability, respectively, Kaluszka and Okolewski (2008) and Cai and Wei (2012) for maximization of the expected utility, and Bernard et al. (2015) for maximization of rank-dependent expected utility. For the problem (1.3) with $\lambda\in[0,1]$, Cai et al. (2016) solved the problem with certain constraints when the functionals ρ_1 and ρ_2 are VaRs; Jiang et al. (2016)) discussed the problem without constraints when the functionals ρ_1 and ρ_2 are VaRs, and Lo (2017a) investigated the problem using the Neyman-Pearson approach. In this paper, we will also solve the problem when the functionals ρ_1 and ρ_2 are TVaRs. Although the approach proposed in Lo (2017a) can solve the problem (1.3) for several special cases, the approach does not work for the problem (1.3) when the functionals ρ_1 and ρ_2 are TVaRs as pointed out in Lo (2017a). In addition, note that there are many Pareto-optimal reinsurance contracts under the setting $(X,\rho_1,\rho_2,\pi,\mathcal{I}_0)$. In this paper, we will also use numerical examples to discuss how to choose the weight λ in (1.2) so that the feasible deals of Pareto-optimal contracts can be made for the practice purpose.

The rest of the paper is organized as follows. In Sections 2, we give the necessary and sufficient conditions for a reinsurance contract to be Pareto-optimal and characterize all Pareto-optimal reinsurance contracts under a more general setting $(X,\rho_1,\rho_2,\pi,\mathcal{I}_0)$. We also obtain the sufficient conditions that guarantee the existence of the solutions to the minimization problem (1.3). In Sections 3, we solve the problem (1.3) explicitly when the functionals ρ_1 and ρ_2 are TVaRs and π is the expected value premium principle. In Section 4, we use two numerical examples to illustrate the solutions derived in Section 3 and discuss how to choose the weight λ in (1.2) to obtain the feasible Pareto-optimal reinsurance contracts for the practice purpose. Some conclusions are drawn in Section 5. Some technical proofs are put in the Appendix.

2 Pareto optimality in reinsurance policy design

2.1 Model assumptions

All random variables in this paper are defined on an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let L^p, $p \in [0, \infty)$, be the set of random variables with finite p-th moment, and L^∞ be the set of essentially bounded random variables. A functional on L^p is said to be L^p-continuous, $p \in [1, \infty]$, if it is continuous with respect to the L^p-norm.

In a reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ with $X \in \mathcal{X}$, to avoid the issue of moral hazard, a reinsurance contract $I \in \mathcal{I}_0$ should satisfy that $I(0) = 0$ and $0 \leqslant I(x) - I(y) \leqslant x - y$ for all $0 \leqslant y \leqslant x$. We denote by $\mathcal{I}$ the set of all contracts that satisfy this property, namely,

$$\mathcal{I} := \{I : \mathbb{R}_+ \to \mathbb{R}_+ \mid I(0) = 0 \text{ and } 0 \leqslant I(x) - I(y) \leqslant x - y, \text{ for all } 0 \leqslant y \leqslant x\}.$$

Hence, in this paper, we have $\mathcal{I}_0 \subset \mathcal{I}$. Note that $\mathcal{I}_0$ does not have to be equal to $\mathcal{I}$. Such a set $\mathcal{I}_0$ can be a finite set of contracts or an infinite set of contracts such as the set of stop-loss contracts, multi-layer contracts, quota-share contracts, or all the contracts in $\mathcal{I}$ with some budget/solvency constraints. Moreover, for any $I \in \mathcal{I}$ and any nonnegative random variable X, the random variables X, $X - I(X)$, and $I(X)$ are comonotonic. Recall that random variables $X_1, \ldots, X_n$ with $n \geqslant 2$ are said to be *comonotonic* if there exist non-decreasing functions $f_1, \ldots, f_n$ and a random variable $Z \in L^0$ such that $X_i = f_i(Z)$ almost surely for $i = 1, \ldots, n$. See Dhaene et al. (2002) for an overview on comonotonicity.

Throughout we let $\mathcal{X}$ be a convex cone of random variables containing L^∞ satisfying $I(X) \in \mathcal{X}_+$ for all $X \in \mathcal{X}_+$ and $I \in \mathcal{I}$, where $\mathcal{X}_+ = \{X \in \mathcal{X} : X \geqslant 0\}$. Note that $\mathcal{X}_+$ is still a convex cone. The set $\mathcal{X}$ is the set of all random losses that are of our interest. In the context of reinsurance, $\mathcal{X}$ may be chosen as L^1, L^∞ or L^0 depending on the specific problems.

For any random loss $X \in \mathcal{X}_+$, a reinsurance contract $I \in \mathcal{I}$, and a premium principle $\pi : \mathcal{X} \to \mathbb{R}$, the two loss random variables C_I and R_I defined in (1.1) are in $\mathcal{X}$, but they may not be in $\mathcal{X}_+$. In particular, if $\pi(I(X)) > 0$, then $C_I \in \mathcal{X}_+$ but R_I may not be in $\mathcal{X}_+$.

For a given $X \in \mathcal{X}$, all random variables involved in a reinsurance contract under the setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$, such as C_I and R_I, are in the set

$$C(X) = \{Y \in \mathcal{X} : Y, X - Y \text{ and } X \text{ are comonotonic}\}. \tag{2.1}$$

Also, we have $I(X) \in C(X)$ for $I \in \mathcal{I}$.

In the following we aim to establish necessary and sufficient conditions for the existence of the Perato-optimal reinsurance policies in a general reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$.

2.2 Necessary and sufficient conditions for Pareto-optimal contracts

To obtain the necessary condition for a reinsurance contract to be Pareto-optimal in a general reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$, we have to make some assumptions on $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$. To do so, we introduce the following definition and notation. A functional ρ is said to be *semilinear* on a set $\mathcal{Y}$ if $\rho(\lambda X + Y) = \lambda\rho(X) + \rho(Y)$ for all $\lambda > 0$, $X, Y \in \mathcal{Y}$. For a reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$, denote

$$K(\lambda) = \underset{I \in \mathcal{I}_0}{\arg\min}\{\lambda\rho_1(C_I) + (1-\lambda)\rho_2(R_I)\}, \quad \lambda \in [0,1], \tag{2.2}$$

$$K^*(0) = \underset{I \in K(0)}{\arg\min}\{\rho_1(C_I)\}, \quad K^*(1) = \underset{I \in K(1)}{\arg\min}\{\rho_2(R_I)\}, \quad \text{and} \quad K^*(\lambda) = K(\lambda), \ \lambda \in (0,1), \tag{2.3}$$

where C_I and R_I are defined in (1.1). Note that $K(0)$ (resp. $K(1)$) is the set of contracts minimizing the objective functional of the reinsurer (resp. insurer) while $K^*(0)$ (resp. $K^*(1)$) is the set of the contracts that are in $K(0)$ (resp. $K(1)$) and minimize the objective functional of the insurer (resp. reinsurer). For $\lambda \in (0,1)$, $K(\lambda)$ is the set of contracts minimizing the convex combination of the objective functionals of the insurer and the reinsurer.

As shown in the following theorem, the sets $K^*(\lambda)$, $\lambda \in [0,1]$, characterize all Pareto-optimal contracts in the reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$. The proof of the following theorem follows the ideas similar to those used in Gerber (1978) for Pareto-optimal risk exchanges.

Theorem 2.1. *Let $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ be a reinsurance setting. If π is semilinear on $\mathcal{C}(X)$, ρ_1, ρ_2 are convex on $\mathcal{C}(X)$, and $\mathcal{I}_0$ is a convex set, then $I^* \in \mathcal{I}_0$ is a Pareto-optimal contract under the setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ if and only if there exists $\lambda \in [0,1]$ such that $I^* \in K^*(\lambda)$, where $K^*(\lambda)$ is defined in (2.3).*

Proof. "$\Longrightarrow$" Define the set $S = \{(\rho_1(C_I), \rho_2(R_I)) : I \in \mathcal{I}_0\} \subset \mathbb{R}^2$. For any set $T \subset \mathbb{R}^2$, we say that $(x^*, y^*) \in T$ is a Pareto-optimal point of T if there is no $(x, y) \in T$ such that $(x, y) \neq (x^*, y^*)$ and $(x, y) \leqslant (x^*, y^*)$; here and below the inequality between vectors are component-wise inequalities. Let $\bar{S}$ be the convex hull of S. The agenda for the proof is the following. (a) First, we verify that for any $(\bar{x}, \bar{y}) \in \bar{S}$, there exists a point $(x, y) \in S$ such that $(x, y) \leqslant (\bar{x}, \bar{y})$. (b) Second, use (a) to show that for any Pareto-optimal point (x^*, y^*) of S, there exists $\lambda \in [0,1]$ such that $(x^*, y^*) \in \arg\min_{(x,y)\in S}\{\lambda x + (1-\lambda)y\}$. (c) Third, use (a) and (b) to prove the necessary conditions for a contract to be Pareto-optimal.

For any $I_1, I_2 \in \mathcal{I}_0$ and $\theta \in [0,1]$, let $I = \theta I_1 + (1-\theta)I_2 \in \mathcal{I}_0$. The convexity of ρ_1 and the semilinearity of π on $\mathcal{C}(X)$ imply

$$\begin{aligned}\theta\rho_1(C_{I_1}) + (1-\theta)\rho_1(C_{I_2}) &\geqslant \rho_1(\theta C_{I_1} + (1-\theta)C_{I_2}) \\ &= \rho_1(X - (\theta I_1(X) + (1-\theta)I_2(X)) + \theta\pi(I_1(X)) + (1-\theta)\pi(I_2(X))) \\ &= \rho_1(X - I(X) + \pi(I(X))).\end{aligned} \tag{2.4}$$

Similarly,

$$\theta\rho_2(R_{I_1}) + (1-\theta)\rho_2(R_{I_2}) \geqslant \rho_2(I(X) - \pi(I(X))). \tag{2.5}$$

Therefore, for any $(\bar{x}, \bar{y}) \in \bar{S}$, there exists $(x, y) = (\rho_1(C_I), \rho_2(R_I)) \in S$ such that $(x, y) \leqslant (\bar{x}, \bar{y})$.

Next we take a Pareto-optimal point (x^*, y^*) of S. If there exists $(\bar{x}, \bar{y}) \in \bar{S}$ such that $(\bar{x}, \bar{y}) \leqslant (x^*, y^*)$ then from the second statement above, we have, there exists $(x, y) \in S$ with $(x, y) \leqslant (\bar{x}, \bar{y})$. From the Pareto-optimality of (x^*, y^*) in S we know $(x, y) = (\bar{x}, \bar{y}) = (x^*, y^*)$. This shows that (x^*, y^*) is a Pareto-optimal point of $\bar{S}$.

Define $T = \{(x, y) \in \mathbb{R}^2 : (x, y) \leqslant (x^*, y^*)\}$. Note that both T and $\bar{S}$ are convex sets, and by the Pareto-optimality of (x^*, y^*) in $\bar{S}$, the interiors of $\bar{S}$ and T are disjoint. By the Hyperplane Separation Theorem (e.g. Theorem 11.3 of Rockafellar (1970)), there exists a vector $(a, b) \in \mathbb{R}^2$, $(a, b) \neq (0, 0)$ such that $\sup_{(x,y)\in T}\{ax + by\} \leqslant \inf_{(x,y)\in\bar{S}}\{ax + by\}$. Note that $\sup_{(x,y)\in T}\{ax + by\} < \infty$ and for any $(x, y) \in T$, we have $(x - 1, y) \in T$. It follows that

$$\sup_{(x,y)\in T}\{ax + by\} \geqslant \sup_{(x,y)\in T}\{a(x - 1) + by\} = \sup_{(x,y)\in T}\{ax + by\} - a,$$

which implies $a \geqslant 0$. Similarly we have $b \geqslant 0$. Therefore $\sup_{(x,y)\in T}\{ax + by\} = ax^* + by^* \leqslant \inf_{(x,y)\in\bar{S}}\{ax + by\}$. This shows that (x^*, y^*) minimizes $ax + by$ over $(x, y) \in \bar{S}$.

Now, suppose that I^* is Pareto-optimal for the reinsurance design problem. Then $(\rho_1(C_{I^*}), \rho_2(R_{I^*}))$ is a Pareto-optimal point of S. The aforementioned arguments suggest that there exist $a, b \geqslant 0$, $a + b > 0$ such that

$$a\rho_1(C_{I^*}) + b\rho_2(R_{I^*}) = \min_{(x,y)\in S}\{ax + by\} = \min_{I\in\mathcal{I}_0}\{a\rho_1(C_I) + b\rho_2(R_I)\}.$$

By setting $\lambda = a/(a + b)$ in the above equation, we conclude that $I^* \in K(\lambda)$ for some $\lambda \in [0, 1]$. If $\lambda \in (0, 1)$, then $K^*(\lambda) = K(\lambda)$ and $I^* \in K^*(\lambda)$. Below suppose $\lambda = 0$ and take any $I \in K(0)$. By the definition of $K(0)$, $\rho_2(R_I) = \rho_2(R_{I^*})$. From the Pareto optimality of I^*, we have $\rho_1(C_{I^*}) \leqslant \rho_1(C_I)$. Therefore, $I^* \in \arg\min_{I\in K(0)}\{\rho_1(C_I)\} = K^*(0)$. The case $\lambda = 1$ is analogous. To summarize, $I^* \in K^*(\lambda)$ for some $\lambda \in [0, 1]$.

