Negative dependence in matrix arrangement problems

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Abstract

Minimizing an arrangement increasing (AI) function with a matrix input over intra-column permutations is a difficult optimization problem of a combinatorial nature. Unlike maximization of AI functions (which is achieved by perfect positive dependence, namely, arranging all columns in an increasing order), minimization is a much more challenging problem due to the lack of a universal definition and construction of compensating arrangements in more than two dimensions. We consider AI functions with a special structure, which facilitates finding close-to-optimal solutions by employing the concept of Σ-countermonotonicity and the (Block) Rearrangement Algorithm. We show that many classical optimization problems, including stochastic crew scheduling and assembly of reliable systems, have objective functions with this structure, and illustrate with a numerical case study. This paves a path to obtaining approximate solutions for problems that have so far been considered intractable.

Keywords: Schur-convexity, negative dependence, scheduling, systems assembly, Archimedean copulas, Rearrangement Algorithm

1 Introduction

In this paper we address a class of matrix arrangement problems. Let S_n be the set of n-permutations, we then write $\mathbf{x}^\pi = (x_{\pi(1)}, \ldots, x_{\pi(n)})$ for a column vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\pi \in S_n$. For a given matrix $X = (\mathbf{x}_1, \ldots, \mathbf{x}_d) \in \mathbb{R}^{n\times d}$ and a target function $\phi : \mathbb{R}^{n\times d} \to \mathbb{R}$, the aim is to minimize or maximize

$$\phi(X^{\boldsymbol{\pi}}) \quad \text{over } \boldsymbol{\pi} \in (S_n)^d, \quad \text{where} \quad X^{\boldsymbol{\pi}} = (\mathbf{x}_1^{\pi_1}, \ldots, \mathbf{x}_d^{\pi_d}) \quad \text{for} \quad \boldsymbol{\pi} = (\pi_1, \ldots, \pi_d). \tag{1}$$

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That is, one is allowed to permute the elements within each column of the matrix, but not to exchange elements between columns. In this paper, any minimization and maximization of an $\mathbb{R}^{n\times d} \to \mathbb{R}$ function refers to the problem in (1).

As one example, we consider an assembly line crew scheduling problem, where n items are to be produced on n assembly lines. To each line we must assign d workers, specialized in different jobs. In total, there are nd workers (n from each of the d specializations), and each has a different random completion time of their job. The objective is to maximize the probability of finishing all n items within a given deadline by assigning the workers to the assembly lines optimally. The matrix X in this example represents some known parameters of the distributions of the completion time for the nd individual workers.

Another example is a systems assembly problem, where n systems are to be assembled, each composed of d components of different types, connected in parallel. There are nd components available (n of each type), with different individual probabilities of failure. The objective is to maximize the sum of system reliabilities, i.e. the expected number of systems that function satisfactorily. The matrix X in this example represents the individual probabilities of failure for the nd components.

In these contexts, a *compensating arrangement* is often desirable, in the sense that the fast workers compensate for the slow workers in each team, or the reliable components compensate for the less reliable ones within each system. The teams or systems, respectively, are represented by the rows of the matrix. The decision variable is a vector of permutations, corresponding to the arrangement of elements within each column of the input matrix.

The above two problems of a stochastic nature will be addressed in Section 4. The main focus of this article is on a general class of objective functions in (1): ones that have the form

$$\phi(X) = g\left(\textstyle\sum_{j=1}^{d} h_j(x_{1j}), \dots, \sum_{j=1}^{d} h_j(x_{nj})\right), \quad X \in \mathbb{R}^{n\times d}, \tag{2}$$

where g is a *Schur-convex* function (see Section 2.2 for the definition) and (throughout) $h_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \dots, d$ are monotone in the same direction. Here and in the following, the terms “monotone”, “increasing” and “decreasing” are used in the non-strict sense. A simple example of such an objective function is

$$\phi_1(X) = \max_{i=1,\dots,n} \left\{\textstyle\sum_{j=1}^{d} x_{ij}\right\}, \quad X \in \mathbb{R}^{n\times d}. \tag{3}$$

To maximize $\phi_1(X^{\boldsymbol{\pi}})$ over $\boldsymbol{\pi} \in (S_n)^d$ is a relatively easy task; the solution is simply to arrange $X^{\boldsymbol{\pi}}$ as a *similarly ordered* matrix (see Section 2.1). However, the minimization of $\phi_1(X^{\boldsymbol{\pi}})$ over $\boldsymbol{\pi} \in (S_n)^d$ is known to be a highly non-trivial task; for $d \geq 3$, the minimization of (3) was shown to be NP-complete in Hsu (1984); Coffman and Yannakakis (1984), see also Haus (2015) for a further analysis of the complexity of this problem. Note that the set of possible arrangements of a given matrix $X \in \mathbb{R}^{n\times d}$ is enormous (of size $(n!)^d$), corresponding to all possible intra-column permutations.

To handle the above minimization problems, we borrow some recent developments from probabilistic dependence modeling. A matrix in $\mathbb{R}^{n\times d}$ can be interpreted as a discrete random vector that is uniformly distributed over n points in $\mathbb{R}^d$, where each column corresponds to a univariate margin and each row - to a possible outcome. Permuting the elements within the columns would therefore alter the joint distribution, but not the (univariate) marginal distributions. Thus, we can interpret different intra-column arrangements of a given matrix as different dependence structures of the corresponding random variable with fixed marginals. The solution to the maximization of $\phi_1(X^{\boldsymbol{\pi}})$ over $\boldsymbol{\pi} \in (S_n)^d$ is naturally translated into dependence modeling as an *extremal positive dependence*. Although not properly formulated, one would expect a solution for the minimization of $\phi_1(X^{\boldsymbol{\pi}})$ to be translated as an *extremal negative dependence*. The extremal positive dependence, known as *comonotonicity* in dependence modeling, is a well-studied and well-understood concept. A considerable amount of research has applied the concept of positive multivariate dependence to optimization problems; see Shaked and Shanthikumar (1990, 1997); Derman et al. (1972); Colangelo et al. (2005). In contrast, negative dependence has attracted less attention, largely due to the difficulties associated even with defining it; see Ebrahimi and Ghosh (1981); Block et al. (1982); Joag-Dev and Proschan (1983) for early examples. The recent paper by Puccetti and Wang (2015) contains an overview of extremal positive and negative dependence concepts, where a clear asymmetry between the two sides of the coin can easily be spotted.

The recent developments in dependence modeling offer some insights into the problem of minimizing $\phi_1(X^{\boldsymbol{\pi}})$ over $\boldsymbol{\pi} \in (S_n)^d$. For example, it is intuitively clear that one needs to find $\boldsymbol{\pi} \in (S_n)^d$ such that the row-sums of $X^{\boldsymbol{\pi}}$ are *as similar as possible*. The concepts of *joint mixability* (Wang and Wang (2011, 2016)) and Σ-*countermonotonicity* (Puccetti and Wang (2015)) are introduced to characterize such dependence scenarios. In the area of financial risk management, a novel matrix rearrangement method (the Rearrangement Algorithm) has recently been introduced for obtaining bounds on risk measures of the aggregate risk, under dependence uncertainty between the individual random variables (risks) with given marginal distributions; see Puccetti and Rüschendorf (2012); Embrechts et al. (2013); Puccetti (2013). This method was successfully applied in a practical case study (Aas and Puccetti, 2014) and has contributed to the regulatory discussion in the financial industry (Embrechts et al., 2014). The above ideas from the risk management context have further applications, beyond computing these so-called *dependence uncertainty* bounds. For example, Boudt et al. (2017) show that k-partitioning and other classical, non-stochastic problems can be formulated as matrix rearrangement problems. The contributions of this paper are twofold. First, we formalize a class of objective functions to which the concept of Σ-countermonotonicity and the new Rearrangement Algorithm can be applied. To our knowledge, this is the most general characterization up to date. Second, we provide explicit examples of stochastic problems in scheduling and reliability engineering that belong to this type, and that have so far not been considered in the literature.

This article aims to expand the usage of concepts and algorithms that originally stem from applications in the financial industry, to problems in other fields. For other examples of such academic cross pollination, see Prékopa (2012); Lee and Prékopa (2013), where multivariate risk measures are examined from an optimization point of view. We also use stochastic orderings to characterize optimal solutions, an approach similar to Yao (1987); Li and You (2014). See also Katehakis and Smit (2012); Shi et al. (2013) for further related stochastic optimization problems.

This paper is organized as follows. In Section 2, we identify and discuss a large class of objective functions associated with the matrix arrangement problem considered in this article. In Section 3, with the help from recent developments in dependence modeling and risk management, we show that the potential minimizers for objective functions of the form (2) belong to a smaller set of matrices having a Σ-countermonotonic structure, leading to a practical sub-optimal solution via efficient algorithms; we also show that Archimedean copulas are suitable objective functions for the matrix arrangement problems we consider. In Section 4, two problems in stochastic scheduling and systems assembly are shown to have objective functions of the form (2).We apply a heuristic which provides arrangements belonging to the smaller set of Σ-countermonotonic arrangements and illustrate, by means of a numerical case study, that often any of these special arrangements provides a close-to-optimal solution.