" $\Longleftarrow$ " Suppose $I^* \in K(\lambda)$ for some $\lambda \in [0, 1]$. For $\lambda \in (0, 1)$, one can obtain from Proposition 1.1 that I^* is Pareto-optimal. If $\lambda = 0$, take $I \in \mathcal{I}_0$ such that $\rho_1(C_I) \leqslant \rho_1(C_{I^*})$ and $\rho_2(R_I) \leqslant \rho_2(R_{I^*})$. By the definition of $K(0)$ and noting that $I^* \in K(0)$, we have $\rho_2(R_I) = \rho_2(R_{I^*})$, thus $I \in K(0)$. Further, by the definition of $K^*(0)$, we have $\rho_1(C_{I^*}) \leqslant \rho_1(C_I)$. Therefore, $\rho_1(C_I) = \rho_1(C_{I^*})$ and $\rho_2(R_I) = \rho_2(R_{I^*})$. This shows that I^* is Pareto-optimal. □

We point out that the assumptions in Theorem 2.1 are easily satisfied by many functionals of ρ_1 and ρ_2, premium principles of π, and feasible sets of $\mathcal{I}_0$, including many practical choices considered in the literature (see discussions below). In addition, in Theorem 2.1, the functionals ρ_1, ρ_2 and π are assumed to satisfy the corresponding properties on the subset $C(\mathcal{X}) \subset \mathcal{X}$. In fact, in many applications, the specified functionals ρ_1, ρ_2 and π can satisfy the corresponding properties globally or on $\mathcal{X}$. We first give the definitions of comonotonic-semilinearity and comonotonic-convexity for a functional as follows.

Definition 2.1. A functional $\rho : \mathcal{X} \to \mathbb{R}$ is said to be *comonotonic-semilinear* if $\rho(\lambda X+Y) = \lambda\rho(X)+\rho(Y)$ for any comonotonic random variables $X, Y \in \mathcal{X}$ and $\lambda > 0$ and to be *comonotonic-convex* if $\rho(\lambda X + (1-\lambda)Y) \leqslant \lambda\rho(X) + (1-\lambda)\rho(Y)$ for any comonotonic random variables $X, Y \in \mathcal{X}$ and $\lambda \in [0, 1]$.

We point out that the property of comonotonic-convexity has been studied and characterized in Song and Yan (2009). Now, we can reformulate Theorem 2.1 based on the global properties of the functionals below.

Corollary 2.2. *Let $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ be a reinsurance setting. If π is comonotonic-semilinear, ρ_1, ρ_2 are comonotonic-convex, and $\mathcal{I}_0$ is a convex set, then $I^* \in \mathcal{I}_0$ is Pareto-optimal under the setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ if and only if there exists $\lambda \in [0, 1]$ such that $I^* \in K^*(\lambda)$, where K^* is defined in (2.3).*

Proof. Note that $C(X) \subset \mathcal{X}$ and every element in $C(X)$ is comonotonic with X. Hence, the assumptions of Corollary 2.2 imply that the assumptions of Theorem 2.1 hold. □

Below we make a few observations on the conditions assumed in Theorem 2.1 and Corollary 2.2.

(i) Comonotonic-convexity is a weaker property than comonotonic-semilinearity or convexity. If functional ρ is comonotonic-semilinear or convex, then it is comonotonic-convex. This property of comonotonic-convexity can be satisfied by many functionals studied in the literature such as distortion risk measures, convex risk measures, convex functionals including concave expected utilities (up to a sign change), and so on.

(ii) The comonotonic-semilinearity of π is essential to Theorem 2.1 and Corollary 2.2, and it cannot be weakened to comonotonic-convexity. The reason is that $\rho(C_I)$ has a positive term $\pi(I(X))$ while $\rho(R_I)$ has a negative term $-\pi(I(X))$. To obtain both inequalities (2.4) and (2.5), one needs to assume that π has a linear structure in these values. The property of comonotonic-semilinearity can be satisfied by the expected value premiums, Wang's premiums, and others.

(iii) In Theorem 2.1, we assume that the set of contracts $\mathcal{I}_0 \subset \mathcal{I}$ is convex. The convex assumption on $\mathcal{I}_0$ allows us to consider the minimization problem (1.3) with constraints if the constraints form a convex subset of I. Interesting examples of such constraints include $\mathcal{I}_0 = \{I \in \mathcal{I} : \ \rho_1(C_I) \leqslant r\}$, where ρ_1 is a convex risk measure and $r \in \mathbb{R}$ is an acceptable risk level under the risk measure ρ_1 (see Cai et al. (2016) and Lo (2017a)), or $\mathcal{I}_0 = \{I \in \mathcal{I} : \ \pi(I) \leqslant p\}$, where π is a convex premium principle and $p \in \mathbb{R}$ is an acceptable budget for the insurer, or $\mathcal{I}_0 = \{I \in \mathcal{I} : \ \mathbb{E}[\pi(I) - I(X)] \geqslant w\}$, where π is a convex premium principle and $w \in \mathbb{R}$ is an acceptable amount for the reinsurer's expected net profit. Also note that $\mathcal{I}$ itself is a convex set.

2.3 Existence of Pareto-optimal reinsurance contracts

From Theorem 2.1, we know that the sets of contracts $K^*(\lambda)$, $\lambda \in [0, 1]$, characterize all Pareto-optimal contracts in a reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$. However, we do not know whether $K^*(\lambda)$ is non-empty. In this section, we show that under the assumptions of compactness of the reinsurance contract set $\mathcal{I}_0$ and (lower-)continuity of the objective functionals, the minimization problem (1.3) has solutions (minimizers), or equivalently, $K^*(\lambda)$ is not empty, for each $\lambda \in [0,1]$.

We first provide a technical lemma. In the following, we say that a set of functions is *pw-closed* if it is closed with respect to point-wise convergence. Note that $\mathcal{I}$ is pw-closed.

Lemma 2.3. *Let $\mathcal{I}_0 \subset \mathcal{I}$ be pw-closed. For any sequence $\{I_n,\, n \in \mathbb{N}\} \subset \mathcal{I}_0$, there exists a subsequence $\{I_{n_k},\, k \in \mathbb{N}\}$ pointwise converging to an $I^* \in \mathcal{I}_0$.*

Proof. Define $\mathcal{G} = \left\{g : [0,\infty) \to [0,1) \mid g(\cdot) = 1 - \frac{1}{I(\cdot)+1},\ I \in \mathcal{I}_0\right\}$. Since any $I \in \mathcal{I}$ is continuous and increasing, so is any $g \in \mathcal{G}$. For any sequence $\{I_n,\, n \in \mathbb{N}\} \subset \mathcal{I}_0$, $\{g_n := 1 - \frac{1}{I_n+1},\, n \in \mathbb{N}\} \subset \mathcal{G}$ is uniformly bounded and by Helly's theorem (see e.g. Klenke (2013)), there exists a function g^* and a subsequence $\{g_{n_k},\, k \in \mathbb{N}\}$ such that $\{g_{n_k},\, k \in \mathbb{N}\}$ pointwise converges to g^*. For any $x \in [0,\infty)$, $I_{n_k}(x) \leqslant x$ and $0 \leqslant g_{n_k}(x) \leqslant 1 - \frac{1}{x+1} < 1$, therefore $\{I_{n_k}(x) = \frac{1}{1-g_{n_k}(x)} - 1,\, k \in \mathbb{N}\}$ converges to $I^*(x) := \frac{1}{1-g^*(x)} - 1$. Since $\mathcal{I}_0$ is closed with respect to pointwise convergence, we have $I^* \in \mathcal{I}_0$. Therefore, there exists a subsequence $\{I_{n_k},\, k \in \mathbb{N}\} \subset \mathcal{I}_0$ pointwise converging to I^*. □

Furthermore, we say that a functional ρ is *as-continuous* on a set $\mathcal{Y} \subset L^0$ if ρ is continuous with respect to almost sure convergence for sequences in $\mathcal{Y}$. We say that a reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ is *proper* if $\inf_{I \in \mathcal{I}_0} \rho_1(C_I) > -\infty$ and $\inf_{I \in \mathcal{I}_0} \rho_2(R_I) > -\infty$. One can easily verify that if ρ_1, ρ_2, π are non-decreasing functionals on $C(X)$, then $\rho_1(C_I) \geqslant \rho_1(\pi(0))$ and $\rho_2(R_I) \geqslant \rho_2(-\pi(X))$, thus both $\rho_1(C_I)$ and $\rho_2(R_I)$ are bounded from below, and hence the corresponding reinsurance setting is proper.

Theorem 2.4. *Let $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ be a proper reinsurance setting. If π, ρ_1, ρ_2 are as-continuous on $C(X)$ and $\mathcal{I}_0$ is pw-closed, then $K^*(\lambda)$ is non-empty for each $\lambda \in [0,1]$.*

Proof. Define the set $S = \{(\rho_1(C_I),\ \rho_2(R_I)) : I \in \mathcal{I}_0\} \subset \mathbb{R}^2$. Since the reinsurance design problem is proper, there exists $M \in \mathbb{R}$ such that $(x,y) \geqslant (M,M)$ for all $(x,y) \in S$, and also note that S is not empty. Next, for $K^*(\lambda)$ to be non-empty, it suffices to verify that S is a closed set. For a sequence of $\{I_n \in \mathcal{I}_0,\, n \in \mathbb{N}\}$ such that $\rho_1(C_{I_n}) \to a$ and $\rho_2(R_{I_n}) \to b$, where $a, b \in \mathbb{R}$, it suffices to show that there exists $I^* \in \mathcal{I}_0$ such that $\rho_1(C_{I^*}) = a$ and $\rho_2(R_{I^*}) = b$. By Lemma 2.3, there exists a subsequence $\{I_{n_k},\, k \in \mathbb{N}\}$ of $\{I_n,\, n \in \mathbb{N}\}$ converging pointwise to, say $I^* \in \mathcal{I}_0$. Therefore, $\{I_{n_k}(X)\}$ converges to $I^*(X)$ almost surely (indeed, for all $\omega \in \Omega$). The limits $\rho_1(C_{I^*}) = a$ and $\rho_2(R_{I^*}) = b$ follow from the assumed continuity of ρ_1, ρ_2 and π. □

Similar to Corollary 2.2, we can replace the condition of the as-continuity on $C(X)$ in Theorem 2.4 by a global condition of L^p-continuity on L^p if X is in L^p as stated in the following corollary.

Corollary 2.5. *Let $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ be a proper reinsurance setting, in which $X \in \mathcal{X} = L^p$ for some $p \in [1,\infty]$. If π, ρ_1, ρ_2 are L^p-continuous on L^p and $\mathcal{I}_0$ is pw-closed, then $K^*(\lambda)$ is non-empty for each $\lambda \in [0,1]$.*

Proof. We need to verify that $\{I_{n_k}(X)\}$ in the proof of Theorem 2.4 converges to $I^*(X)$ in L^p. This is implied by the the dominated convergence theorem, noting that $I_{n_k}(X)$ is dominated by $X \in L^p$. □

Below we make a few observations on the conditions assumed in Theorem 2.4 and Corollary 2.5.

(i) For $X \in L^p$, $p \in [1, \infty]$, the as-continuity on $\mathcal{C}(X)$ in Theorem 2.1 is weaker than L^p-continuity in Corollary 2.5.

(ii) If ρ_1 and ρ_2 are monetary risk measures (monotone and cash-invariant; see Section 2.4), then they are L^∞-continuous. Thus, for monetary risk measures, the continuity assumption can be removed if X is bounded.

(iii) If ρ_1 and ρ_2 are finite-valued convex risk measures on L^p, $p \in [1, \infty]$, then they are L^p-continuous, see e.g. Kaina and Rüschendorf (2009). Hence, all finite-valued convex risk measures satisfy the conditions for ρ_1 and ρ_2 in Theorems 2.1 and 2.4.

2.4 Special cases: VaR and TVaR

In this section we have a closer look at the two popular risk measures, VaR and TVaR, and put them into the framework of Theorems 2.1 and 2.4. The Value-at-Risk (VaR) of a random variable X at a confidence level $\alpha \in (0, 1)$ is defined as $\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leqslant x) \geqslant \alpha\}$ for $X \in L^0$, and the Tail-Value-at-Risk (TVaR) of a random variable X at a confidence level $\alpha \in (0, 1)$ is defined as $\mathrm{TVaR}_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_q(X)\mathrm{d}q$ for $X \in L^1$. Note that $\mathrm{TVaR}_\alpha(X)$ can also be written as

$$\mathrm{TVaR}_\alpha(X) = \mathrm{VaR}_\alpha(X) + \frac{1}{1-\alpha}\mathbb{E}\left[(X - \mathrm{VaR}_\alpha(X))_+\right], \quad X \in L^1. \tag{2.6}$$

For $\alpha \in (0, 1)$, both VaR_α and TVaR_α, considered as functionals mapping a set $\mathcal{X} = L^0$ or $\mathcal{X} = L^1$ to $\mathbb{R}$, are comonotonic-semilinear, and they are *monetary* in the sense of satifying:

(a) Monotonicity: $\rho(X) \leqslant \rho(Y)$ if $X \leqslant Y$, $X, Y \in \mathcal{X}$;

(b) Cash-invariance: $\rho(X + c) = \rho(X) + c$ for any $c \in \mathbb{R}$ and $X \in \mathcal{X}$.

In addition, TVaR is also convex and subadditive:

(c) Subadditivity: $\rho(X + Y) \leqslant \rho(X) + \rho(Y)$, for all $X, Y \in \mathcal{X}$;

(d) Convexity: $\rho(\lambda X + (1 - \lambda)Y) \leqslant \lambda\rho(X) + (1 - \lambda)\rho(Y)$ for all $\lambda \in [0, 1]$ and $X, Y \in \mathcal{X}$.

Now we put VaR and TVaR into the context of Theorems 2.1 and 2.4. Since TVaR_α, $\alpha \in (0, 1)$ is L^1-continuous and comonotonic-semilinear, for any X in L^1, TVaR_α satisfies the conditions for ρ_1 and ρ_2 in Theorems 2.1 and 2.4. For the case of VaR, for any X in L^0, noting that $\mathrm{VaR}_\alpha(I(X)) = I(\mathrm{VaR}_\alpha(X))$ for any continuous and increasing function I, VaR_α is continuous with respect to the almost sure convergence $I_{n_k}(X)$ to $I^*(X)$. Thus, for any X in L^0, VaR_α satisfies the conditions for ρ_1 and ρ_2 in Theorems 2.1 and 2.4. We summarize our findings above on VaR and TVaR in the proposition below. Write $\mathcal{R}_1 = \{\mathrm{VaR}_\alpha : \alpha \in (0, 1)\}$ and $\mathcal{R}_2 = \{\mathrm{TVaR}_\alpha : \alpha \in (0, 1)\}$.

Proposition 2.6. *Suppose that $\rho_1, \rho_2 \in \mathcal{R}_1 \cup \mathcal{R}_2$, $X \in L^0$ ($X \in L^1$ if at least one of ρ_1, ρ_2 is in $\mathcal{R}_2$), π is an additive and as-continuous functional on $\mathcal{C}(X)$ and $\mathcal{I}_0$ is convex and pw-closed. Then, the following assertions hold.*

(i) $I^* \in \mathcal{I}_0$ is Pareto-optimal under the setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$ if and only if $I^* \in K^*(\lambda)$ for some $\lambda \in [0,1]$.

(ii) For each $\lambda \in [0,1]$, $K^*(\lambda)$ is non-empty.

3 Pareto-optimal reinsurance contracts under TVaRs

In this section, we solve the minimization problem (1.3) when the functionals ρ_1 and ρ_2 are TVaRs and find the explicit forms of Pareto-optimal reinsurance contracts. More precisely, in this section, in the reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I}_0)$, we choose the feasible set to be $\mathcal{I}_0 = \mathcal{I}$. Furthermore, assume that for any reinsurance contract $I \in \mathcal{I}$, the reinsurance premium $\pi(I(X))$ is determined by the expected value principle, namely $\pi(I(X)) = (1+\theta)\mathbb{E}[I(X)]$, where θ is a positive risk loading factor. Suppose that the insurer and the reinsurer use TVaR_α and TVaR_β, respectively, to measure their own risk, where $\alpha, \beta \in (0,1)$. Thus, the problem (1.3) reduces to the following minimization problem

$$\min_{I\in\mathcal{I}}\{\lambda\,\mathrm{TVaR}_\alpha\,(X - I(X) + \pi\,(I\,(X))) + (1-\lambda)\,\mathrm{TVaR}_\beta(I(X) - \pi\,(I\,(X)))\}, \tag{3.1}$$

where $\lambda \in [0,1]$.

We use the following notation henceforth

$$m = m(\lambda) = \frac{\lambda}{1-\alpha} + (1-2\lambda)(1+\theta), \tag{3.2}$$

$$p = p(\lambda) = 1 - (1-\lambda)/m, \tag{3.3}$$

$$q = q(\lambda) = 1 - \frac{\lambda}{\frac{1-\lambda}{1-\beta} - (1-2\lambda)(1+\theta)}. \tag{3.4}$$

The following Theorems 3.1 and 3.2 give explicit solutions to the problem (3.1). Theorem 3.1 deals with the case $\alpha \leqslant \beta$ and Theorem 3.2 handles the case $\alpha \geqslant \beta$. The idea of the proof is as follows. First, for any reinsurance contract $I \in \mathcal{I}$, denote the objective function in the problem (3.1) by $V(I) = \lambda\,\mathrm{TVaR}_\alpha\,(X - I(X) + \pi\,(I\,(X))) + (1-\lambda)\,\mathrm{TVaR}_\beta(I(X) - \pi\,(I\,(X)))$, we can find a two-parameter piecewise linear contract $\hat{I} \in \mathcal{I}$ such that $V(\hat{I}) \leqslant V(I)$, where $\hat{I}$ is determined by the two parameters $\xi_a := I(\mathrm{VaR}_\alpha(X))$ and $\xi_b := I(\mathrm{VaR}_\beta(X))$. Thus, the infinite-dimensional minimization problem (3.1) is reduced to a feasible two-dimensional minimization problem $\min_{(\xi_a,\,\xi_b)} V(\hat{I})$. Second, through a detailed analysis of the property of $V(\hat{I})$ as a two-variable function of $(\xi_a,\,\xi_b)$, we obtain the explicit solutions for the two-dimensional minimization problem $\min_{(\xi_a,\,\xi_b)} V(\hat{I})$. The proofs of the two theorems are similar. We only give the proof of Theorem 3.1 and omit the proof for Theorem 3.2.

Theorem 3.1. *For $0 < \alpha \leqslant \beta < 1$, $\lambda \in [0,1]$, and a non-negative integrable ground-up loss random variable X, optimal reinsurance contracts $I^* = I^*_\lambda$ to problem (3.1) are given as follows:*

(i) *If* $0 \leqslant \lambda < \frac{1}{2}$ *and* $\frac{1-\lambda}{1-\beta} > m$*, then*

$$I^*(x) = \begin{cases} x \wedge \mathrm{VaR}_{p(\lambda)}(X), & \text{if } (1-\alpha)(1+\theta) \geqslant 1; \\ x \wedge \mathrm{VaR}_{\theta/1+\theta}(X), & \text{if } (1-\alpha)(1+\theta) < 1. \end{cases}$$

(ii) *If* $0 \leqslant \lambda < \frac{1}{2}$ *and* $\frac{1-\lambda}{1-\beta} = m$*, then* $I^*(x) = \left(x \wedge \mathrm{VaR}_{p(\lambda)}(X)\right) \mathbb{I}_{\{x \leqslant \mathrm{VaR}_\beta(X)\}} + I(x)\, \mathbb{I}_{\{x > \mathrm{VaR}_\beta(X)\}}$*, where* I *can be any function such that* $I^* \in \mathcal{I}$*.*

(iii) *If* $0 \leqslant \lambda < \frac{1}{2}$ *and* $\frac{1-\lambda}{1-\beta} < m$*, then* $I^*(x) = x$*.*

(iv) *If* $\lambda = \frac{1}{2}$*, then*

$$I^*(x) = \begin{cases} I(x) \wedge I(\mathrm{VaR}_\alpha(X)), & \text{if } \alpha < \beta; \\ I(x), & \text{if } \alpha = \beta, \end{cases}$$

where I *can be any function such that* $I^* \in \mathcal{I}$*.*

(v) *If* $\frac{1}{2} < \lambda < 1$ *and* $\frac{1-\lambda}{1-\beta} > m > 0$*, then*

$$I^*(x) = \begin{cases} 0, & \text{if } (1-\alpha)(1+\theta) \geqslant 1; \\ (x - \mathrm{VaR}_{\theta/1+\theta}(X))_+ \wedge \left(\mathrm{VaR}_{p(\lambda)}(X) - \mathrm{VaR}_{\theta/1+\theta}(X)\right), & \text{if } (1-\alpha)(1+\theta) < 1. \end{cases}$$

(vi) *If* $\frac{1}{2} < \lambda < 1$ *and* $\frac{1-\lambda}{1-\beta} = m > 0$*, then*

$$I^*(x) = \left[(x - \mathrm{VaR}_{\theta/1+\theta}(X))_+ \wedge \left(\mathrm{VaR}_\beta(X) - \mathrm{VaR}_{\theta/1+\theta}(X)\right)\right] \mathbb{I}_{\{x \leqslant \mathrm{VaR}_\beta(X)\}} + \ I(x)\, \mathbb{I}_{\{x > \mathrm{VaR}_\beta(X)\}},$$

where I *can be any function such that* $I^* \in \mathcal{I}$*.*

(vii) *If* $\frac{1}{2} < \lambda \leqslant 1$ *and* $\frac{1-\lambda}{1-\beta} < m$*, then* $I^*(x) = (x - \mathrm{VaR}_{\theta/1+\theta}(X))_+$*.*

(viii) *If* $\frac{1}{2} < \lambda < 1$ *and* $m = 0$*, then* $I^*(x) = 0$*.*

(ix) *If* $\lambda = 1$ *and* $m = 0$*, then* $I^*(x) = I(x)\, \mathbb{I}_{\{x > \mathrm{VaR}_\alpha(X)\}}$*, where* I *can be any function such that* $I^* \in \mathcal{I}$*.*

(x) *If* $\frac{1}{2} < \lambda \leqslant 1$ *and* $m < 0$*, then* $I^*(x) = 0$*.*

Proof. The proof of each case is similar. We only give the proof of case (i) here, from which the reader can grasp the main idea of the proof. The proof for the rest cases is put in the Appendix.

For any $I \in \mathcal{I}$, define $V(I) = \lambda \mathrm{TVaR}_\alpha\,(X - I(X) + \pi\,(I\,(X))) + (1-\lambda)\mathrm{TVaR}_\beta(I(X) - \pi\,(I\,(X)))$. Since $X - I(X)$ and $I(X)$ are comonotonic, by comonotonic additivity and cash invariance of TVaR, we have

$$V(I) = \lambda \mathrm{TVaR}_\alpha\,(X) - \lambda \mathrm{TVaR}_\alpha\,(I(X)) + (1-\lambda)\mathrm{TVaR}_\beta(I(X)) + (2\lambda - 1)(1+\theta)\mathbb{E}\,[I(X)]\,.$$

With the expression (2.6) for TVaR, we have

$$\begin{aligned} V(I) = {} & \lambda \mathrm{TVaR}_\alpha\,(X) - \lambda\Big\{\mathrm{VaR}_\alpha(I(X)) + \frac{1}{1-\alpha}\mathbb{E}\,[(I\,(X) - \mathrm{VaR}_\alpha\,(I\,(X)))_+]\Big\} \\ & + (1-\lambda)\Big\{\mathrm{VaR}_\beta(I(X)) + \frac{1}{1-\beta}\mathbb{E}\left[\left(I(X) - \mathrm{VaR}_\beta\,(I(X))\right)_+\right]\Big\} + (2\lambda-1)(1+\theta)\mathbb{E}\,[I(X)]\,. \end{aligned} \tag{3.5}$$

Note that for any random variable Y, $\mathbb{E}[Y] = \int_0^1 \mathrm{VaR}_r(Y)\mathrm{d}r$. Thus, $V(I)$ can be rewritten as

$$\begin{aligned}
V(I) &= \lambda \mathrm{TVaR}_\alpha(X) - \lambda I(\mathrm{VaR}_\alpha(X)) - \frac{\lambda}{1-\alpha}\int_0^1 [I(\mathrm{VaR}_r(X)) - I(\mathrm{VaR}_\alpha(X))]_+ \mathrm{d}r \\
&\quad + (1-\lambda) I(\mathrm{VaR}_\beta(X)) + \frac{1-\lambda}{1-\beta}\int_0^1 \left[I(\mathrm{VaR}_r(X)) - I(\mathrm{VaR}_\beta(X))\right]_+ \mathrm{d}r \\
&\quad + (2\lambda-1)(1+\theta)\int_0^1 I(\mathrm{VaR}_r(X))\mathrm{d}r \\
&= \lambda \mathrm{TVaR}_\alpha(X) - (1-2\lambda)(1+\theta)\int_0^\alpha I(\mathrm{VaR}_r(X))\mathrm{d}r - m\int_\alpha^\beta I(\mathrm{VaR}_r(X))\mathrm{d}r \\
&\quad + \left(\frac{1-\lambda}{1-\beta} - m\right)\int_\beta^1 I(\mathrm{VaR}_r(X))\mathrm{d}r. \qquad (3.6)
\end{aligned}$$

Let $\xi_a = I(\mathrm{VaR}_\alpha(X))$ and $\xi_b = I\left(\mathrm{VaR}_\beta(X)\right)$. Clearly $\xi_a \leqslant \xi_b$ and $\mathrm{VaR}_\alpha(X) - \xi_a \leqslant \mathrm{VaR}_\beta(X) - \xi_b$ as $I(x)$ and $x - I(x)$ are nondecreasing for all $x \geqslant 0$ and $\alpha \leqslant \beta$. Note that $0 \leqslant \xi_a \leqslant \mathrm{VaR}_\alpha(X)$ and $0 \leqslant \xi_b \leqslant \mathrm{VaR}_\beta(X)$ since $0 \leqslant I(x) \leqslant x$ for all $x \geqslant 0$. Recall the definition of m in (3.2). Equality (3.5) reduces to

$$\begin{aligned}
V(I) &= \lambda \mathrm{TVaR}_\alpha(X) + (1-2\lambda) I(\mathrm{VaR}_\alpha(X)) + (1-\lambda)\left[I\left(\mathrm{VaR}_\beta(X)\right) - I(\mathrm{VaR}_\alpha(X))\right] \\
&\quad - \frac{\lambda}{1-\alpha}\int_{I(\mathrm{VaR}_\alpha(X))}^\infty \mathbb{P}(I(X) > z)\mathrm{d}z + \frac{1-\lambda}{1-\beta}\int_{I(\mathrm{VaR}_\beta(X))}^\infty \mathbb{P}(I(X) > z)\mathrm{d}z \\
&\quad + (2\lambda-1)(1+\theta)\int_0^\infty \mathbb{P}(I(X) > z)\mathrm{d}z \\
&= \lambda \mathrm{TVaR}_\alpha(X) + (1-2\lambda)\xi_a + (1-\lambda)(\xi_b - \xi_a) - (1-2\lambda)(1+\theta)\int_0^{\xi_a} \mathbb{P}(I(X) > z)\mathrm{d}z \\
&\quad - m\int_{\xi_a}^{\xi_b} \mathbb{P}(I(X) > z)\mathrm{d}z + \left(\frac{1-\lambda}{1-\beta} - m\right)\int_{\xi_b}^\infty \mathbb{P}(I(X) > z)\mathrm{d}z. \qquad (3.7)
\end{aligned}$$

(i) If $0 \leqslant \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} > m$, then $m > 0$. For the above $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \xi_a; \\ \xi_a & \text{if } \xi_a \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) \leqslant x \leqslant \xi_b - \xi_a + \mathrm{VaR}_\alpha(X); \\ \xi_b & \text{if } x \geqslant \xi_b - \xi_a + \mathrm{VaR}_\alpha(X). \end{cases} \qquad (3.8)$$