2 Notation and preliminaries

An excellent reference for the concepts defined in this section is Marshall et al. (2011); we shall mostly follow the notation from this textbook. For a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $\mathbf{x}^{\downarrow} = (x_{[1]}, \ldots, x_{[n]})$ and $\mathbf{x}^{\uparrow} = (x_{[n]}, \ldots, x_{[1]})$ be the decreasing and increasing arrangements of $\mathbf{x}$, respectively. Vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are called *similarly ordered* if $(\mathbf{x}^{\pi}, \mathbf{y}^{\pi}) = (\mathbf{x}^{\uparrow}, \mathbf{y}^{\uparrow})$ for some $\pi \in S_n$, and *oppositely ordered* if $(\mathbf{x}^{\pi}, \mathbf{y}^{\pi}) = (\mathbf{x}^{\uparrow}, \mathbf{y}^{\downarrow})$ for some $\pi \in S_n$. These notions correspond, respectively, to *comonotonicity* and *countermonotonicity* of random variables in probability theory (the matrix representation is relevant in a finite sample space; however, these concepts are also defined for general random variables). In the following, we introduce arrangement increasing functions and the related notions of Schur-convexity, supermodularity, and total positivity. The concept of arrangement increasing functions will be useful to formalize the broader class of functions which are maximized, respectively, minimized by matrix arrangements corresponding to extremal positive, respectively, negative dependence. We shall see however, that the former case is trivial, while the latter requires considerable restrictions to be tractable.

2.1 Arrangement increasing functions

Boland and Proschan (1988) consider functions $\phi : (\mathbb{R}^n)^d \to \mathbb{R}$ of d vector arguments in $\mathbb{R}^n$ (equivalently, functions $\phi : \mathbb{R}^{n\times d} \to \mathbb{R}$ with a matrix argument, writing $X = (\mathbf{x}_1, \dots, \mathbf{x}_d)$), which increase in value as the components of the vector arguments become more similarly arranged. To rigorously define "more similarly arranged", they introduce an equivalence relation $=_\mathrm{a}$ and a preordering $\preccurlyeq_\mathrm{a}$ between matrices. If $X, Y \in \mathbb{R}^{n\times d}$ and $(\mathbf{x}_1^\pi, \dots, \mathbf{x}_d^\pi) = Y$ for some $\pi \in S_n$, then $X =_\mathrm{a} Y$. An operation called *basic rearrangement* consists of replacing the entries in two rows k, l ($1 \le k < l \le n$) of a matrix X by their coordinate-wise minimum $(x_{k1} \wedge x_{l1}, \dots, x_{kd} \wedge x_{ld})$ and maximum $(x_{k1} \vee x_{l1}, \dots, x_{kd} \vee x_{ld})$, respectively. If matrix X can be transformed into matrix $X' =_\mathrm{a} Y$ using a sequence of basic rearrangements, then we write $X \preccurlyeq_\mathrm{a} Y$; this defines the *arrangement preordering* on $\mathbb{R}^{n\times d}$.

Definition 1. A function $\phi : \mathbb{R}^{n\times d} \to \mathbb{R}$ is called *arrangement increasing (AI)* if, for any $X, Y \in \mathbb{R}^{n\times d}$,

$$X \preccurlyeq_\mathrm{a} Y \qquad \text{implies} \qquad \phi(X) \le \phi(Y).$$

Hence, arrangement increasing functions preserve the arrangement preordering. Three useful classes of AI functions are provided by the following results from Boland and Proschan (1988); see also Marshall et al. (2011, Section 6.F).

(A) If ϕ has the form $\phi(X) = g(\mathbf{x}_1 + \dots + \mathbf{x}_n) = g(\sum_{j=1}^d x_{1j}, \dots, \sum_{j=1}^d x_{nj})$, then ϕ is AI if and only if g is Schur-convex.

(B) If ϕ has the form $\phi(X) = \sum_1^n g(x_{i1}, \dots, x_{id})$, then ϕ is AI if and only if g is supermodular.

(C) If ϕ has the form $\phi(X) = \prod_1^n g(x_{i1}, \dots, x_{id})$, then ϕ is AI if and only if g is MTP_2.

These examples motivate the relevance of the properties of Schur-convexity (defined in Section 2.2), and supermodularity and MTP_2 (defined in Appendix A.1). We will frequently refer to the above three types of AI functions throughout the paper. Type (A) functions are indeed our main objectives in (2), albeit in (2) one allows further monotone transformations $h_1, \dots, h_d$. Taking the logarithm of a type (C) function yields a type (B) function, and functions of the form

$$\phi(X) = \sum_{i=1}^n f\left(\textstyle\sum_{j=1}^d x_{ij}\right), \quad X \in \mathbb{R}^{n\times d}, \tag{4}$$

where $f : \mathbb{R} \to \mathbb{R}$ is convex, belong to the intersection of types (A) and (B). The types (A)-(C) by no means provide a complete classification of all AI functions, but merely identify three large, possibly overlapping classes.

It is straightforward to obtain the arrangement that maximizes an AI function ϕ:

$$\phi(X) \le \phi(\mathbf{x}_1^\uparrow, \dots, \mathbf{x}_d^\uparrow) = \phi(\mathbf{x}_1^\downarrow, \dots, \mathbf{x}_d^\downarrow) \quad \text{for any} \quad X \in \mathbb{R}^{n\times d}. \tag{5}$$

This follows from the fact that any matrix X can be transformed into a comonotonic one by applying at most $n(n-1)/2$ basic rearrangements. Arrangement increasing functions ϕ of two vectors, $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ (case $d = 2$), were first considered in Hollander et al. (1977) under the name *decreasing in transposition*. For two vector arguments it is also easy to obtain the minimizing arrangement - it is the countermonotonic i.e. oppositely ordered arrangement:

$$\phi(\mathbf{x}^{\downarrow}, \mathbf{y}^{\uparrow}) = \phi(\mathbf{x}^{\uparrow}, \mathbf{y}^{\downarrow}) \leq \phi(\mathbf{x}, \mathbf{y}) \quad \text{for any} \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

For higher dimensions d, however, the minimizing arrangement is not easily constructed, because an oppositely ordered arrangement is only defined for $d = 2$. Therefore, while maximizing AI functions is trivial, reversing the direction of the optimization problem often leads to a much more difficult problem. In the following, we investigate in more depth the classes of AI functions which can indeed be (approximately) minimized by applying recent developments in dependence modeling.

2.2 Majorization order and Schur-convex functions

In this section, we define the majorization preorder and Schur-convexity - concepts that are relevant to the type (A) of AI functions and, in particular, our main objective function (2). For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we say that $\mathbf{x}$ *majorizes* $\mathbf{y}$, written $\mathbf{x} \succcurlyeq_{\mathrm{m}} \mathbf{y}$, if

$$\sum_{i=1}^{k} x_{[i]} \geq \sum_{i=1}^{k} y_{[i]} \quad \text{for} \quad k = 1, \ldots, n-1, \quad \text{and} \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_j.$$

Definition 2. A function $g : D \subset \mathbb{R}^n \to \mathbb{R}$ is called *Schur-convex* if, for any $\mathbf{x}, \mathbf{y} \in D$,

$$\mathbf{x} \succcurlyeq_{\mathrm{m}} \mathbf{y} \qquad \text{implies} \qquad g(\mathbf{x}) \geq g(\mathbf{y}).$$

A function $g : D \subset \mathbb{R}^n \to \mathbb{R}$ is called *strictly Schur-convex* if, for any $\mathbf{x}, \mathbf{y} \in D$,

$$\mathbf{x} \succcurlyeq_{\mathrm{m}} \mathbf{y} \text{ and } \mathbf{y} \not\succcurlyeq_{\mathrm{m}} \mathbf{x} \qquad \text{imply} \qquad g(\mathbf{x}) > g(\mathbf{y}).$$

Reversing the inequality for g would define a *Schur-concave* function. Note that Schur-convexity requires that the function is symmetric (arguments are exchangeable). From Definition 2, it follows that a majorization-least[1] element in some set $D \subset \mathbb{R}^n$ minimizes Shur-convex functions over the set D. For example, if the set D consists of the row-sum vectors of matrices in $\{X^{\boldsymbol{\pi}} : \boldsymbol{\pi} \in (S_n)^d\}$ (column-rearrangements of $X \in \mathbb{R}^{n \times d}$), Day (1972) shows that the minimizing arrangement for $d = 2$ is the countermonotonic one (with row-sum vector $\mathbf{x}^* = \mathbf{x}_1^{\uparrow} + \mathbf{x}_2^{\downarrow}$). A comprehensive account of the applications of majorization order in statistics, probability and reliability theory can be found in Marshall et al. (2011, Chapters 11-13).

Examples of Schur-convex functions include (see Proposition 1 below)

[1] A *least element* v of a preordered set $(A, \succ)$ satisfies $u \succ v$ for all $u \in A$. For some preordered sets, such an element may not exist.

a) $g(\mathbf{x}) = \max\{x_1, \dots, x_n\}$,

b) $g(\mathbf{x}) = \sum_{i=1}^n f(x_i)$, where f is convex,

c) $g(\mathbf{x}) = \prod_{i=1}^n f(x_i)$, where f is log-convex.

d) $g(\mathbf{x}) = \sum_{i=1}^n w_i x_{[i]}$, where $w_1 \geq \dots \geq w_n \geq 0$.

e) $g(\mathbf{x}) = \sum_{i=1}^n w_i f(x_{[i]})$, where $w_1 \geq \dots \geq w_n \geq 0$ and f is increasing and convex.

Observe that substituting b) into a type (A) of AI functions, we obtain a function of the form (4). Note that, applying an increasing transformation to a Schur-convex function would preserve the Schur-convexity, however, such transformations do not affect optimization problems. By taking a logarithm of c), we arrive at b). Functions of the form d) are a special case of *L-statistics*, i.e. linear combinations of order statistics. The form in e) is the most general, in the sense that it includes a), b) and d) as special cases, and also c) after a log-transform. The Schur-convexity of functions in e) is characterized in the following proposition.

Proposition 1. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be of the form*

$$g(\mathbf{x}) = \sum_{i=1}^n w_i f(x_{[i]}), \quad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \tag{6}$$

where $w_1, \dots, w_n \geq 0$ and $f : \mathbb{R} \to \mathbb{R}$ is increasing. The following are equivalent.

(i) g is Schur-convex.

(ii) g is convex.

(iii) $w_1 \geq \dots \geq w_n$ and f is convex.

Functions of the form (6) belong to the family of *Rank-Dependent Expected Utilities* (RDEU); see Quiggin (1993). The majorization order, translated into probability theory, is equivalent to *convex order* between discrete random variables. From there, the equivalence of (i) (called *strong risk aversion* for RDEU) and (iii) is given in Chew et al. (1987); see also Schmidt and Zank (2008). A precise formulation of (i)⇔(ii)⇔(iii) is given in Theorem 5.1 of Mao and Wang (2015).

The maximum operator is a special case of (6), taking $f = \mathrm{id}$ and weights $(1, 0, \dots, 0)$. In Quantitative Risk Management, the *Expected Shortfall* at some level $\alpha \in (0, 1)$ of a random loss, whose distribution is represented by a vector $\mathbf{x}$ (as explained in the Introduction), is of the form (6) by taking equal weights $1/(1-\alpha)$ for the first $n(1-\alpha)$ entries (for simplicity assume $n\alpha \in \mathbb{Z}$). Matrix arrangement problems where type (A) objectives with $g = \max$ or $g = -\min(\cdot) = \max(-\cdot)$ are minimized, have been investigated in Puccetti and Rüschendorf (2012), and the Expected Shortfall case in Puccetti (2013). In the above examples, the matrix would correspond to a random vector with a discrete uniform distribution, the sum of columns - to the aggregate loss, and the maximization/minimization problems would correspond to determining the so-called dependence uncertainty bounds (recall that different arrangements of the input matrix represent different dependence structures between the components of the random vector). We do not further pursue these types of objectives in this article.

3 A heuristic solution to matrix arrangement problems

3.1 Sigma-countermonotonicity

As explained earlier, matrix arrangement problems are challenging problems of a combinatorial nature. The set of possible arrangements of a matrix $X \in \mathbb{R}^{n \times d}$ is of size $(n!)^d$, so finding the solution by brute force search is not possible in practice. Hence, reducing the set of possible optimizers by identifying a necessary property would be advantageous. Puccetti and Wang (2015) introduce the property of Σ-countermonotonicity for general multivariate random variables. Here, we give the finite (discrete) formulation.

Definition 3. A matrix $X \in \mathbb{R}^{n \times d}$ is said to be Σ-*countermonotonic*, if $\sum_{j \in J} \mathbf{x}_j$ and $\sum_{j \notin J} \mathbf{x}_j$ are oppositely ordered for all nonempty $J \subsetneq \{1, \dots, d\}$.

A weaker property than Σ-countermonotonic is *column oppositely ordered* (COO), which corresponds to considering only singleton sets $J = \{j\}$, $j = 1, \dots, d$ in Definition 3. In order to apply these properties in the minimization problem with the objective (2), we first transform the entries of the j^{th} column of the input matrix X by the function h_j, $j = 1, \dots, d$, and denote the resulting matrix by $H(X)$ (or simply H), where $(H)_{ij} = h_j(x_{ij})$. Then, we can work with arrangements of H instead, defining

$$\widetilde{\phi}(H) := g\left(\sum_{j=1}^d (H)_{1j}, \dots, \sum_{j=1}^d (H)_{nj}\right) \qquad (= \phi(X)).$$

The following theorem provides the theoretical basis for restricting the minimization problem with objective $\widetilde{\phi}$ to Σ-countermonotonic arrangements of H.

Theorem 2. *For a given matrix $X \in \mathbb{R}^{n \times d}$ and a function ϕ of the form (2), there exists an arrangement $\boldsymbol{\pi}^* \in (S_n)^d$ that minimizes $\phi(X^{\boldsymbol{\pi}^*})$, such that $H(X^{\boldsymbol{\pi}^*}) = H(X)^{\boldsymbol{\pi}^*}$ is Σ-countermonotonic. Furthermore, if g in (2) is strictly Schur-convex, then $H(X)^{\boldsymbol{\pi}^*}$ is Σ-countermonotonic for all minimizers $\boldsymbol{\pi}^*$.*

The proof of Theorem 2 is similar to those of Propositions 2.4 and 2.6 of Puccetti and Rüschendorf (2015). One difference is that we use the stronger property of Σ-countermonotonicity instead of COO. The other difference is permitting general monotone transformations h_j in (2). This allows

Theorem 2 to include more general objective functions than the above mentioned Proposition 2.4, while still including all known explicit and non-trivial objectives considered in Proposition 2.6 of Puccetti and Rüschendorf (2015); see Section 3.3 for details and a further explicit class of objective functions for which Theorem 2 applies and that, to our knowledge, has so far not been considered in the context of minimizing AI functions.

Proof of Theorem 2. Since there are finitely many arrangements of the matrix X, there exists a minimizing arrangement X^*. If the corresponding $H(X^*) = (\mathbf{h}_1, \dots, \mathbf{h}_d)$ is not Σ-countermonotonic, by definition, there exists a nonempty $J \subsetneq \{1, \dots, d\}$ such that $\mathbf{a} = \sum_{j \in J} \mathbf{h}_j$ and $\mathbf{b} = \sum_{j \notin J} \mathbf{h}_j$ are not oppositely ordered. Define $\widetilde{\phi}_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\widetilde{\phi}_2(\mathbf{x}, \mathbf{y}) = g(x_1 + y_1, \dots, x_n + y_n)$$

and note that $\widetilde{\phi}(H) = \widetilde{\phi}_2(\sum_{j \in J} \mathbf{h}_j, \sum_{j \notin J} \mathbf{h}_j)$ for any J. Let $\pi_a, \pi_b \in S_n$ be such that $\mathbf{a}^{\pi_a} = \mathbf{a}^{\uparrow}$ and $\mathbf{b}^{\pi_b} = \mathbf{b}^{\downarrow}$. Denote $\widetilde{H} = (\widetilde{\mathbf{h}}_1, \dots, \widetilde{\mathbf{h}}_d)$, where $\widetilde{\mathbf{h}}_j = \mathbf{h}_j^{\pi_a}$ for $j \in J$ and $\widetilde{\mathbf{h}}_j = \mathbf{h}_j^{\pi_b}$ for $j \notin J$. Since g is Schur-convex and $\mathbf{a} + \mathbf{b} \succcurlyeq_{\mathrm{m}} \mathbf{a}^{\uparrow} + \mathbf{b}^{\downarrow}$ (Day (1972)), we have $\widetilde{\phi}_2(\mathbf{a}, \mathbf{b}) \geq \widetilde{\phi}_2(\mathbf{a}^{\uparrow}, \mathbf{b}^{\downarrow})$ (Definition 2). Thus $\widetilde{\phi}(H) \geq \widetilde{\phi}(\widetilde{H})$. For any arrangement, we can also compute a score function

$$V(H) := \textstyle\sum_{i=1}^n (h_{i1} + \ldots + h_{id})^2.$$

Since $\mathbf{a}$ and $\mathbf{b}$ are not oppositely ordered by assumption, it follows that $V(H) > V(\widetilde{H})$. As the set of arrangements of H is finite and each iteration from H to $\widetilde{H}$ strictly decreases the value of the score function V, after a bounded number of iterations we obtain a matrix $\hat{H}$ that is Σ-countermonotonic. The corresponding arrangement $\hat{X}$ is still optimal for the objective function ϕ, as the iterations do not increase its value $\phi(\hat{X}) = \widetilde{\phi}(\hat{H})$. Furthermore, if g in (2) is strictly Schur-convex, the same argument shows that any minimizer H^* must be Σ-countermonotonic to begin with. □

In particular, the above result means that instead of minimizing $\phi(X)$ over all arrangements of X, we can minimize $\widetilde{\phi}(\hat{H})$ over Σ-countermonotonic arrangements $\hat{H}$ of $H(X)$. In the following section, we connect the theoretical properties of Σ-countermonotonicity and COO with practical algorithms for obtaining arrangements that satisfy these properties.