The relationship between $I(x)$ and $\hat{I}(x)$ is illustrated by Figure 1. One can show that $\hat{I}(x) \in \mathcal{I}$ and $V(I) \geqslant V(\hat{I})$ for any $I \in \mathcal{I}$. Indeed, from Figure 1, we conclude that for $0 \leqslant x \leqslant \mathrm{VaR}_\beta(X)$, $I(x) \leqslant \hat{I}(x)$, and for $x \geqslant \mathrm{VaR}_\beta(X)$, $I(x) \geqslant \hat{I}(x)$. Moreover, since $-(1-2\lambda)(1+\theta) < 0$, $-m < 0$, and $\frac{1-\lambda}{1-\beta} - m > 0$, we have

$$\begin{aligned}
-(1-2\lambda)(1+\theta)\int_0^\alpha \hat{I}(\mathrm{VaR}_r(X))\mathrm{d}r &\leqslant -(1-2\lambda)(1+\theta)\int_0^\alpha I(\mathrm{VaR}_r(X))\mathrm{d}r, \\
-m\int_\alpha^\beta \hat{I}(\mathrm{VaR}_r(X))\mathrm{d}r &\leqslant -m\int_\alpha^\beta I(\mathrm{VaR}_r(X))\mathrm{d}r, \\
\left(\frac{1-\lambda}{1-\beta} - m\right)\int_\beta^1 \hat{I}(\mathrm{VaR}_r(X))\mathrm{d}r &\leqslant \left(\frac{1-\lambda}{1-\beta} - m\right)\int_\beta^1 I(\mathrm{VaR}_r(X))\mathrm{d}r.
\end{aligned}$$

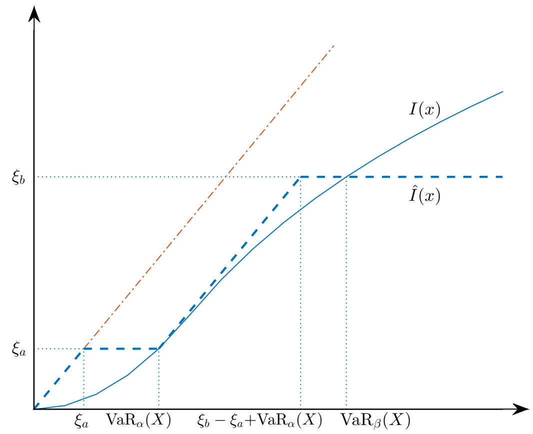

Figure 1: Relationship between $I(x)$ and $\hat{I}(x)$ in case (i)

Hence, it follows immediately from (3.6) that $V(I) \geqslant V(\hat{I})$, where the inequality is strict if I and $\hat{I}$ are not identical almost everywhere, which means that the optimal reinsurance contract can only take the form of (3.8) in case (i). The equivalence of (3.6) and (3.7) implies that $\min_{I\in\mathcal{I}} V(I) = \min_{\{\xi_a,\xi_b\}} V(\hat{I})$, where $V(\hat{I})$ is the expression in (3.7).

Next, it remains to find the values of ξ_a and ξ_b such that $V(\hat{I})$ is minimized. Let $s = \xi_a$ and $t = \xi_b - \xi_a$. Clearly $0 \leqslant s \leqslant \mathrm{VaR}_\alpha(X)$ and $0 \leqslant t \leqslant \mathrm{VaR}_\beta(X) - \mathrm{VaR}_\alpha(X)$. Since

$$\mathbb{P}\left(\hat{I}(X) > x\right) = \begin{cases} \mathbb{P}\left(X > x\right) & \text{if } 0 \leqslant x < \xi_a; \\ \mathbb{P}\left(X > x + \mathrm{VaR}_\alpha(X) - \xi_a\right) & \text{if } \xi_a \leqslant x < \xi_b; \\ 0 & \text{if } x \geqslant \xi_b, \end{cases}$$

we have

$$\begin{aligned} V(\hat{I}) = \lambda \mathrm{TVaR}_\alpha(X) + (1-2\lambda)s - (1-2\lambda)(1+\theta)\int_0^s \mathbb{P}(X > z)\mathrm{d}z \\ + (1-\lambda)t - m\int_{\mathrm{VaR}_\alpha(X)}^{t+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z. \end{aligned} \tag{3.9}$$

Denote by

$$\begin{aligned} f(s) &= \lambda \mathrm{TVaR}_\alpha(X) + (1-2\lambda)s - (1-2\lambda)(1+\theta)\int_0^s \mathbb{P}(X > z)\mathrm{d}z, \\ g(t) &= (1-\lambda)t - m\int_{\mathrm{VaR}_\alpha(X)}^{t+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z, \\ \mathcal{A} &= \{s \in \mathbb{R} \mid 0 \leqslant s \leqslant \mathrm{VaR}_\alpha(X)\}, \\ \mathcal{B} &= \left\{t \in \mathbb{R} \mid 0 \leqslant t \leqslant \mathrm{VaR}_\beta(X) - \mathrm{VaR}_\alpha(X)\right\}, \\ \mathcal{C} &= \left\{(s,t) \in \mathbb{R}^2 \mid s \in \mathcal{A}, t \in \mathcal{B}\right\}. \end{aligned}$$

Then $V(\hat{I}) = f(s) + g(t)$. Lebesgue differentiation theorem implies that f and g are continuous in s and t, respectively. Suppose that there exist $s^* \in \mathcal{A}$ and $t^* \in \mathcal{B}$ such that $\min_{s\in\mathcal{A}} f(s) = f(s^*)$ and

$\min_{t\in\mathcal{B}} g(t) = g(t^*)$. Then $\min_{(s,t)\in\mathcal{C}} V(\hat{I}) = f(s^*) + g(t^*)$ because $\min_{(s,t)\in\mathcal{C}} V(\hat{I}) = \min_{(s,t)\in\mathcal{C}}\{f(s) + g(t)\} \geqslant \min_{s\in\mathcal{A}} f(s) + \min_{t\in\mathcal{B}} g(t) = f(s^*) + g(t^*) \geqslant \min_{(s,t)\in\mathcal{C}}\{f(s)+g(t)\} = \min_{(s,t)\in\mathcal{C}} V(\hat{I})$. Therefore, it remains to find s^* and t^*, or the corresponding ξ_a^* and ξ_b^*, where $\xi_a^* = s^*$ and $\xi_b^* = t^* + s^*$.

If $(1-\alpha)(1+\theta) \geqslant 1$, then $p \geqslant \alpha$ since $m = \frac{\lambda}{1-\alpha} + (1-2\lambda)(1+\theta) \geqslant \frac{\lambda}{1-\alpha} + (1-2\lambda)\frac{1}{1-\alpha} = \frac{1-\lambda}{1-\alpha}$. For $0 \leqslant s_1 < s_2 < \mathrm{VaR}_\alpha(X)$, as $\lambda < 1/2$ and $\mathbb{P}(X \leqslant s_2) < \alpha$, we have

$$\begin{aligned} f(s_1) - f(s_2) &= (1-2\lambda)\Big((1+\theta)\int_{s_1}^{s_2} \mathbb{P}(X > z)\mathrm{d}z - (s_2 - s_1)\Big) \\ &\geqslant (1-2\lambda)(1+\theta)(s_2 - s_1)\left[\mathbb{P}\left(X > s_2\right) - \theta^*\right] > 0, \end{aligned}$$

which, together with the continuity of f, implies that $f(s)$ is strictly decreasing for $s \in \mathcal{A}$ and $\xi_a^* = s^* = \mathrm{VaR}_\alpha(X)$. On the other hand, $\frac{1-\lambda}{1-\beta} > m$ implies $\beta > 1 - \frac{1-\lambda}{m}$, that is, $\beta > p$. Thus, $\mathrm{VaR}_p(X) - \mathrm{VaR}_\alpha(X) \in \mathcal{B}$. Note that if $\mathrm{VaR}_\alpha(X) = \mathrm{VaR}_\beta(X)$, then $\mathrm{VaR}_p(X) = \mathrm{VaR}_\alpha(X)$ since $\alpha \leqslant p < \beta$. For $0 \leqslant t_1 < t_2 < \mathrm{VaR}_p(X) - \mathrm{VaR}_\alpha(X)$, as $m > 0$ and $\mathbb{P}(X \leqslant t_2 + \mathrm{VaR}_\alpha(X)) < p = 1 - \frac{1-\lambda}{m}$, we have

$$\begin{aligned} g(t_1) - g(t_2) &= m\int_{t_1+\mathrm{VaR}_\alpha(X)}^{t_2+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z - (1-\lambda)(t_2 - t_1) \\ &\geqslant \Big(\mathbb{P}(X > t_2 + \mathrm{VaR}_\alpha(X)) - (1-\lambda)/m\Big)(t_2 - t_1)m \; > \; \Big(1 - p - \frac{1-\lambda}{m}\Big)(t_2 - t_1)m = 0. \end{aligned}$$

Therefore, $g(t)$ is strictly decreasing for $0 \leqslant t \leqslant \mathrm{VaR}_p(X) - \mathrm{VaR}_\alpha(X)$. Similarly, for $\mathrm{VaR}_p(X) - \mathrm{VaR}_\alpha(X) < t_1 < t_2 \leqslant \mathrm{VaR}_\beta(X) - \mathrm{VaR}_\alpha(X)$, as $m > 0$, we have

$$\begin{aligned} g(t_1) - g(t_2) &= m\int_{t_1+\mathrm{VaR}_\alpha(X)}^{t_2+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z - (1-\lambda)(t_2 - t_1) \\ &\leqslant \Big(\mathbb{P}(X > t_1 + \mathrm{VaR}_\alpha(X)) - (1-\lambda)/m\Big)(t_2 - t_1)m \\ &\leqslant \Big(\mathbb{P}(X > \mathrm{VaR}_p(X)) - (1-\lambda)/m\Big)(t_2 - t_1)m \\ &= \Big(1 - (1-\lambda)/m - \mathbb{P}(X \leqslant \mathrm{VaR}_p(X))\Big)(t_2 - t_1)m \leqslant 0. \end{aligned}$$

Thus, $g(t)$ is increasing for $\mathrm{VaR}_p(X) - \mathrm{VaR}_\alpha(X) \leqslant t \leqslant \mathrm{VaR}_\beta(X) - \mathrm{VaR}_\alpha(X)$. Thus $t^* = \mathrm{VaR}_p(X) - \mathrm{VaR}_\alpha(X)$ minimizes g, $\xi_a^* = \mathrm{VaR}_\alpha(X)$, $\xi_b^* = \mathrm{VaR}_p(X)$, and the optimal reinsurance contract is $I^*(x) = x \wedge \mathrm{VaR}_p(X)$.

If $(1-\alpha)(1+\theta) < 1$, then $p < \alpha$. Similarly, we obtain $\xi_a^* = s^* = \mathrm{VaR}_{\theta/1+\theta}(X)$ and $t^* = 0$. Therefore, $\xi_a^* = \xi_b^* = \mathrm{VaR}_{\theta/1+\theta}(X)$ and the optimal reinsurance contract is $I^*(x) = x \wedge \mathrm{VaR}_{\theta/1+\theta}(X)$. □

Remark 3.1. When $\lambda = 1$, cases (vii), (ix) and (x) of Theorem 3.1 recover Theorem 3.3 of Cheung et al. (2014). Theorem 3.3 of Cheung et al. (2014) holds under the assumption that the ground-up loss random variable X has a continuous and strictly increasing distribution function, whereas our Theorem 3.1 does not require such an assumption.

Remark 3.2. From Theorem 3.1, we see that the shapes of Pareto-optimal reinsurances depend on the weight coefficient λ, the confidence levels α and β, and the loading factor θ. We point out that the confidence levels α and β used by the insurer and reinsurer in their VaRs also represent their respective solvency requirements. In fact, the higher confidence levels mean the higher/tougher solvency requirements. Note that if we denote by $\alpha^* = \frac{\theta}{1+\theta}$ and $\beta^* = 1 - (1-\lambda)/m$, then the condition $\frac{1-\lambda}{1-\beta} > (=, <)\, m$ in Theorem 3.1 is equivalent to $\beta > (=, <)\, \beta^*$ for $m > 0$; and the condition $(1-\alpha)(1+\theta) \geqslant (<)\, 1$ is equivalent to $\alpha \leqslant (>)\, \alpha^*$. Thus, we can give intuitive explanations of the Pareto-optimal shapes in Theorem 3.1 and explain how the shapes change with the model parameters for the case that the insurer has a lower solvency requirement than the reinsurer has (or $\alpha \leqslant \beta$) as follows.

When the insure's risk is weighted less than the reinsure's one in the objective function (3.1) (or $0 \leqslant \lambda < 1/2$), Theorem 3.1 (i)-(iii) imply that if the reinsurer has a tougher solvency requirement (or $\beta > \beta^*$), the limited reinsurances are Pareto-optimal, namely, the reinsurer has a limit on the risk she/he would like to take; if the reinsurer's solvency requirement is at the "threshold level" β^* (or $\beta = \beta^*$), the one-layer unlimited reinsurance is Pareto-optimal, namely, the reinsurer can take more risk from the one-layer unlimited reinsurance than from the corresponding limited reinsurance; and if the reinsurer has a more relaxed solvency requirement (or $\beta < \beta^*$), the full reinsurance is Pareto-optimal, namely, the reinsurer will like to take all the risk from the insurer.

When the risks of the insurer and reinsurer are weighted equally (or $\lambda = 1/2$), Theorem 3.1 (iv) shows that if the solvency requirements of the insurer and reinsurer are at the same standard (or $\alpha = \beta$), any feasible reinsurance is Pareto-optimal; and if the solvency requirement of the reinsurer is higher than the insurer's one (or $\alpha < \beta$), the limited reinsurance is Pareto-optimal.