3.2 The Rearrangement Algorithm

A procedure for obtaining a COO arrangement of a given matrix, called the *Rearrangement Algorithm* (RA), is described in Puccetti and Rüschendorf (2012); Embrechts et al. (2013); Puccetti (2013). The original application of the RA is computing dependence uncertainty bounds for risk measures of sums of random variables with given univariate distributions. In the risk management context, the algorithm has proven to be easily applicable, and to reliably give close-to-optimal bounds (testing in cases where the analytical solution is known). Improvements on the RA, e.g. regarding the stopping conditions, are given in Hofert et al. (2015); for other recent updates, see `sites.google.com/site/RearrangementAlgorithm`.

Recently, a modification called the *Block Rearrangement Algorithm* has been introduced, which finds a Σ-countermonotonic arrangement of a given matrix; see Remark 4.1 in Bernard et al. (2015)[2], and further analysis in Bernard and McLeish (2016); Boudt et al. (2017). The basic iteration of this algorithm is indeed as described in the proof of Theorem 2. Thus, for type (A) AI functions and, more generally, objectives in (2), the Block RA can directly be applied to obtain an approximate solution.

We remark that theoretical results on efficiency and run-time analysis of the RA and its variations are not available, and they seem to be mathematically intractable at the moment. Earlier heuristic algorithms for optimizing (3) were given in Hsu (1984); Coffman and Yannakakis (1984), including a known run-time and a 3/2-optimality guarantee; however, the typical performance is also close to this guarantee.

3.3 Further objectives compatible with negative dependence

In this section we discuss further types of AI objectives for which the Rearrangement Algorithm may be of use. Puccetti and Rüschendorf (2015) generalize the RA for a special class of type (B) functions, in which the supermodular row-aggregation function $g : \mathbb{R}^d \to \mathbb{R}$ is *decomposable*.

Definition 4. A supermodular function $g : \mathbb{R}^d \to \mathbb{R}$ is *decomposable* if it is coordinate-wise monotone, and there exist supermodular functions $g^{(2)} : \mathbb{R}^2 \to \mathbb{R}$ and $g^{(d-1)} : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$g(z_1, \dots, z_d) = g^{(2)}(z_j, g^{(d-1)}(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_d)), \qquad j = 1, \dots, d. \tag{7}$$

For a decomposable supermodular function g, minimizing type (B) AI functions is indeed a two-dimensional problem. Expression (7) may seem to considerably extend the scope of applications beyond our main focus, functions of the form (2). However, the only explicit examples given by Puccetti and Rüschendorf (2015) satisfying the decomposition (7) are the sum, the product, the minimum and (minus) the maximum operators.

Remark 1. It is perhaps useful to note that minimizing the sum of row-minima has a trivial solution: putting the n smallest elements of the matrix, one in each row. Analogously, for the sum of (minus) row-maxima, we put the n largest elements in different rows; see also Section 4.1 in Puccetti and Rüschendorf (2012). Hence, while these examples cannot be directly expressed in the form (2), it is

[2] It seems to the authors that there is a parallel development for Σ-countermonotonicity: in Puccetti and Wang (2015) as a probabilistic concept (continuous and discrete settings) and in Bernard et al. (2015) and Boudt et al. (2017) as an algorithmic concept (discrete setting). The term Σ-*countermonotonicity* is proposed by Puccetti and Wang (2015).

straightforward to obtain the solution, so we do not further discuss these special cases of submodular functions.

Here we identify another useful class of functions that are submodular and have the required structure (7): the class of multivariate distributions with an Archimedean copula. This is a flexible subclass of supermodular functions, it includes the product function, and corresponding optimization problem is non-trivial. For the reader's convenience, we provide the definition of Archimedean copulas in Appendix A.2. For further details on Archimedean copulas, and copulas in general, see McNeil and Nešlehová (2009); Joe (2014). For another example where the Archimedean copula enables tractable results on stochastic orderings, see the portfolio allocation problem in Li and You (2014).

All distribution functions are supermodular, but those that admit an Archimedean copula also have the structure (7) (after a marginal transformation), as shown in the following proposition.

Proposition 3. *Let $F:\mathbb{R}^d\to[0,1]$ be a distribution function with an Archimedean copula. Then F admits the following form:*

$$F(z_1,\ldots,z_d)=g(F_1(z_1),\ldots,F_d(z_d)) \tag{8}$$

for some decomposable supermodular function g, where F_j denote the margins of F, $j=1,\ldots,d$. Furthermore, F admits the form

$$F(z_1,\ldots,z_d)=f(\textstyle\sum_{j=1}^d h_j(z_j)), \tag{9}$$

for some convex function f and decreasing functions h_j, $j=1,\ldots,d$,

Proof. Sklar's theorem states that any multivariate cdf F can be expressed in terms of its margins F_j, $j=1,\ldots,d$ and a copula $C:[0,1]^d\to[0,1]$, that is,

$$F(z_1,\ldots,z_d)=C(F_1(z_1),\ldots,F_d(z_d)).$$

Let ψ be the generator of the Archimedean copula C and let $C_\psi^{(k)}$ be the k-variate Archimedean copula with generator ψ. Observe that $\psi^{-1}\circ\psi(z)=z$ for $z\in\{x:\psi(x)>0\}$, since ψ is strictly decreasing on its support; see Embrechts and Hofert (2013) for properties of inverses. By definition,

$$\begin{aligned}C_\psi(u_1,\ldots,u_d)&=\psi(\psi^{-1}(u_1)+\ldots+\psi^{-1}(u_d))\\&=\psi(\psi^{-1}(u_1)+\psi^{-1}\circ\psi(\psi^{-1}(u_2)+\ldots+\psi^{-1}(u_d)))\\&=C_\psi^{(2)}(u_1,C_\psi^{(d-1)}(u_2,\ldots,u_n)).\end{aligned}$$

By symmetry, for each $j=1,\ldots,d$, we have

$$F(z_1,\ldots,z_d)=C_\psi^{(2)}(F_j(z_j),C_\psi^{(d-1)}(F_1(z_1),\ldots,F_{j-1}(z_{j-1}),F_{j+1}(z_{j+1}),\ldots,F_d(z_d)), \tag{10}$$

which admits the form (8), since any copula is supermodular. Taking $h_j = \psi^{-1} \circ F_j$, $j = 1, \dots, d$ and $f = \psi$ in (10), we obtain (9). As ψ is the generator of an Archimedean copula, f is convex and h_j are decreasing (see Definition 7 in Appendix A.2). $\square$

From representation (9) in Proposition 3, it follows that in order to minimize

$$\phi(X) = \sum_{i=1}^{n} F(x_{i1}, \dots, x_{id}), \quad X \in \mathbb{R}^{n\times d}, \tag{11}$$

where F has an Archimedean copula, we can apply the Block RA to the matrix $H(X)$ to obtain a Σ-countermonotonic arrangement; we shall see in Section 4.2 that this often provides an approximate solution.

A straightforward corollary of Proposition 3 is that AI functions of type (C) that have the form

$$\phi(X) = \prod_{i=1}^{n} F(x_{i1}, \dots, x_{id}), \quad X \in \mathbb{R}^{n\times d}, \tag{12}$$

where F has an Archimedean copula, are also compatible with Σ-countermonotonicity and the Block RA (after applying a logarithmic transformation). The fact that Archimedean distributions are MTP_2 is shown in Müller and Scarsini (2005), see also Appendix A.2.

Interestingly, we notice that although we attempted to expand the class of objectives for which Σ-countermonotonicity and the Block RA can be applied, the only tractable examples of type (B) and (C) AI functions also admit the form (2), with Schur function g of type b) (see page 7).

We conclude this section with a brief discussion regarding a possible extension to more general distribution functions, beyond Archimedean. Notice that the Σ-countermonotonicity condition for F with an Archimedean copula is equivalent to $F_J(x_{iJ})$ and $F_{J^c}(x_{iJ^c})$, $i = 1, \dots, n$, being oppositely ordered for all nonempty $J \subsetneq \{1, \dots, d\}$ (where $F_J = F_{j_1,\dots,j_k}$ are multivariate marginal distributions and $x_{iJ} = (x_{ij_1}, \dots, x_{ij_k})$ for $J = \{j_1, \dots, j_k\}$). One may wonder whether the necessity of this structure remains true for minimizing (11) with an arbitrary distribution function F (not necessarily associated with an Archimedean copula). As the following example demonstrates, this is in general not true; restricting to the class of distributions associated with an Archimedean copula is important for our approach to work.

Consider a distribution on $\{0,1\}^3$ as given in Table 1, and suppose that the input matrix and the corresponding marginal distributions are

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad \begin{matrix} F_1(0) = 0.5 & \quad F_{2,3}(0,1) = 0.7 \\ F_1(1) = 1 & \quad F_{2,3}(1,0) = 0.6 \end{matrix}$$

The marginal probabilities are oppositely ordered, and the corresponding objective value of type (B) is $F(0,0,1) + F(1,1,0) = 0.3 + 0.6 = 0.9$. Arranging the first column *similarly*, however, would give a smaller objective value, $F(1,0,1) + F(0,1,0) = 0.7 + 0.1 = 0.8$. In fact, this is the minimizing

Table 1: Probability mass function and cdf of a distribution of (X_1, X_2, X_3) on $\{0,1\}^3$.

$\mathrm{P}(X_i = x_i, i = 1, 2, 3)$

	$x_1 = 0$		$x_1 = 1$	
$x_2 \backslash x_3$	0	1	0	1
0	0	0.3	0.4	0
1	0.1	0.1	0.1	0

$\mathrm{P}(X_i \leq x_i, i = 1, 2, 3)$

	$x_1 = 0$		$x_1 = 1$	
$x_2 \backslash x_3$	0	1	0	1
0	0	0.3	0.4	**0.7**
1	**0.1**	0.5	0.6	1

arrangement, moreover $F_{1,2}$ and F_3 are also similarly ordered for this arrangement. This shows that for general dependence structures, the arrangement that minimizes the sum of joint cdfs may not have oppositely ordered marginal cdfs. Note that the arrangement that *maximizes* the sum of cdfs is the comonotonic one, with $F(0,0,0) + F(1,1,1) = 0 + 1 = 1$. This is true for any choice of the distribution function, since AI functions are always maximized by the comonotonic arrangement; see (5).

4 Scheduling and systems assembly problems

In the following subsections, we provide two classical examples of practical problems: the assembly line crew scheduling and the systems assembly problems. These problems, on their first appearance, may not directly translate to problems discussed in Sections 2-3. However, with some careful analysis, we show that they indeed belong to the class of minimization problems with an objective function of the form (2). Thereby, we apply the (Block) RA to obtain approximate solutions following the methodology outlined in Section 3 and the numerical results are discused.

4.1 Assembly line crew scheduling problem

The deterministic version of the assembly line crew scheduling (ALCS) problem was introduced in Hsu (1984); in the following, we describe a stochastic version of this problem, which includes more real-world applications. ALCS considers the production of n items on n parallel assembly lines (one item on each line), where each item requires d different operations. There are n workers specialized in each of the d operations, hence nd workers in total. The time taken by worker i of specialty j to complete the operation is a random variable $T_{ij} \sim F_{ij}$, where F_{ij} is a distribution function (df) that models the completion time of operation j. All the rvs T_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, d$ are assumed to be independent. We are allowed to assign the workers within each specialty to the n assembly lines in any order. Writing $T = (T_{ij})_{n \times d}$, the assignment of workers corresponds to permuting the entries within each column (representing specialty) of T. Denote by $C_i = \sum_{j=1}^{d} T_{ij}$ the *completion time* of the i^{th} item, and by $C_{\max} = \max_{i=1}^{n} C_i$ the *makespan* of the project. We will consider three different objectives for this stochastic problem.

- Maximize the probability of meeting a deadline $D \in \mathbb{R}$,

$$\mathrm{P}(C_{\max} \le D) = \prod_{i=1}^{n} \mathrm{P}(C_i \le D) = \prod_{i=1}^{n} \big(\ast_{j=1}^{d} F_{ij}\big)(D), \tag{13}$$

  where $F \ast G$ denotes the df of the sum of independent variables with dfs F and G, respectively.

- Minimize the expected makespan

$$\mathrm{E}[C_{\max}] = \int_{t=0}^{\infty} \mathrm{P}(C_{\max} > t)\,\mathrm{d}t. \tag{14}$$

- Maximize the expected number of items finished within the deadline $D \in \mathbb{R}$,

$$\mathrm{E}\left[\sum_{i=1}^{n} \mathbf{1}_{\{C_i \le D\}}\right] = \sum_{i=1}^{n} \mathrm{P}(C_i \le D) = \sum_{i=1}^{n} \big(\ast_{j=1}^{d} F_{ij}\big)(D). \tag{15}$$

Suppose the individual completion time of worker i of specialty j follows the Normal distribution with mean θ_{ij} and variance σ_j^2, namely $F_{ij}(\cdot) = \Phi((\cdot - \theta_{ij})/\sigma_j)$. Let $\Theta = (\theta_{ij})_{n \times d}$ be the matrix of location parameters, and let $\sigma_+^2 = \sum_{j=1}^{d} \sigma_j^2$, $\theta_{i+} = \sum_{j=1}^{d} \theta_{ij}$. Then $C_i \sim \mathcal{N}(\theta_{i+}, \sigma_+^2)$ for $i = 1, \ldots, n$. The assumption of homogeneity in σ_j within each column allows us to translate the arrangement problem of T to that of Θ. Note that the same approach would also be applicable if the workers on the same assembly line had dependent completion times, modeled by a multivariate Normal distribution with covariance matrix Σ, by taking $\sigma_+^2 = \mathbf{1}^\top \Sigma \mathbf{1}$. Since the logarithm is an increasing function, maximizing (13) is equivalent to maximizing

$$\log(\mathrm{P}(C_{\max} \le D)) = \sum_{i=1}^{n} \log \Phi((D - \theta_{i+})/\sigma_+). \tag{16}$$

Furthermore, the Normal df is log-concave (see Bagnoli and Bergstrom (2005) for this and other examples), so (16) is a sum of a concave function evaluated at row-sums of Θ.

Hence, the objective (13) and its log-counterpart (16) are Schur-concave, corresponding to the cases c) and b) (with a negative sign, to switch from convex to concave) in Section 2.2, respectively. Thus, an arrangement of Θ which makes the row-sum vector "small" in majorization order yields a larger objective value (13). Moreover, the objective is then large for all deadlines $D = t$ simultaneously (i.e. large in stochastic order), hence the objective (14), being the integral of tail probabilities, is "small" for this permutation. Therefore, a row-sum vector that is small in majorization order is desirable with respect to both objectives (13) and (14). The objective (15) does not obey the majorization order, because Φ is not convex on its domain. For example, suppose that $\theta_{ij} = i$ for

$i = 1, \dots, 3$ and $j = 1, 2$; $\sigma_+^2 = 1$ and $D = 4$. Then the countermonotonic arrangement is

$$\Theta = \begin{pmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{pmatrix} \Rightarrow \sum_{i=1}^{3} \mathrm{P}(C_i \le 4) = 1.5, \qquad \text{while} \qquad \Theta = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 3 \end{pmatrix} \Rightarrow \sum_{i=1}^{3} \mathrm{P}(C_i \le 4) = 1.71.$$

An alternative heuristic for functions that are convex on the left and concave on the right would be to take the $k \in \{0, 1, \dots, n\}$ slowest workers in each specialty and arrange them comonotonically, while the fastest $(n - k)$ workers Σ-countermonotonically (which is optimal in the example above). However, we do not pursue this idea further and leave it for future research.

For a simple numerical example, we choose dimensions $n = 15$, $d = 4$; note that even for such small dimensions a brute force search of all $(15!)^3 \approx 10^{36}$ arrangements is out of the question. We take $\sigma_j^2 = j$, $j = 1, \dots, d$ and generate the matrix Θ of location parameters by sampling $\theta_{ij} \sim 5\sigma_j(1+\mathrm{Beta}(2,5))$; the entries of this matrix are then fixed for the remainder of the case study (the choice of the parameters is such that the completion times have a negligible probability of being negative). Thereafter, we consider 1000 random (intra-column) arrangements of the matrix Θ and compute the value of objectives (13)-(15) with deadline $D = 45$. A histogram of the objective (13) values is given in Figure 1. Taking each random arrangement as the input, we apply the (Block) RA to obtain a Σ-countermonotonic or COO matrix, and again compute the corresponding probability of meeting the project deadline. We observe that in all cases the resulting arrangements consistently yield high probabilities of success, supporting the assertion that Σ-countermonotonic matrices are often close-to-optimal. The COO matrices are also close, but show slightly more variation (see Figure 1, right panel, as well as Table 2), since it is a larger set of arrangements. Table 3 in the Appendix B shows a Σ-countermonotonic arrangement of Θ with the corresponding probabilities of exceeding the deadline.

Table 2: Range (minimum and maximum over 1000 experiments) of the objective functions (13)-(15) for Σ-countermonotonic, COO and random arrangements for the matrix Θ for the assembly line crew scheduling example with $n = 15$, $d = 4$ and deadline $D = 45$. The last column shows the results for the comonotonic arrangement of Θ (worst case).