On the other hand, when the insurer's risk is weighted more than the reinsurer's one (or $1/2 < \lambda \leqslant 1$) and the reinsurer has a tougher solvency requirement (or $\beta > \beta^*$), Theorem 3.1 (v) means that if the insurer has a more relaxed solvency requirement (or $\alpha < \alpha^*$), no reinsurance is Pareto-optimal, namely, the insurer would like to take all the underlying risk X; and if the insurer has a higher solvency requirement (or $\alpha \geqslant \alpha^*$), the limited stop-loss reinsurance is Pareto-optimal. Furthermore, when the insurer's risk is weighted more than the reinsurer's one (or $1/2 < \lambda \leqslant 1$), Theorem 3.1 (vi)-(vii) show that if the reinsurer's solvency requirement is at the "threshold level" β^* (or $\beta = \beta^*$), the two-layer unlimited reinsurance is Pareto-optimal, namely, the reinsurer can take more risk from the two-layer unlimited reinsurance than from the corresponding limited stop-loss reinsurance; if the reinsurer has a more relaxed solvency requirement (or $\beta < \beta^*$), the unlimited stop-loss reinsurance is Pareto-optimal.

In addition, denote by $\hat{\alpha} = 1 + \lambda/((1-2\lambda)(1+\theta))$. Note that under the assumption of $\frac{1}{2} < \lambda \leqslant 1$, the condition $m = (<)\, 0$ in Theorem 3.1(viii)-(x) is equivalent to $\alpha = (<)\, \hat{\alpha}$. Moreover, $I^*(x) = 0$ is a Pareto-optimal solution for the case of Theorem 3.1(ix) when $I(x)$ is chosen as $I(x) = 0$. Thus, Theorem 3.1 (viii)-(x) mean that if the insurer's risk is weighted more than the reinsurer's one (or $\frac{1}{2} < \lambda \leqslant 1$) and the insurer has a more relaxed solvency requirement (or $\alpha \leqslant \hat{\alpha}$), then no reinsurance is Pareto-optimal, namely, the insurer would like to take all the underlying risk X.

The results of Theorem 3.1 imply that the insurer/reinsurer would like to take more risks in a Pareto-optimal reinsurance if her/his risks are weighted more than the other party and she/he has a relaxed solvency requirement. These findings are consistent with the intuitive sense of a Pareto-optimal reinsurance arrangement. Similar remarks also apply to Theorem 3.2 and thus are omitted.

Theorem 3.2. *For $0 < \beta \leqslant \alpha < 1$, $\lambda \in [0,1]$ and a non-negative integrable ground-up loss random variable X, optimal reinsurance contracts $I^* = I^*_\lambda$ to problem (3.1) are given as follows:*

(i) If $\lambda = 0$ and $(1-\beta)(1+\theta) > 1$, then $I^*(x) = x$.

(ii) If $\lambda = 0$ and $(1-\beta)(1+\theta) = 1$, then $I^*(x) = x\,\mathbb{I}_{\{x \leqslant \mathrm{VaR}_\beta(X)\}} + I(x)\,\mathbb{I}_{\{x > \mathrm{VaR}_\beta(X)\}}$, where I can be any function such that $I^* \in \mathcal{I}$.

(iii) If $0 \leqslant \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} > m$, then $I^*(x) = x \wedge \mathrm{VaR}_{\theta/1+\theta}(X)$.

(iv) If $0 < \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} = m$, then $I^*(x) = \left(x \wedge \mathrm{VaR}_{\theta/1+\theta}(X)\right)\mathbb{I}_{\{x \leqslant \mathrm{VaR}_\alpha(X)\}} + I(x)\,\mathbb{I}_{\{x > \mathrm{VaR}_\alpha(X)\}}$, where I can be any function such that $I^* \in \mathcal{I}$.

(v) If $0 < \lambda < \frac{1}{2}$ and $(1-2\lambda)(1+\theta) < \frac{1-\lambda}{1-\beta} < m$, then

$$I^*(x) = \begin{cases} x, & \text{if } (1-\beta)(1+\theta) \geqslant 1; \\ x \wedge \mathrm{VaR}_{\theta/1+\theta}(X) + \left(x - \mathrm{VaR}_{q(\lambda)}(X)\right)_+, & \text{if } (1-\beta)(1+\theta) < 1. \end{cases}$$

(vi) If $0 < \lambda < \frac{1}{2}$ and $(1-2\lambda)(1+\theta) = \frac{1-\lambda}{1-\beta}$, then

$$I^*(x) = x\,\mathbb{I}_{\left\{x \leqslant \mathrm{VaR}_\beta(X) \text{ or } x \geqslant \mathrm{VaR}_\alpha(X)\right\}} + I(x)\,\mathbb{I}_{\left\{\mathrm{VaR}_\beta(X) \leqslant x \leqslant \mathrm{VaR}_\alpha(X)\right\}},$$

where I can be any function such that $I^* \in \mathcal{I}$.

(vii) If $0 < \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} < (1-2\lambda)(1+\theta)$, then $I^*(x) = x$.

(viii) If $\lambda = \frac{1}{2}$, then

$$I^*(x) = \begin{cases} I(x)\,\mathbb{I}_{\{x \leqslant \mathrm{VaR}_\beta(X)\}} + \left(x - \mathrm{VaR}_\beta(X) + I(\mathrm{VaR}_\beta(X))\right)\mathbb{I}_{\{x > \mathrm{VaR}_\beta(X)\}}, & \text{if } \alpha > \beta; \\ I(x), & \text{if } \alpha = \beta, \end{cases}$$

where I can be any function such that $I^* \in \mathcal{I}$.

(ix) If $\frac{1}{2} < \lambda \leqslant 1$ and $\frac{1-\lambda}{1-\beta} > m$,, then $I^*(x) = 0$.

(x) If $\frac{1}{2} < \lambda \leqslant 1$ and $\frac{1-\lambda}{1-\beta} = m$, then $I^*(x) = I(x)\,\mathbb{I}_{\{x > \mathrm{VaR}_\alpha(X)\}}$, where I can be any function such that $I^* \in \mathcal{I}$.

(xi) If $\frac{1}{2} < \lambda \leqslant 1$ and $\frac{1-\lambda}{1-\beta} < m$, then

$$I^*(x) = \begin{cases} \left(x - \mathrm{VaR}_{q(\lambda)}(X)\right)_+, & \text{if } (1-\beta)(1+\theta) \geqslant 1; \\ \left(x - \mathrm{VaR}_{\theta/1+\theta}(X)\right)_+, & \text{if } (1-\beta)(1+\theta) < 1. \end{cases}$$

4 Mutually acceptable Pareto-optimal reinsurance contracts

Under a reinsurance setting $(X, \rho_1, \rho_2, \pi, \mathcal{I})$ with $\rho_1 = \mathrm{TVaR}_\alpha$, $\rho_2 = \mathrm{TVaR}_\beta$, and $\pi(I(X)) = (1 + \theta)\,\mathbb{E}(I(X))$ for $I \in \mathcal{I}$, by Proposition 1.1 or Theorem 2.1, we know that for any $\lambda \in (0, 1)$, the reinsurance contract $I^* = I^*_\lambda$ given in Theorems 3.1 or 3.2 is a Pareto-optimal contract for the case of $\alpha \leqslant \beta$ or the case of $\alpha > \beta$. However, an interesting and practical question is that what the Pareto-optimal contracts I^*_λ for $\lambda \in (0, 1)$ are the mutually acceptable ones in the sense that the Pareto-optimal contracts I^*_λ could be accepted by both the insurer and the reinsurer. To address this issue, let us recall that one of the main reasons for an insurer to buy a reinsurance contract is to reduce its risk measure (required reserves/capitals), while the goal of a reinsurer as the seller of a contract is to make profits. Before reinsurance, the risk of the insurer is X and its risk measure is $\mathrm{TVaR}_\alpha(X)$. Under a Pareto-optimal contract I^*_λ, the risk of the insurer is $C_{I^*_\lambda} = C_{I^*_\lambda}(X) = X - I^*_\lambda(X) + \pi(I^*_\lambda(X))$, and the insurer expects its risk measure to be reduced at least $100(1 - \gamma)\%$ under the Pareto-optimal reinsurance I^*_λ, namely

$$\mathrm{TVaR}_\alpha(C_{I^*_\lambda}(X)) \leqslant \gamma\,\mathrm{TVaR}_\alpha(X) \tag{4.1}$$

for $0 < \gamma < 1$. On the other hand, under the Pareto-optimal reinsurance I^*_λ, the reinsurer has an expected gross income $\mathbb{E}(\pi(I^*_\lambda(X)))$ and an expected net profit $\mathbb{E}(\pi(I^*_\lambda(X)) - I^*_\lambda(X))$, and the reinsurer would like the expected net profit at least to be $100\sigma\%$ of the expected gross income, namely

$$\mathbb{E}(\pi(I^*_\lambda(X)) - I^*_\lambda(X)) \geqslant \sigma\,\mathbb{E}(\pi(I^*_\lambda(X))) \tag{4.2}$$

for $0 < \sigma < 1$. In addition, the reinsurer also has a concern about the TVaR of its risk. Assume that the reinsurer wishes that under a Pareto-optimal contract I^*, the maximum TVaR of its risk is not bigger than $100\kappa\%$ of the TVaR of X if the reinsurer acts as the insurer and has the ground-up loss X, namely

$$\mathrm{TVaR}_\beta(R_{I^*_\lambda}(X)) \leqslant \kappa\,\mathrm{TVaR}_\beta(X) \tag{4.3}$$

for $0 < \kappa < 1$. Therefore, the mutually acceptable Pareto-optimal reinsurance contracts I^*_λ for $\lambda \in (0, 1)$ should be those such that all of the three conditions (4.1), (4.2), and (4.3) hold.

In the rest of this section, we will use two examples to illustrate how to determine the mutually acceptable Pareto-optimal reinsurance contracts $I^*_{\lambda^*}$ among the available Pareto-optimal reinsurance contracts I^*_λ with $\lambda \in (0, 1)$, such that all of the three conditions (4.1), (4.2), and (4.3) hold.

Note that if $\pi(I(X)) = (1 + \theta)\,\mathbb{E}(I(X))$ for $I \in \mathcal{I}$, then $\mathbb{E}(\pi(I^*_\lambda(X)) - I^*_\lambda(X)) = \frac{\theta}{1+\theta}\,\mathbb{E}(\pi(I^*_\lambda(X))$ for any $\lambda \in (0, 1)$ and (4.2) holds if and only if $\sigma \leqslant \frac{\theta}{1+\theta}$.

In the following two examples, we let the safety loading factor be $\theta = 0.2$. Thus, $\theta^* = 1/(1 + \theta) = 0.8333$. Furthermore, let $\sigma \leqslant \frac{\theta}{1+\theta} = 0.16667$. Thus, (4.2) holds for any I^*_λ. Moreover, we let $\kappa = 0.8$ and discuss the impacts of the parameter γ and the distribution of the ground-up loss random variable X on the mutually acceptable Pareto-optimal contracts by setting $\gamma = 0.5, 0.7, 0.8$ and assuming that X has an exponential distribution and a Pareto distribution, respectively.

Example 4.1. Suppose that the ground-up loss X follows an exponential distribution with distribution function $F(x) = 1 - e^{-0.001x}$ for $x \geqslant 0$. Then $\mathbb{E}(X) = 1000$, $\mathrm{VaR}_\alpha(X) = -1000\ln(1-\alpha)$, and $\mathrm{TVaR}_\alpha(X) = 1000\,[1 - \ln(1-\alpha)]$. Thus, $\mathrm{VaR}_{\theta/1+\theta}(X) = 182.32$.

If $\alpha < \beta$ with $\alpha = 0.95$ and $\beta = 0.99$, then $(1-\alpha)(1+\theta) < 1$. When $\lambda = 0.5$, which is the case (iv) of Theorem 3.1, by taking $I(x) = x \wedge \mathrm{VaR}_{\theta/1+\theta}(X) = x \wedge 182.32$ in (iv) of Theorem 3.1, we have that the Pareto-optimal reinsurance can be $I^*(x) = x \wedge 182.32$. When $\lambda = 0.84$, which is the case (vi) of Theorem 3.1, by taking $I(x) = \mathrm{VaR}_{p(0.84)}(X) - 182.32$ in (vi) of Theorem 3.1, we see that the Pareto-optimal reinsurance can be $I^*(x) = (x - 182.32)_+ \wedge (\mathrm{VaR}_{p(0.84)}(X) - 182.32)$. When $\lambda \in (0,\ 0.5)$, $\lambda \in (05,\ 0.84)$, and $\lambda \in (0.84,\ 1)$, the Pareto-optimal reinsurance contracts are of cases (i), (v), and (vii) of Theorem 3.1, respectively. Therefore, the Pareto-optimal reinsurance contracts are

$$I^*_\lambda(x) = \begin{cases} x \wedge 182.32 & \text{if } \lambda \in (0,\ 0.5]; \\ (x - 182.32)_+ \wedge (\mathrm{VaR}_{p(\lambda)}(X) - 182.32) & \text{if } \lambda \in (0.5,\ 0.84]; \\ (x - 182.32)_+ & \text{if } \lambda \in (0.84,\ 1). \end{cases} \tag{4.4}$$

It is easy to verify that when $\lambda \in (0.5,\ 0.84]$, $\mathrm{TVaR}_\alpha(C_{I^*_\lambda}(X))$ is decreasing in λ, while $\mathrm{TVaR}_\beta(R_{I^*_\lambda}(X))$ and $\mathbb{E}(\pi(I^*_\lambda(X)))$ are increasing in λ. In addition, they are all constants for $\lambda \in (0, 0.5]$ and $\lambda \in (0.84, 1)$, respectively. The values of $\mathrm{TVaR}_\alpha(X)$, $\mathrm{TVaR}_\alpha(C_{I^*_\lambda}(X))$, $\mathrm{TVaR}_\beta(X)$, $\mathrm{TVaR}_\beta(R_{I^*_\lambda}(X))$, and $\mathbb{E}[\pi(I^*_\lambda(X))]$ are the key for one to find $\lambda^* \in (0, 1)$ such that the Pareto-optimal contracts $I^*_{\lambda^*}$ satisfy (4.1)-(4.3). These key values are presented in Table 1.