Objective	Block RA	RA	random	comonotonic
$\max \mathrm{P}(C_{\max} \le 45)$	0.6691 0.6723	0.6670 0.6723	0.0430 0.5686	0.0101
$\max \sum_{i=1}^{15} \mathrm{P}(C_i \le 45)$	14.6037 14.6081	14.6007 14.6081	13.1289 14.4545	12.6084
$\min \mathrm{E}[C_{\max}]$	44.3512 44.3660	44.3513 44.3760	44.8328 49.0833	50.3584

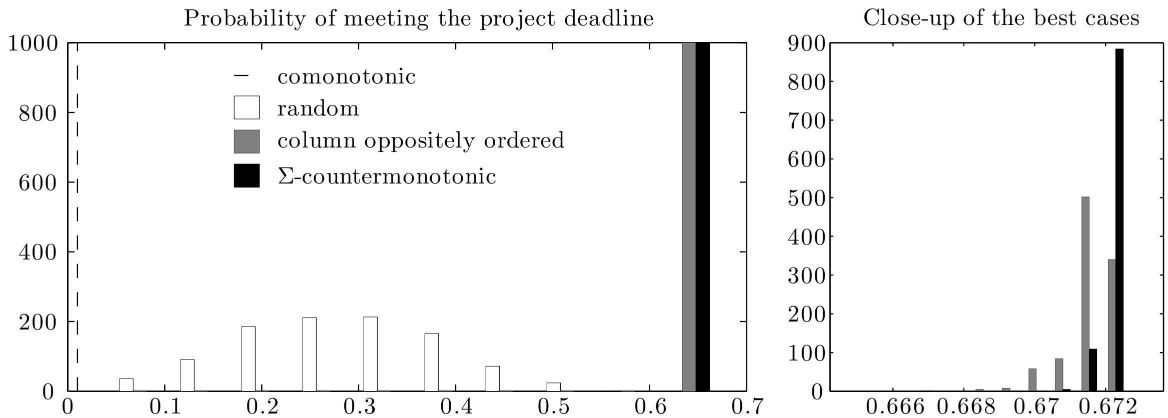

Figure 1: *Left panel:* Histogram of $\mathrm{P}(C_{\max} \leq 45)$ corresponding to 1000 random initial arrangements of Θ, as well as the result after applying the (Block) Rearrangement Algorithm in order to obtain Σ-countermonotonic or column oppositely ordered matrices. Worst case is given by the comonotonic arrangement. *Right panel:* Close-up of the best cases.

Figure 2 shows the expected number of items finished on time (left panel) and the expected makespan (right panel); see also Table 2 for the range of observed values for these objectives. Again, we notice that the Σ-countermonotonic and COO arrangements consistently give solutions that are significantly better than the random arrangements, with the Σ-countermonotonic ones showing less variation. Although the objective $\max \sum_{i=1}^{15} \mathrm{P}(C_i \leq 45)$ in (15) does not respect majorization order, from Table 2 and Figure 2, the Σ-countermonotonic and COO arrangements perform quite well, suggesting that they may have a wider range of applications than the ones described in Section 2, and this is left for future exploration.

4.2 Systems assembly problem

In reliability theory, a system that is composed of d individual components, each of which may be functioning properly of not, is characterized by its *structure function* $\rho : \{0,1\}^d \to \{0,1\}$. Denoting the state of component j by $x_j \in \{0,1\}$ (representing failed or functional, respectively), $\rho(x_1, \ldots, x_d) \in \{0,1\}$ returns the state of the system. For example, a *series system* is described by

$$\rho(x_1, \ldots, x_d) = \min\{x_1, \ldots, x_d\} = \prod_{j=1}^{d} x_j = \begin{cases} 0 & \text{if } x_j = 0 \text{ for some } j, \\ 1 & \text{if } x_j = 1 \text{ for all } j. \end{cases}$$

The state of the components is usually not known in advance and is modeled by a random vector $(X_1, \ldots, X_d)$ taking values in $\{0,1\}^d$. Then, the *reliability* of the system is given by

$$R = \mathrm{E}[\rho(X_1, \ldots, X_d)] = \mathrm{P}(\rho(X_1, \ldots, X_d) = 1).$$

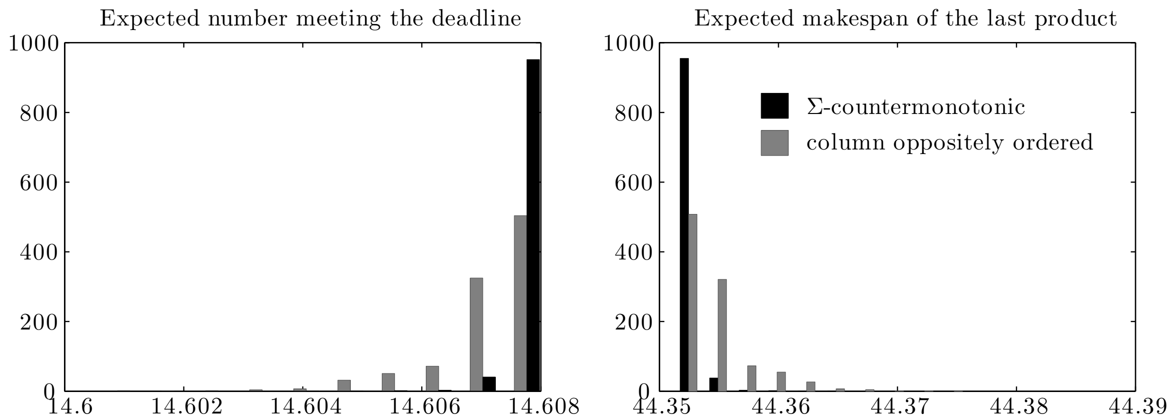

Figure 2: *Left panel:* Histogram of $\sum_{i=1}^{15} \mathrm{P}(C_i \leq 45)$, based on 1000 random initial arrangements of Θ, after applying the (Block) Rearrangement Algorithm in order to obtain Σ-countermonotonic or column oppositely ordered matrices, respectively. *Right panel:* Histogram for the objective $\mathrm{E}[C_{\max}]$.

First, we describe a series systems assembly problem following the setup of Derman et al. (1972), and then proceed to our example with *parallel systems*. Suppose that a system consists of d components, each of a different type, and there are n components of each type available. This enables the assembly of n systems. The state of the components $(X_{i1}, \ldots, X_{id})$ within system i is determined by a vector $(Z_{i1}, \ldots, Z_{id}) \sim F_{\mathbf{Z}}$ of external shocks and individual "quality" parameters a_{ij} for each of the components. Specifically, $X_{ij} = \mathbf{1}_{\{Z_{ij} \leq a_{ij}\}}$, $i = 1, \ldots, n$, $j = 1, \ldots, d$. Hence, the reliability of system i is

$$R_i = \mathrm{P}(Z_{i1} \leq a_{i1}, \ldots, Z_{id} \leq a_{id}) = F_{\mathbf{Z}}(a_{i1}, \ldots, a_{id}).$$

We can freely choose the order of components of each type when assembling the systems, which corresponds to rearranging elements within the columns $\mathbf{a}_1, \ldots, \mathbf{a}_d$ of the input matrix $(A)_{ij} = a_{ij}$, so that each row corresponds to an assembled system. The expected number of systems that perform satisfactorily is

$$\sum_{i=1}^{d} R_i = \sum_{i=1}^{d} F_{\mathbf{Z}}(a_{i1}, \ldots, a_{id}) \tag{17}$$

Recall from Section A.1 that cumulative distribution functions are supermodular, therefore the objective function (17) is arrangement increasing, in particular, of the type (B). This yields the main result in Derman et al. (1972), that the comonotonic arrangement $A^* = (\mathbf{a}_1^{\uparrow}, \ldots, \mathbf{a}_d^{\uparrow})$ of components maximizes the expected number of systems that perform satisfactorily. They also note that under a further assumption that the probabilities that components within a system work are independent ($F_{\mathbf{Z}}$ has the independence copula), the comonotonic arrangement also maximizes the probability that at least k systems perform satisfactorily, $1 \leq k \leq n$.

We now consider the assembly of parallel systems, where at least one component is required to work for a system to perform satisfactorily. In this case, the structure function is

$$\rho^{\|}(x_1,\dots,x_d) = \max\{x_1,\dots,x_d\} = 1 - \prod_{j=1}^{d}(1-x_j) = \begin{cases} 0 & \text{if } x_j = 0 \text{ for all } j, \\ 1 & \text{if } x_j = 1 \text{ for some } j. \end{cases}$$

Using the same model for component states $X_{ij} = \mathbf{1}_{\{Z_{ij} \le a_{ij}\}}$ as in the series example, the reliability of system i in the parallel case is

$$R_i^{\|} = 1 - \mathrm{P}(Z_{i1} > a_{i1},\dots,Z_{id} > a_{id}) = 1 - S_{\mathbf{Z}}(a_{i1},\dots,a_{id}), \tag{18}$$

where $S_{\mathbf{Z}}$ denotes the so-called *survival function* of $(Z_{i1},\dots,Z_{id})$. Maximizing the reliability (18) is equivalent to *minimizing* a survival function. Since the survival function is also a distribution function, it is supermodular. Therefore, maximizing the expected number of functioning systems, $\sum_{i=1}^{d} R_i^{\|}$, is equivalent to minimizing the sum of supermodular functions (type (B) of AI functions). Derman et al. (1972) note that in the special case of independence copula and $d = 2$ types of components, the countermonotonic arrangement is optimal. Prasad et al. (1991) consider a related problem, and also only give the solution for the case $d = 2$. In Derman et al. (1974) another variation with independent component failures is considered, and a pairwise component interchange heuristic is proposed.