If $\alpha > \beta$ with $\alpha = 0.99$ and $\beta = 0.95$, then $(1-\beta)(1+\theta) < 1$. When $\lambda = 0.1599$, which is the case (iv) of Theorem 3.2, the Pareto-optimal reinsurance can be $I^*(x) = x \wedge 182.32$ by taking $I(x) = \mathrm{VaR}_{\theta/1+\theta}(X) = 182.32$ in (iv) of Theorem 3.2. When $\lambda = 0.5$, which is the case (viii) of Theorem 3.2, then $\beta = q(1/2)$ and the Pareto-optimal reinsurance can be $I^*(x) = x \wedge 182.32 + \big(x - \mathrm{VaR}_{q(1/2)}(X)\big)_+$ by taking $I(x) = x \wedge \mathrm{VaR}_{\theta/1+\theta}(X) = x \wedge 182.32$ in (viii) of Theorem 3.2. When $\lambda \in (0, 0.1599)$, $\lambda \in (0.1599, 0.5)$, and $\lambda \in (0.5, 1)$ and the Pareto-optimal contracts are of cases (iii), (v), and(xi) of Theorem 3.2, respectively. Thus, the Pareto-optimal reinsurance contracts are

$$I^*_\lambda(x) = \begin{cases} x \wedge 182.32 & \text{if } \lambda \in (0,\ 0.1599]; \\ x \wedge 182.32 + \big(x - \mathrm{VaR}_{q(\lambda)}(X)\big)_+ & \text{if } \lambda \in (0.1599,\ 0.5]; \\ (x - 182.32)_+ & \text{if } \lambda \in (0.5,\ 1). \end{cases} \tag{4.5}$$

When $\lambda \in (0.1599,\ 0.5]$, $\mathrm{TVaR}_\alpha(C_{I^*_\lambda}(X)))$ is decreasing in λ, while $\mathrm{TVaR}_\beta(R_{I^*_\lambda}(X))$ and $\mathbb{E}(\pi(I^*_\lambda(X)))$ are increasing in λ. The key values in this case are given in Table 2.

If $\alpha = \beta = 0.95$, then $(1-\alpha)(1+\theta) < 1$. By Theorem 3.1 or 3.2, the Pareto-optimal reinsurance contracts are

$$I^*_\lambda(x) = \begin{cases} x \wedge 182.32 & \text{if } \lambda \in (0,\ 0.5); \\ ax,\ a \in [0, 1] & \text{if } \lambda = 0.5; \\ (x - 182.32)_+ & \text{if } \lambda \in (0.5,\ 1). \end{cases} \tag{4.6}$$

We point out that for the case $\alpha = \beta$ and $\lambda = 1/2$, the Pareto-optimal contract can be any contract in $\mathcal{I}$. To simplify the discussion of how to determine the mutually acceptable Pareto-optimal contracts, we consider all of quota-share reinsurances and find the mutually acceptable quota-share reinsurances as the mutually acceptable Pareto-optimal contracts for the case $\alpha = \beta$ and $\lambda = 1/2$. The corresponding key values are given in Table 3.

Based on those values given in Tables 1-3 and the forms of Pareto-optimal reinsurance contracts given in (4.4)-(4.6), we can easily find $\lambda^* \in (0, 1)$ such that the corresponding Pareto-optimal reinsurance contracts $I^*_{\lambda^*}$ satisfy (4.1)-(4.3). Such values of λ^* are summarized in Table 4.

From Table 4 and (4.4), we see that if $\alpha < \beta$ with $\alpha = 0.95$ and $\beta = 0.99$, then the limited stop-loss reinsurances $I^*_{\lambda^*}(x) = (x - 182.32)_+ \wedge (\mathrm{VaR}_{p(\lambda^*)}(X) - 182.32)$ and the stop-loss reinsurance $I^*(x) = (x - 182.32)_+$ are the mutually acceptable Pareto-optimal reinsurance contracts for all the three cases of γ, where the values of λ^* for each of the three cases of γ are given in Table 4.

If $\alpha > \beta$ with $\alpha = 0.99$ and $\beta = 0.95$, then, from Table 4 and (4.5), we find that the stop-loss contract $I^*(x) = (x - 182.32)_+$ is the mutually acceptable Pareto-optimal reinsurance for all the three cases of γ. Besides, the contracts $I^*_{\lambda^*}(x) = x \wedge 182.32 + (x - \mathrm{VaR}_{q(\lambda^*)}(X))_+$ are also the mutually acceptable Pareto-optimal reinsurance contracts for the cases of $\gamma = 0.7, 0.8$, where the values of λ^* for each of the two cases of γ are given in Table 4.

If $\alpha = \beta = 0.95$, then, from Table 4 and (4.6), we see that the stop-loss reinsurance $I^*(x) = (x - 182.32)_+$ and the quota-share reinsurances $I^*(x) = a^* x$ are the mutually acceptable Pareto-optimal reinsurance contracts for all the three cases of γ, where the values of a^* for each case are given in Table 4. □

Example 4.2. Suppose that the ground loss X follows a Pareto distribution with distribution function $1 - \left(\frac{2000}{x+2000}\right)^3$ for any $x \geqslant 0$. Thus $\mathbb{E}(X) = 1000$, which is the same mean as the exponential distribution assumed in Example 4.1. In addition, $\mathrm{VaR}_\alpha(X) = 2000((1-\alpha)^{-1/3} - 1)$ and $\mathrm{TVaR}_\alpha(X) = 3000(1-\alpha)^{-1/3} - 2000$. Hence, $\mathrm{VaR}_{\theta/1+\theta}(X) = 125.32$. By using the arguments similar to those for Example 4.1, we obtain the (mutually acceptable) Pareto-optimal reinsurances for the Pareto distribution as follows.

If $\alpha < \beta$ with $\alpha = 0.95$ and $\beta = 0.99$, then $(1-\alpha)(1+\theta) < 1$. By Theorem 3.1, the Pareto-optimal reinsurance contracts are

$$I^*_\lambda(x) = \begin{cases} x \wedge 125.32 & \text{if } \lambda \in (0,\ 0.5]; \\ (x - 125.32)_+ \wedge (\mathrm{VaR}_{p(\lambda)}(X) - 125.32) & \text{if } \lambda \in (0.5,\ 0.84]; \\ (x - 125.32)_+ & \text{if } \lambda \in (0.84,\ 1). \end{cases} \tag{4.7}$$

When $\lambda \in (0.5, 0.84]$, $\mathrm{TVaR}_\alpha(C_{I^*_\lambda}(X))$ is decreasing in λ, while $\mathrm{TVaR}_\beta(R_{I^*_\lambda}(X))$ and $\mathbb{E}(\pi(I^*_\lambda(X)))$ are increasing in λ. The key values for this case are given in Table 5.

If $\alpha > \beta$ with $\alpha = 0.99$ and $\beta = 0.95$, then $(1-\beta)(1+\theta) < 1$. By Theorem 3.2, the Pareto-optimal reinsurance contracts are

$$I^*_\lambda(x) = \begin{cases} x \wedge 125.32 & \text{if } \lambda \in (0,\ 0.1599]; \\ x \wedge 125.32 + \left(x - \mathrm{VaR}_{q(\lambda)}(X)\right)_+ & \text{if } \lambda \in (0.1599,\ 0.5]; \\ (x - 125.32)_+ & \text{if } \lambda \in (0.5,\ 1). \end{cases} \tag{4.8}$$

When $\lambda \in (0.1599, 0.5]$, $\mathrm{TVaR}_\alpha(C_{I^*_\lambda}(X))$ is decreasing in λ, while $\mathrm{TVaR}_\beta(R_{I^*_\lambda}(X))$ and $\mathbb{E}(\pi(I^*_\lambda(X)))$ are increasing in λ. The key values are given in Table 6.

Table 1: Key values with an exponential ground-up loss and $\alpha = 0.95 < \beta = 0.99$

	$\mathrm{TVaR}_\alpha(X)$	$\mathrm{TVaR}_\alpha((C_{I^*_\lambda}(X))$	$\mathrm{TVaR}_\beta(X)$	$\mathrm{TVaR}_\beta((R_{I^*_\lambda}(X))$	$\mathbb{E}(\pi(I^*_\lambda(X)))$
$\lambda \in (0,\ 0.5]$	3995.73	4013.41	5605.17	-17.68	200
$\lambda \in (0.5,\ 0.84]$	3995.73	(2122.32, 1370.51]	5605.17	(1873.41, 3433.86]	(940, 987.99]
$\lambda \in (0.84,\ 1)$	3995.73	1182.32	5605.17	4422.85	1000

Table 2: Key values with an exponential ground-up loss and $\alpha = 0.99 > \beta = 0.95$

	$\mathrm{TVaR}_\alpha(X)$	$\mathrm{TVaR}_\alpha(C_{I^*_\lambda}(X))$	$\mathrm{TVaR}_\beta(X)$	$\mathrm{TVaR}_\beta((R_{I^*_\lambda}(X))$	$\mathbb{E}(\pi(I^*_\lambda(X)))$
$\lambda \in (0,\ 0.1599]$	5605.17	5622.85	3995.73	-17.68	200
$\lambda \in (0.1599,\ 0.5]$	5605.17	(4634.55, 3073.41]	3995.73	(170.33, 922.32]	(212.04, 260]
$\lambda \in (0.5,\ 1)$	5605.17	1182.32	3995.73	2813.41	1000

Table 3: Key values with an exponential ground-up loss and $\alpha = \beta = 0.95$

	$\mathrm{TVaR}_\alpha(X)$	$\mathrm{TVaR}_\alpha(C_{I^*_\lambda}(X))$	$\mathrm{TVaR}_\beta(X)$	$\mathrm{TVaR}_\beta((R_{I^*_\lambda}(X))$	$\mathbb{E}(\pi(I^*_\lambda(X)))$
$\lambda \in (0,\ 0.5)$	3995.73	4013.41	3995.73	-17.68	200
$\lambda = 0.5,\ I^*_\lambda(X) = a\,x,\ a \in [0,1]$	3995.73	[3995.73, 1200]	3995.73	[0, 2795.73]	[0, 1200]
$\lambda \in (0.5,\ 1)$	3995.73	1182.32	3995.73	2813.41	1000

Table 4: Mutually acceptable Pareto-optimal reinsurance contracts $I^*_{\lambda^*}$ with an exponential ground-up loss

$\kappa = 0.8$	$\alpha = 0.95,\ \beta = 0.99$	$\alpha = 0.99,\ \beta = 0.95$	$\alpha = \beta = 0.95$
$\gamma = 0.5$	$\lambda^* \in [0.5376,\ 1)$	$\lambda^* \in [0.5,\ 1)$	$\lambda^* \in (0.5,\ 1)$ or $\lambda^* = 0.5$ with $a^* \in [0.7146, 1]$
$\gamma = 0.7$	$\lambda^* \in (0.5,\ 1)$	$\lambda^* \in [0.2845,\ 1)$	$\lambda^* \in (0.5,\ 1)$ or $\lambda^* = 0.5$ with $a^* \in [0.4288, 1]$
$\gamma = 0.8$	$\lambda^* \in (0.5,\ 1)$	$\lambda^* \in [0.1817,\ 1)$	$\lambda^* \in (0.5,\ 1)$ or $\lambda^* = 0.5$ with $a^* \in [0.2858, 1]$

If $\alpha = \beta = 0.95$, then $(1-\alpha)(1+\theta) < 1$. By Theorem 3.1 or 3.2 , the Pareto-optimal reinsurance contracts are

$$I^*_\lambda(x) = \begin{cases} x \wedge 125.32 & \text{if } \lambda \in (0,\ 0.5); \\ ax,\ a \in [0,1] & \text{if } \lambda = 0.5; \\ (x - 125.32)_+ & \text{if } \lambda \in (0.5,\ 1). \end{cases} \tag{4.9}$$

The corresponding key values are given in Table 7.

Based on those values given in Table 5-7, and the forms of Pareto-optimal reinsurance contracts given in (4.7)-(4.9), we can easily find $\lambda^* \in (0,1)$ such that the corresponding Pareto-optimal reinsurance contracts $I^*_{\lambda^*}$ satisfy (4.1)-(4.3). Such values of λ^* are summarized in Table 8.

If $\alpha < \beta$ with $\alpha = 0.95$ and $\beta = 0.99$, from Table 8 and (4.7), we see that the limited stop-loss reinsurances $I^*_{\lambda^*}(x) = (x - 125.32)_+ \wedge (\mathrm{VaR}_{p(\lambda^*)}(X) - 125.32)$ are the mutually acceptable Pareto-optimal reinsurances for all the three cases of γ, where the values of λ^* for each case are given in Table 8.