Using the insights from Section 3, we are able to consider dimensions higher than 2, and relax the assumption of independence. In particular, we suppose that for the survival function $S_{\mathbf{Z}}$ of the external factors, the dependence is modeled using an Archimedean copula (see Section A.2). For a concrete example, we consider the Clayton copula C_θ with parameter $\theta > 0$ and generator

$$\psi(t) = (1+\theta t)^{-1/\theta}, \qquad t \in [0,\infty), \qquad \psi^{-1}(u) = (u^{-\theta}-1)/\theta, \qquad u \in (0,1].$$

Note that the "survival" function $S_{\mathbf{Z}}$ in (18) actually gives the probability of *system failure*, therefore we will work with the marginal and joint probabilities of component failures. For $i = 1,\dots,n$ and $j = 1,\dots,d$, define $U_{ij} = 1 - F_j(Z_{ij})$ and $p_{ij} = 1 - F_j(a_{ij})$, where F_j is the j^{th} margin of $F_{\mathbf{Z}}$ (assumed continuous for simplicity, so that $U_{ij} \sim \mathrm{UNIF}(0,1)$). Applying Sklar's theorem, we can take $(P)_{ij} = p_{ij}$ as the input matrix, and express the probability that the i^{th} system fails as

$$1 - R_i^{\|} = \mathrm{P}(U_{i1} \le p_{i1},\dots,U_{id} \le p_{id}) = C_\theta(p_{i1},\dots,p_{id}).$$

Clayton copula induces a tail dependence with *lower-tail dependence index* $\lambda_L = 2^{-1/\theta}$; see Nelsen (2006, Section 5.4). For the present model this implies that the probability of joint failures for reliable components (with small marginal probability p_{ij} of failure) is higher than in the independent case; see Figure 3 for a bivariate sample of $(U_1, U_2) \sim C_\theta$, as well as the conditional probability $\mathrm{P}(U_2 \le 0.1 | U_1 \le p)$.

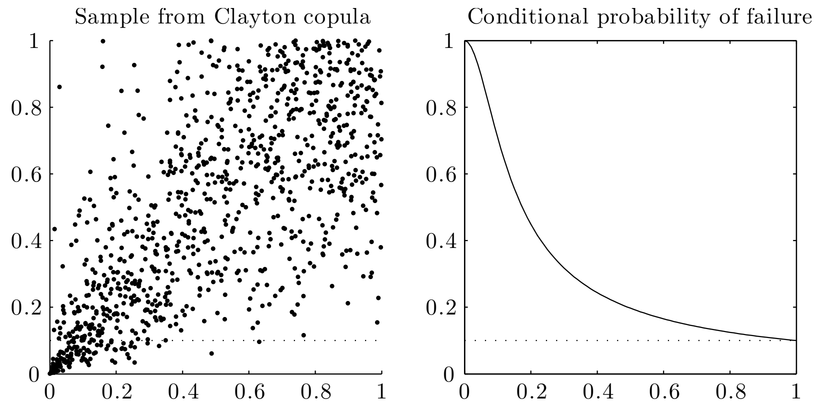

Figure 3: *Left panel:* 1000 sample points from a bivariate Clayton copula with parameter $\theta = 2$. *Right panel:* Conditional failure probability $\mathrm{P}(U_2 \leq 0.1 \,|\, U_1 \leq p) = C_\theta(p, 0.1)/p$ as a function of $p \in [0, 1]$ on the horizontal axis.

For a numerical example, we generate the input matrix $P \in \mathbb{R}^{20\times 5}$ by sampling the component failure probabilities $p_{ij} \overset{\text{iid}}{\sim} \text{UNIF}(0, 1)$; we fix the entries of this matrix for the remainder of the experiment. Following the approach from Section 3.3, we transform P by applying ψ^{-1} to its entries, thus obtaining $H(P)$. Thereafter, we consider 1000 random arrangements of the matrix $H(P)$ and compute the expected number of failed systems. A histogram of the results is given in Figure 4. Taking each random arrangement as the input, we apply the (Block) RA to obtain a Σ-countermonotonic or COO matrix, respectively, and again compute the corresponding expected number of failed systems. We observe that in all cases the resulting arrangements consistently yield highly reliable systems, supporting the hypothesis that Σ-countermonotonic matrices are close-to-optimal. The COO matrices are also close, but show more variation (see Figure 4, right panel), since it is a larger set or arrangements. Table 4 in the Appendix B shows a Σ-countermonotonic arrangement of $H(P)$ with the corresponding failure probabilities.

5 Conclusions

While maximizing an arrangement increasing (AI) function over intra-column rearrangements of a given matrix is trivial, the corresponding minimization problem is in general intractable. We have shown that for a subclass of AI functions with a special structure, the minimizing arrangement has to satisfy a specific property of negative dependence. This can be exploited to obtain close-to-optimal arrangements using a practical heuristic called the (Block) Rearrangement Algorithm; this is possible due to the special structure of the considered AI functions. We also show that this subclass of AI functions includes the objective functions of many classical optimization problems, also stochastic ones, and give explicit examples. The numerical case study seems to support the

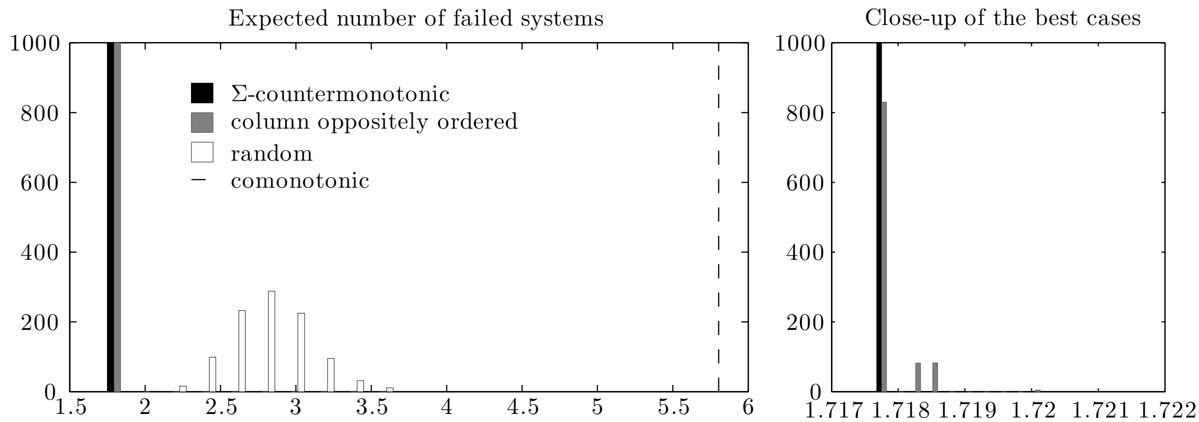

Figure 4: *Left panel:* Histogram of $\sum_{i=1}^{20}(1-R_i^{\|})$ corresponding to 1000 random initial arrangements of $H(P)$, as well as the result after applying the (Block) Rearrangement Algorithm in order to obtain Σ-countermonotonic or column oppositely ordered matrices. With the Block RA, the result was always 1.7176 (up to four decimal digits), with the standard RA - in the range $[1.7176, 1.7202]$, for the random arrangements - in the range $[1.9465, 3.6781]$, and equal to 5.8037 for the comonotonic arrangement. *Right panel:* Close-up of the best cases.

intuition that any arrangement with the mentioned countermonotonicity property gives a close-to-optimal objective value.

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A Relevant concepts

A.1 Supermodularity and total positivity

In this section, we define the properties of submodularity and MTP_2, which are relevant to arrangement increasing functions of types (B) and (C), respectively.

Definition 5. A function $g : D \subset \mathbb{R}^d \to \mathbb{R}$ is said to be *supermodular* if it satisfies

$$g(\mathbf{z} \wedge \mathbf{w}) + g(\mathbf{z} \vee \mathbf{w}) \geq g(\mathbf{z}) + g(\mathbf{w}) \quad \text{for all} \quad \mathbf{z}, \mathbf{w} \in D,$$

where $\wedge$ and $\vee$ denote the component-wise minimum and maximum, respectively.

Supermodular functions are sometimes called *L-superadditive*, where L stands for "lattice". Reversing the inequality in Definition 5 yields *submodularity*. In reliability theory, Block et al. (1989) explains the intuition that the supermodular/submodular property for a structure function describes whether the system is more series-like or more parallel-like; see Section 4.2 for further details. Examples of supermodular functions $(z_1, \ldots, z_d) \mapsto g(z_1, \ldots, z_d)$ include

a) $f(z_1 + \ldots + z_d)$, if and only if f is convex,

b) for a twice differentiable f, $f(\prod_1^d z_j)$, if and only if $f'(z) + zf''(z) \geq 0$,

c) $f(\min_{j=1}^d z_j)$ and $f(-\max_{j=1}^d z_j)$, where f is non-decreasing,

d) elementary symmetric polynomials,

e) cumulative distribution functions.

Observe that if we substitute a) into type (B) of AI functions, we obtain a function of type (A). Expressing $f(\prod_1^d z_j) = f(\exp\{\log(z_1) + \ldots + \log(z_d)\})$, we effectively reduce example b) to a); in particular, if the conditions for f in b) hold, then $f \circ \exp$ is convex, hence we have again recovered an objective of the structure (2). If we substitute e) into type (B) of AI functions, the sum of cdfs can be interpreted as the expected number of events that occur; see the example in Section 4.2.

Lorentz (1953) shows that the comonotonic arrangement maximizes the sum of a supermodular function applied to the rows of a matrix; this can also be seen by applying a sequence of basic rearrangements (see Section 2.1). Furthermore, for $d = 2$ columns, the minimizing arrangement is the countermonotonic one; see Marshall et al. (2011, Sections 6.D-E) for these and further properties and examples. For $d \geq 3$ columns, finding the minimizing arrangement is more challenging; this is related to the fact that the "opposite" operation of a basic rearrangement is no longer well-defined.

Multivariate total positivity of order 2 (MTP_2) is the log-analogue of supermodularity, namely, a positive-valued function g is MTP_2 if and only if $\log \circ g$ is supermodular.

Definition 6. A function $g : D \subset \mathbb{R}^d \to \mathbb{R}_+$ is said to be *multivariate totally positive (MTP_2)*, if

$$g(\mathbf{z} \wedge \mathbf{w})g(\mathbf{z} \vee \mathbf{w}) \geq g(\mathbf{z})g(\mathbf{w}) \quad \text{for all} \quad \mathbf{z}, \mathbf{w} \in D.$$

If g represents a density function, then the MTP_2 property is often seen as a characteristic of multivariate positive dependence; see Karlin and Rinott (1980a); Hu et al. (2003); Colangelo et al. (2005). Reversing the inequality in Definition 6 defines a *multivariate reverse regular of order 2* (MRR_2) function. This property is considered as a characteristic of negative dependence; see Karlin and Rinott (1980b); Block et al. (1982); Joag-Dev and Proschan (1983). In statistics, the product of densities represents the likelihood of independent (vector-valued) observations. Alternatively, g may represent a cumulative distribution function, in which case the product is the joint probability of independent events occurring. A useful result Joe (1997, p.55) states: if g is an MTP_2 density, then its cdf and survival function are also MTP_2. For applications of MTP_2 in reliability theory, see Shaked and Shanthikumar (1990).

It was observed by John Napier in *Mirifici Logarithmorum Canonis Descriptio* (1614) that by applying the logarithm, we can transform multiplication into addition, which is easier to work with. This is also well-known to statisticians, who typically prefer to use log-likelihoods. Hence, it may be easier to transform type (C) AI functions into type (B) by taking the logarithm, and then continue working with sums of supermodular functions.

A.2 Archimedean copula

In this section, we recall the definitions of d-monotonicity and Archimedean copulas from McNeil and Nešlehová (2009).

Definition 7. A function $\psi : [0, \infty) \to \mathbb{R}$ is called *d-monotone*, where $d \geq 2$, if it is differentiable up to the order $d-2$ and the derivatives satisfy

$$(-1)^k \psi^{(k)}(x) \geq 0, \quad k = 0, \dots, d-2,$$

for any $x \geq 0$ and further if $(-1)^{d-2}\psi^{(d-2)}$ is decreasing and convex. If ψ has derivatives of all orders and if $(-1)^k \psi^{(k)}(x) \geq 0$ for any $x \geq 0$, then ψ is called *completely monotone*.

Proposition 4. *Let $\psi : [0, \infty) \to [0, 1]$ be a d-monotone function such that $\psi(0) = 1$ and $\lim_{x\to\infty} \psi(x) = 0$. Then $C_\psi : [0, 1]^d \to [0, 1]$ defined by*

$$C_\psi(u_1, \dots, u_d) = \psi(\psi^{-1}(u_1) + \dots + \psi^{-1}(u_d)) \tag{19}$$

is a copula. If ψ is completely monotone, then (19) is a copula for any $d \geq 2$.

Any copula of the form (19) is called *Archimedean*, and the function ψ is the *generator* of the copula. Note that in the literature, the notation of ψ and ψ^{-1} is sometimes swapped, e.g. in Nelsen (2006).

A useful property which connects Archimedean copulas to the optimization with objectives of type (C) is given in Müller and Scarsini (2005) and stated below.

Lemma 5. *If ψ is completely monotone, then $C_\psi : [0, 1]^d \to [0, 1]$ is MTP_2 for any $d \geq 2$.*

In fact, this result is proved by showing that $\log \circ \psi$ is a convex function for completely monotone ψ.

B Numerical results

Table 3: A Σ-countermonotonic arrangement for the matrix Θ (obtained using the Block Rearrangement Algorithm) for the assembly line crew scheduling example (Section 4.1) with $n = 15$, $d = 4$. The last four columns show row-sums θ_{i+} and $\mathrm{P}(C_i > 45) = 1 - \Phi((\theta_{i+} - 45)/\sigma_+)$ for a Σ-countermonotonic and comonotonic arrangement of Θ, respectively.

Σ-countermonotonic arrangement of Θ				Σ-countermonotonic Θ		comonotonic Θ	
				Row-sums	$\mathrm{P}(C_i > 45)$	Row-sums	$\mathrm{P}(C_i > 45)$
5.73	8.39	11.84	12.83	38.79	0.025	31.60	$1.13 \cdot 10^{-5}$
6.90	8.38	10.41	13.13	38.82	0.025	32.86	$6.15 \cdot 10^{-5}$
6.21	9.26	13.06	10.30	38.83	0.026	33.69	$1.75 \cdot 10^{-4}$
5.32	10.27	8.93	14.31	38.83	0.026	35.17	$9.44 \cdot 10^{-4}$
7.29	8.02	10.14	13.39	38.84	0.026	35.43	0.00124
7.71	9.43	9.87	11.84	38.84	0.026	36.07	0.00238
7.38	7.66	11.68	12.12	38.84	0.026	36.52	0.00366
7.14	7.78	11.76	12.17	38.85	0.026	38.42	0.0187
7.70	7.76	11.34	12.05	38.85	0.026	39.64	0.0450
6.12	9.46	12.67	10.60	38.85	0.026	40.12	0.0614
7.13	7.68	12.19	11.86	38.86	0.026	41.51	0.135
7.28	9.42	9.73	12.44	38.87	0.026	42.29	0.195
5.96	7.62	14.66	10.69	38.93	0.027	43.80	0.352
6.02	7.30	9.65	15.98	38.95	0.028	46.20	0.648
5.16	7.21	9.64	16.96	38.98	0.028	49.60	0.927
			$\mathrm{P}(C_{\max} \leq 45)$	0.6721		0.0101	
			$\sum_{i=1}^{15} \mathrm{P}(C_i \leq 45)$	14.6079		12.6084	
			$\mathrm{E}[C_{\max}]$	44.3519		50.3584	

Table 4: A Σ-countermonotonic arrangement for the matrix $H(P)$ (obtained using the Block Rearrangement Algorithm) for the systems assembly example (Section 4.2) with $n = 20$, $d = 5$ and Clayton($\theta = 2$) copula. The last four columns show the row-sums and individual system failure probabilities for a Σ-countermonotonic and comonotonic arrangement of $H(P)$, respectively ($R_i^{||}$ is the reliability of the i^{th} system).

Σ-countermonotonic arrangement of $H(P)$					Σ-countermonotonic		comonotonic	
					Row-sums	$1 - R_i^{\parallel}$	Row-sums	$1 - R_i^{\parallel}$
6.23	4.48	1.48	12.56	2.12	26.86	0.14	0.20	0.85
6.04	9.94	1.26	9.02	1.48	27.74	0.13	0.48	0.71
13.13	3.26	4.10	5.64	1.67	27.80	0.13	1.00	0.58
3.41	3.18	14.22	5.03	2.10	27.94	0.13	1.76	0.47
3.15	2.67	1.09	1.95	20.98	29.83	0.13	2.24	0.43
2.11	2.13	0.91	2.92	21.77	29.84	0.13	2.97	0.38
1.49	23.69	1.06	2.38	1.29	29.91	0.13	3.96	0.33
1.48	2.06	0.86	32.05	1.23	37.67	0.11	4.34	0.32
1.09	1.05	0.81	34.58	1.05	38.58	0.11	4.89	0.30
0.79	0.82	39.44	1.75	0.83	43.63	0.11	5.89	0.28
0.57	0.68	0.60	43.07	0.78	45.69	0.10	7.19	0.25
0.48	87.49	0.30	1.69	0.74	90.71	0.07	11.06	0.21
0.34	0.51	139.28	1.24	0.62	141.98	0.06	12.03	0.20
0.32	0.51	0.30	0.95	152.77	154.85	0.06	19.38	0.16
0.31	0.30	0.10	171.77	0.50	172.99	0.05	34.77	0.12
0.28	0.28	222.46	0.78	0.40	224.20	0.05	80.25	0.08
0.15	0.16	265.89	0.42	0.25	266.86	0.04	186.50	0.05
2637.05	0.10	0.04	0.12	0.22	2637.54	0.01	302.68	0.04
0.14	0.08	10927.86	0.02	0.09	10928.19	0.01	496.25	0.03
0.04	0.07	0.01	12971.12	0.05	12971.30	0.01	26776.29	0.00
				$\sum_{i=1}^{20}(1 - R_i^{\parallel})$	1.7176		5.8037	