If $\alpha > \beta$ with $\alpha = 0.99$ and $\beta = 0.95$, then, from Table 8 and (4.8), we find that the contracts $I^*_{\lambda^*}(x) = x \wedge 125.32 + (x - \mathrm{VaR}_{q(\lambda^*)}(X))_+$ are the mutually acceptable Pareto-optimal reinsurance contracts

Table 5: Key values with a Pareto ground-up loss when $\alpha = 0.95 < \beta = 0.99$

	$\mathrm{TVaR}_\alpha(X)$	$\mathrm{TVaR}_\alpha(C_{I_\lambda^*}(X))$	$\mathrm{TVaR}_\beta(X)$	$\mathrm{TVaR}_\beta((R_{I_\lambda^*}(X))$	$\mathbb{E}[\pi(I_\lambda^*(X))]$
$\lambda \in (0, 0.5]$	6143.25	6155.27	11924.77	-12.02	137.34
$\lambda \in (0.5, 0.84]$	6143.25	(3739.53, 2061.18]	11924.77	(2403.72, 6147.84]	(899.79, 1006.92]
$\lambda \in (0.84, 1)$	6143.25	1187.98	11924.77	10736.79	1062.66

Table 6: Key values with a Pareto ground-up loss when $\alpha = 0.99 > \beta = 0.95$

	$\mathrm{TVaR}_\alpha(X)$	$\mathrm{TVaR}_\alpha(C_{I_\lambda^*}(X))$	$\mathrm{TVaR}_\beta(X)$	$\mathrm{TVaR}_\beta((R_{I_\lambda^*}(X))$	$\mathbb{E}[\pi(I_\lambda^*(X))]$
$\lambda \in (0,\ 0.1599]$	11924.77	11936.79	6143.25	-12.02	137.34
$\lambda \in (0.1599,\ 0.5]$	11924.77	(7349.98, 3603.72]	6143.25	(860.77, 2539.53]	(193.05, 300.21]
$\lambda \in (0.5,\ 1)$	11924.77	1187.98	6143.25	4955.27	1062.66

Table 7: Key values a Pareto ground-up loss when $\alpha = \beta = 0.95$

	$\mathrm{TVaR}_\alpha(X)$	$\mathrm{TVaR}_\alpha(C_{I_\lambda^*}(X))$	$\mathrm{TVaR}_\beta(X)$	$\mathrm{TVaR}_\beta((R_{I_\lambda^*}(X))$	$\mathbb{E}[\pi(I^*(X))]$
$\lambda \in (0,\ 0.5)$	6143.25	6155.27	6143.25	-12.02	137.34
$\lambda = 0.5,\ I_\lambda^*(X) = a \cdot x,\ a \in [0, 1]$	6143.25	[6143.25, 1200]	6143.25	[0, 4943.25]	[0, 1200]
$\lambda \in (0.5,\ 1)$	6143.25	1187.98	6143.25	4955.27	1062.66

Table 8: Mutually acceptable Pareto-optimal reinsurance contracts $I_{\lambda^*}^*$ with a Pareto ground-up loss

$\kappa = 0.8$	$\alpha = 0.95,\ \beta = 0.99$	$\alpha = 0.99,\ \ \beta = 0.95$	$\alpha = \beta = 0.95$
$\gamma = 0.5$	$\lambda^* \in [0.6173,\ 0.84]$	$\lambda^* \in [0.2392,\ 0.5]$	$\lambda^* = 0.5$ with $a^* \in [0.6214,\ 0.9942]$
$\gamma = 0.7$	$\lambda^* \in (0.5,\ 0.84]$	$\lambda^* \in (0.1599,\ 0.5]$	$\lambda^* = 0.5$ with $a^* \in [0.3728,\ 0.9942]$
$\gamma = 0.8$	$\lambda^* \in (0.5,\ 0.84]$	$\lambda^* \in (0.1599,\ 0.5]$	$\lambda^* = 0.5$ with $a^* \in [0.2486,\ 0.9942]$

for all the three cases of γ, where the values of λ^* for each case are given in Table 8.

If $\alpha = \beta = 0.95$, then, from Table 8 and (4.9), we see that the quota-share reinsurances $I^*(x) = a^*\, x$ are the mutually acceptable Pareto-optimal reinsurance contracts for all the three cases of γ, where the values of a^* for each case are given in Table 8. □

Both Tables 4 and 8 show that the higher of the insurer's requirement (such as a smaller value of γ), the less of the choices of the mutually acceptable Pareto-optimal reinsurances or the best values of λ^*. Moreover, the distribution of the ground-up loss random variable and the confidence levels of TVaRs also have significant influences in the mutually acceptable Pareto-optimal contracts. If $\alpha \leqslant \beta$, which means that the TVaR standard of the reinsurer is not lower than the insurer, then the riskier of the ground-up loss (such as the Pareto loss), the less of the choices of the mutually acceptable Pareto-optimal contracts or the best values of λ^*. However, if $\alpha > \beta$ or the insurer has a higher standard on TVaR than the reinsurer, then a more riskier ground-up loss (the Pareto loss) will result in a more conservative mutually acceptable Pareto-optimal contract (such as the reinsurance with a limit) for the reinsurer, while for a less riskier ground-up loss (the exponential loss), an unlimited contract such as the stop-loss reinsurance can be the mutually acceptable Pareto-optimal contract for the reinsurer. All these observations or findings reflect the conflicting interests between the insurer and the reinsurer.

In addition, we also point out that if an insurer or an insurance has a 'greedy' requirement in a reinsurance contract, such as a very small value of γ or a very large value of σ and κ in Examples 4.1 and 4.2, then the mutually acceptable Pareto-optimal reinsurance contracts may not exist. Indeed, the insurer and the reinsurer are not able to make a deal of reinsurance if any of the two parties has a 'greedy' requirement in a reinsurance contract.

5 Conclusions

In this paper, we give a comprehensive study of Pareto-optimal reinsurance arrangements and show that under general model settings and assumptions, a Pareto-optimal reinsurance contract is an optimizer of the convex combination of both parties' preferences, and such optimizers always exist. This result helps to justify many existing research techniques on the joint optimization problems for an insurer and a reinsurer. Moreover, we show how to solve an optimal reinsurance problem by minimizing the convex combination of TVaRs of the insurer's and the reinsurer's losses and to find the mutually acceptable Pareto-optimal reinsurance contracts in the sense that both the insurer's aim and the reinsurer's goal can be satisfied.

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A Complete proof of Theorem 3.1

Proof. (i) This case is proved in Section 3.

(ii) If $0 \leqslant \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} = m$, then $m > 0$. For any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \xi_a; \\ \xi_a & \text{if } \xi_a \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) \leqslant x \leqslant \xi_b - \xi_a + \mathrm{VaR}_\alpha(X); \\ \xi_b & \text{if } \xi_b - \xi_a + \mathrm{VaR}_\alpha(X) \leqslant x \leqslant \mathrm{VaR}_\beta(X); \\ \tilde{I}(x) & \text{if } x > \mathrm{VaR}_\beta(X), \end{cases}$$

where $\tilde{I}$ can be any function such that $\hat{I} \in \mathcal{I}$.

The conditions $0 \leqslant \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} = m$ imply $(1-\alpha)(1+\theta) \geqslant 1$. Then $\xi_a^* = \mathrm{VaR}_\alpha(X)$ and $\xi_b^* = \mathrm{VaR}_p(X)$. The optimal reinsurance contract is $I^*(x) = (x \wedge \mathrm{VaR}_p(X))\mathbb{I}_{\{x \leqslant \mathrm{VaR}_\beta(X)\}} + \tilde{I}(x)\mathbb{I}_{\{x > \mathrm{VaR}_\beta(X)\}}$.

(iii) If $0 \leqslant \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} < m$, then $m > 0$. For any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \xi_a; \\ \xi_a & \text{if } \xi_a \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) \leqslant x \leqslant \mathrm{VaR}_\alpha(X) + \xi_b - \xi_a; \\ \xi_b & \text{if } \mathrm{VaR}_\alpha(X) + \xi_b - \xi_a \leqslant x \leqslant \mathrm{VaR}_\beta(X); \\ x - \mathrm{VaR}_\beta(X) + \xi_b & \text{if } x \geqslant \mathrm{VaR}_\beta(X). \end{cases}$$

One can show that $\hat{I}(x) \in \mathcal{I}$, $V(I) \geqslant V(\hat{I})$ from (3.6), and

$$\mathbb{P}\left(\hat{I}(X) > x\right) = \begin{cases} \mathbb{P}\left(X > x\right) & \text{if } 0 \leqslant x < \xi_a; \\ \mathbb{P}\left(X > x + \mathrm{VaR}_\alpha(X) - \xi_a\right) & \text{if } \xi_a \leqslant x < \xi_b; \\ \mathbb{P}\left(X > x + \mathrm{VaR}_\beta(X) - \xi_b\right) & \text{if } x \geqslant \xi_b. \end{cases}$$

Let $s = \xi_a$ and $t = \xi_b - \xi_a$. It follows from (3.7)

$$\begin{aligned} V(\hat{I}) = {} & \lambda \mathrm{TVaR}_\alpha(X) + (1-2\lambda)s + (1-\lambda)t - (1-2\lambda)(1+\theta) \int_0^s \mathbb{P}(X > z)\mathrm{d}z \\ & - m \int_{\mathrm{VaR}_\alpha(X)}^{t+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z + \left(\frac{1-\lambda}{1-\beta} - m\right) \int_{\mathrm{VaR}_\beta(X)}^\infty \mathbb{P}(X > z)\mathrm{d}z. \end{aligned}$$

Let

$$g(t) = (1-\lambda)t - m \int_{\mathrm{VaR}_\alpha(X)}^{t+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z.$$

Then for $0 < t_1 < t_2 < \mathrm{VaR}_\beta(X) - \mathrm{VaR}_\alpha(X)$, as $\mathbb{P}\left(X \leqslant t_2 + \mathrm{VaR}_\alpha(X)\right) < \beta$, we have

$$\begin{aligned} g(t_1) - g(t_2) &= (1-\lambda)(t_1 - t_2) + m \int_{t_1+\mathrm{VaR}_\alpha(X)}^{t_2+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z \\ &\geqslant (t_2 - t_1)\left(m\mathbb{P}\left(X > t_2 + \mathrm{VaR}_\alpha(X)\right) - (1-\lambda)\right) > 0. \end{aligned}$$

The conditions $0 \leqslant \lambda < \frac{1}{2}$ and $\frac{1-\lambda}{1-\beta} < m$ imply $(1-\alpha)(1+\theta) > 1$, $s^* = \mathrm{VaR}_\alpha(X)$ and $t^* = \mathrm{VaR}_\beta(X) - \mathrm{VaR}_\alpha(X)$ minimize $V(\hat{I})$. Therefore, $\xi_a^* = \mathrm{VaR}_\alpha(X)$ and $\xi_b^* = \mathrm{VaR}_\beta(X)$ minimize $V(\hat{I})$ and the optimal reinsurance contract is $I^*(x) = x$.

(iv) If $\lambda = \frac{1}{2}$, then $m = \frac{1}{2(1-\alpha)}$ and $p = \alpha$. Furthermore, if $\alpha = \beta$, then $V(I) = \frac{1}{2}\mathrm{TVaR}_\alpha(X)$ for any $I \in \mathcal{I}$; if $\alpha < \beta$, then for any $I \in \mathcal{I}$,

$$V(I) = \frac{1}{2}\mathrm{TVaR}_\alpha(X) - \frac{1}{2(1-\alpha)} \int_\alpha^\beta I(\mathrm{VaR}_r(X))\mathrm{d}r + \frac{1}{2}\left(\frac{1}{1-\beta} - \frac{1}{1-\alpha}\right) \int_\beta^1 I(\mathrm{VaR}_r(X))\mathrm{d}r, \quad \text{(A.1)}$$

or equivalently

$$\begin{aligned} V(I) = {} & \frac{1}{2}\mathrm{TVaR}_\alpha(X) + \frac{1}{2}(\xi_b - \xi_a) - \frac{1}{2(1-\alpha)} \int_{\xi_a}^{\xi_b} \mathbb{P}(I(X) > z)\mathrm{d}z \\ & + \frac{1}{2}\left(\frac{1}{1-\beta} - \frac{1}{1-\alpha}\right) \int_{\xi_b}^\infty \mathbb{P}(I(X) > z)\mathrm{d}z. \end{aligned} \quad \text{(A.2)}$$

where $\xi_a = I(\mathrm{VaR}_\alpha(X))$ and $\xi_b = I\left(\mathrm{VaR}_\beta(X)\right)$. Define

$$\hat{I}(x) = \begin{cases} \tilde{I}(x) & \text{if } 0 \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) \leqslant x \leqslant \xi_b - \xi_a + \mathrm{VaR}_\alpha(X); \\ \xi_b & \text{if } x \geqslant \xi_b - \xi_a + \mathrm{VaR}_\alpha(X), \end{cases}$$

where $\tilde{I}$ can be any function such that $\hat{I} \in \mathcal{I}$. According to (A.1), it is easy to show that $V(I) \geqslant V(\hat{I})$. By (A.2), we have

$$V(\hat{I}) = \frac{1}{2}\mathrm{TVaR}_\alpha(X) + \frac{1}{2}(\xi_b - \xi_a) - \frac{1}{2(1-\alpha)}\int_{\mathrm{VaR}_\alpha(X)}^{\mathrm{VaR}_\alpha(X)+\xi_b-\xi_a} \mathbb{P}(X > z)\mathrm{d}z.$$

Let $t = \xi_b - \xi_a$ and $g(t) = t - \frac{1}{(1-\alpha)}\int_{\mathrm{VaR}_\alpha(X)}^{t+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z$. For $t_1, t_2 \in [0, \mathrm{VaR}_\beta(X) - \mathrm{VaR}_\alpha(X)]$ and $t_1 \geqslant t_2$,

$$g(t_2)-g(t_1) = t_2-t_1+\frac{1}{1-\alpha}\int_{t_2+\mathrm{VaR}_\alpha(X)}^{t_1+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z \leqslant (t_1-t_2)\left[\frac{1}{1-\alpha}\mathbb{P}(X > t_2 + \mathrm{VaR}_\alpha(X)) - 1\right] \leqslant 0.$$

Therefore, g is increasing in $t \in [0, \mathrm{VaR}_\beta(X) - \mathrm{VaR}_\alpha(X)]$ and $t^* = 0$ minimizes g. The optimal solution is $I^*(x) = \tilde{I}(x) \wedge \tilde{I}(\mathrm{VaR}_\alpha(X))$, where $\tilde{I}$ can be any function such that $I^* \in \mathcal{I}$.

(v) If $\frac{1}{2} < \lambda < 1$ and $\frac{1-\lambda}{1-\beta} > m > 0$, for any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \mathrm{VaR}_\alpha(X) - \xi_a; \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) - \xi_a \leqslant x \leqslant \xi_b - \xi_a + \mathrm{VaR}_\alpha(X); \\ \xi_b & \text{if } x \geqslant \xi_b - \xi_a + \mathrm{VaR}_\alpha(X). \end{cases} \tag{A.3}$$

We can show that $\hat{I}(x) \in \mathcal{I}$, $V(I) \geqslant V(\hat{I})$ from (3.6), and

$$\mathbb{P}\left(\hat{I}(X) > x\right) = \begin{cases} \mathbb{P}(X > x + \mathrm{VaR}_\alpha(X) - \xi_a) & \text{if } 0 \leqslant x < \xi_b; \\ 0 & \text{if } x \geqslant \xi_b. \end{cases}$$

Let $s = \xi_a$ and $t = \xi_b - \xi_a$. It follows from (3.7)

$$V(\hat{I}) = \lambda\mathrm{TVaR}_\alpha(X) + (1-2\lambda)s + (1-\lambda)t - (1-2\lambda)(1+\theta)\int_{\mathrm{VaR}_\alpha(X)-s}^{\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z$$
$$- m\int_{\mathrm{VaR}_\alpha(X)}^{t+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z.$$

If $(1-\alpha)(1+\theta) \geqslant 1$, then $\theta/1+\theta \geqslant \alpha$ and $s^* = 0$. Moreover,

$$(1-\alpha)(1+\theta) \geqslant 1 \iff p \leqslant \alpha. \tag{A.4}$$

Indeed, recall that $p = 1 - \frac{1-\lambda}{m}$ and $m = \frac{\lambda}{1-\alpha} + (1-2\lambda)(1+\theta) > 0$. It is equivalent to show that $(1-\alpha)m \leqslant 1-\lambda$, i.e., $\lambda + (1-2\lambda)(1+\theta)(1-\alpha) \leqslant 1-\lambda$, which is $(1-2\lambda)(1+\theta)(1-\alpha) \leqslant 1-2\lambda$, or $(1+\theta)(1-\alpha) \geqslant 1$ since $1-2\lambda < 0$. Thus, $p \leqslant \alpha$. One can show that $t^* = 0$. As a result, $\xi_a^* = 0$ and $\xi_b^* = 0$. Thus, $I^*(x) = 0$ is an optimal contract.

If $(1-\alpha)(1+\theta) < 1$, then $\theta/1+\theta < \alpha$ and from (A.4), we know $p > \alpha > \theta/1+\theta$ and $\frac{1-\lambda}{1-\beta} > m$ implies $p < \beta$. Therefore, $s^* = \mathrm{VaR}_\alpha(X) - \mathrm{VaR}_{\theta/1+\theta}(X)$ and $t^* = \mathrm{VaR}_p(X) - \mathrm{VaR}_\alpha(X)$ minimize $V(\hat{I})$, which implies $\xi_a^* = \mathrm{VaR}_\alpha(X) - \mathrm{VaR}_{\theta/1+\theta}(X)$ and $\xi_b^* = \mathrm{VaR}_p(X) - \mathrm{VaR}_{\theta/1+\theta}(X)$. Hence, $I^*(x) = (x - \mathrm{VaR}_{\theta/1+\theta}(X))_+ \wedge \left(\mathrm{VaR}_p(X) - \mathrm{VaR}_{\theta/1+\theta}(X)\right)$ is an optimal reinsurance contract.

(vi) If $\frac{1}{2} < \lambda < 1$ and $\frac{1-\lambda}{1-\beta} = m > 0$ and note that $\xi_b - \xi_a + \mathrm{VaR}_\alpha(X) \leqslant \mathrm{VaR}_\beta(X)$, for any $I \in \mathcal{I}$, define $\hat{I}(x)$ the same as in (A.3) for $x \leqslant \mathrm{VaR}_\beta(X)$ and define $\hat{I}(x) = \tilde{I}(x)$ for $x \geqslant \mathrm{VaR}_\beta(X)$, where $\tilde{I}$ can be any function such that $\hat{I} \in \mathcal{I}$. If $\alpha = \beta$, then $\frac{1-\lambda}{1-\beta} = m$ implies $(1-\alpha)(1+\theta) = 1$ and $p = \alpha$, the optimal reinsurance contract is $I^*(x) = \tilde{I}(x)\mathbb{I}_{\{x > \mathrm{VaR}_\beta(X)\}}$. If $\alpha < \beta$, then $\frac{1-\lambda}{1-\beta} = m$ implies $(1-\alpha)(1+\theta) < 1$, the optimal contract is

$$I^*(x) = \left[(x - \mathrm{VaR}_{\theta/1+\theta}(X))_+ \wedge \left(\mathrm{VaR}_p(X) - \mathrm{VaR}_{\theta/1+\theta}(X)\right)\right]\mathbb{I}_{\{x \leqslant \mathrm{VaR}_\beta(X)\}} + \tilde{I}(x)\mathbb{I}_{\{x > \mathrm{VaR}_\beta(X)\}}. \tag{A.5}$$

Hence, in either case, the optimal contract is given in (A.5). Note $p = \beta$ since $\frac{1-\lambda}{1-\beta} = m$.

(vii) If $\frac{1}{2} < \lambda \leqslant 1$ and $\frac{1-\lambda}{1-\beta} < m$, for any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \mathrm{VaR}_\alpha(X) - \xi_a; \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) - \xi_a \leqslant x \leqslant \xi_b - \xi_a + \mathrm{VaR}_\alpha(X); \\ \xi_b & \text{if } \xi_b - \xi_a + \mathrm{VaR}_\alpha(X) \leqslant x \leqslant \mathrm{VaR}_\beta(X); \\ x - \mathrm{VaR}_\beta(X) + \xi_b & \text{if } x \geqslant \mathrm{VaR}_\beta(X). \end{cases}$$

One can show that $\hat{I}(x) \in \mathcal{I}$, $V(I) \geqslant V(\hat{I})$ from (3.6), and

$$\mathbb{P}\left(\hat{I}(X) > x\right) = \begin{cases} \mathbb{P}\left(X > x + \mathrm{VaR}_\alpha(X) - \xi_a\right) & \text{if } 0 \leqslant x < \xi_b; \\ \mathbb{P}\left(X > x + \mathrm{VaR}_\beta(X) - \xi_b\right) & \text{if } x \geqslant \xi_b. \end{cases}$$

Let $s = \xi_a$ and $t = \xi_b - \xi_a$. It follows from (3.7)

$$\begin{aligned} V(\hat{I}) = {} & \lambda \mathrm{TVaR}_\alpha(X) + (1-2\lambda)s + (1-\lambda)t - (1-2\lambda)(1+\theta)\int_{\mathrm{VaR}_\alpha(X)-s}^{\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z \\ & - m\int_{\mathrm{VaR}_\alpha(X)}^{t+\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z + \left(\frac{1-\lambda}{1-\beta} - m\right)\int_{\mathrm{VaR}_\beta(X)}^{\infty} \mathbb{P}(X > z)\mathrm{d}z. \end{aligned}$$

As $1-\beta \leqslant \mathbb{P}(X > t + \mathrm{VaR}_\alpha(X)) \leqslant 1-\alpha$ and $m > 0$, one can show that $t^* = \mathrm{VaR}_\beta(X) - \mathrm{VaR}_\alpha(X)$. The conditions $\frac{1}{2} < \lambda \leqslant 1$ and $\frac{1-\lambda}{1-\beta} < m$ imply $(1-\alpha)(1+\theta) < 1$, thus $\theta/1+\theta < \alpha \leqslant \beta$. $s^* = \mathrm{VaR}_\alpha(X) - \mathrm{VaR}_{\theta/1+\theta}(X)$ minimizes f. Hence, $\xi_a^* = \mathrm{VaR}_\alpha(X) - \mathrm{VaR}_{\theta/1+\theta}(X)$ and $\xi_b^* = \mathrm{VaR}_\beta(X) - \mathrm{VaR}_{\theta/1+\theta}(X)$ minimize $V(\hat{I})$. Thus, $I^*(x) = (x - \mathrm{VaR}_{\theta/1+\theta}(X))_+$ is an optimal reinsurance contract.

(viii) If $\frac{1}{2} < \lambda < 1$ and $m = 0$, then $\frac{1-\lambda}{1-\beta} - m > 0$. For any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \mathrm{VaR}_\alpha(X) - \xi_a; \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) - \xi_a \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ \tilde{I}(x) & \text{if } \mathrm{VaR}_\alpha(X) \leqslant x \leqslant \mathrm{VaR}_\beta(X); \\ \xi_b & \text{if } x \geqslant \mathrm{VaR}_\beta(X), \end{cases}$$

where $\tilde{I}$ can be any function such that $\hat{I} \in \mathcal{I}$. It follows from (3.7)

$$V(\hat{I}) = \lambda \mathrm{TVaR}_\alpha(X) + (1-2\lambda)\xi_a + (1-\lambda)(\xi_b - \xi_a) - (1-2\lambda)(1+\theta)\int_{\mathrm{VaR}_\alpha(X)-\xi_a}^{\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z.$$

Let $s = \xi_a$ and $t = \xi_b - \xi_a$. Then $t^* = 0$ as $1 - \lambda > 0$. Let

$$f(s) = \lambda \mathrm{TVaR}_\alpha(X) + (1-2\lambda)s - (1-2\lambda)(1+\theta)\int_{\mathrm{VaR}_\alpha(X)-s}^{\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z.$$

For $0 < s_1 < s_2 \leqslant \mathrm{VaR}_\alpha(X)$,

$$\begin{aligned} f(s_2) - f(s_1) &= (2\lambda - 1)\left[(1+\theta)\int_{\mathrm{VaR}_\alpha(X)-s_2}^{\mathrm{VaR}_\alpha(X)-s_1} \mathbb{P}(X > z)\mathrm{d}z - (s_2 - s_1)\right] \\ &\geqslant (2\lambda-1)(s_2 - s_1)\left[(1+\theta)\mathbb{P}(X > \mathrm{VaR}_\alpha(X) - s_1) - 1\right] \\ &> (2\lambda-1)(s_2 - s_1)\left[(1+\theta)(1-\alpha) - 1\right] > 0, \end{aligned}$$

where the last inequality holds since $m = 0$ implies $(1-\alpha)(1+\theta) = \frac{\lambda}{2\lambda-1} > 1$. Therefore, $s^* = 0$ is the unique minimizer of f. Thus, $\xi_a^* = \xi_b^* = 0$ and the optimal contract is $I^* = 0$.

(ix) If $\lambda = 1$ and $m = 0$, then $(1-\alpha)(1+\theta) = 1$. For any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \mathrm{VaR}_\alpha(X) - \xi_a; \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) - \xi_a \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ \tilde{I}(x) & \text{if } x \geqslant \mathrm{VaR}_\alpha(X), \end{cases}$$

where $\tilde{I}$ can be any function such that $\hat{I} \in \mathcal{I}$. It follows from (3.7)

$$V(\hat{I}) = \mathrm{TVaR}_\alpha(X) - \xi_a + (1+\theta)\int_{\mathrm{VaR}_\alpha(X)-\xi_a}^{\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z.$$

It is easy to show that $\xi_a^* = 0$ and the optimal reinsurance contract is $I^* = \tilde{I}(x)\mathbb{I}_{\{x \geqslant \mathrm{VaR}_\alpha(X)\}}$.

(x) If $\frac{1}{2} < \lambda \leqslant 1$ and $m < 0$, then $\frac{1-\lambda}{1-\beta} > m$. For any $I \in \mathcal{I}$, define

$$\hat{I}(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant \mathrm{VaR}_\alpha(X) - \xi_a; \\ x - \mathrm{VaR}_\alpha(X) + \xi_a & \text{if } \mathrm{VaR}_\alpha(X) - \xi_a \leqslant x \leqslant \mathrm{VaR}_\alpha(X); \\ \xi_a & \text{if } \mathrm{VaR}_\alpha(X) \leqslant x \leqslant \mathrm{VaR}_\beta(X) - (\xi_b - \xi_a); \\ x - \mathrm{VaR}_\beta(X) + \xi_b & \text{if } \mathrm{VaR}_\beta(X) - (\xi_b - \xi_a) \leqslant x \leqslant \mathrm{VaR}_\beta(X); \\ \xi_b & \text{if } x \geqslant \mathrm{VaR}_\beta(X). \end{cases} \tag{A.6}$$

One can show that $\hat{I}(x) \in \mathcal{I}$, $V(I) \geqslant V(\hat{I})$ from (3.6),, and

$$\mathbb{P}\left(\hat{I}(X) > x\right) = \begin{cases} \mathbb{P}\left(X > x + \mathrm{VaR}_\alpha(X) - \xi_a\right) & \text{if } 0 \leqslant x < \xi_a; \\ \mathbb{P}\left(X > x + \mathrm{VaR}_\beta(X) - \xi_b\right) & \text{if } \xi_a < x < \xi_b; \\ 0 & \text{if } x \geqslant \xi_b. \end{cases}$$

Let $s = \xi_a$ and $t = \xi_b - \xi_a$. It follows from (3.7)

$$V(\hat{I}) = \lambda \mathrm{TVaR}_\alpha(X) + (1-2\lambda)s + (1-\lambda)t - (1-2\lambda)(1+\theta)\int_{\mathrm{VaR}_\alpha(X)-s}^{\mathrm{VaR}_\alpha(X)} \mathbb{P}(X > z)\mathrm{d}z$$
$$- m\int_{\mathrm{VaR}_\beta(X)-t}^{\mathrm{VaR}_\beta(X)} \mathbb{P}(X > z)\mathrm{d}z.$$

Note that $m < 0$. The conditions $\frac{1}{2} < \lambda \leqslant 1$ and $m < 0$ imply $(1-\alpha)(1+\theta) > 1$. Then $s^* = 0$ and $t^* = 0$. Hence, $\xi_a^* = \xi_b^* = 0$ and $I^*(x) = 0$ is an optimal reinsurance contract. □

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