Collective Risk Models with Dependence Uncertainty

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Abstract

We bring the recently developed framework of dependence uncertainty into collective risk models, one of the most classic models in actuarial science. We study the worst-case values of the Value-at-Risk (VaR) and the Expected Shortfall (ES) of the aggregate loss in collective risk models, under two settings of dependence uncertainty: (i) the counting random variable (claim frequency) and the individual losses (claim sizes) are independent, and the dependence of the individual losses is unknown; (ii) the dependence of the counting random variable and the individual losses is unknown. Analytical results for the worst-case values of ES are obtained. For the loss from a large portfolio of insurance policies, an asymptotic equivalence of VaR and ES is established. Our results can be used to provide approximations for VaR and ES in collective risk models with unknown dependence. Approximation errors are obtained in both cases.

Key-words: collective risk model; Value-at-Risk; Expected Shortfall; dependence uncertainty; asymptotic equivalence

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1 Introduction

The question we address in this paper comes from a practical challenge of measuring large insurance portfolios using a risk measure under model uncertainty at the level of the dependence among individual claims and the number of claims.

The aggregate loss of an insurance company (the total amount paid on all claims occurring over a fixed period) is often modelled by a sum of random variables,

$$S_N = Y_1 + \cdots + Y_N, \tag{1.1}$$

where $Y_1, Y_2, \dots$ are non-negative random variables and N (random or deterministic) takes values in non-negative integers. Nowadays the simple model (1.1) is taught in practically every undergraduate actuarial science course on loss models; see, for instance, standard textbooks Kaas et al. (2008) and Klugman et al. (2012).

When N is a non-random positive integer, (1.1) is called an *individual risk model*, in which $Y_1, Y_2, \dots$ represent losses from each individual policy and N is the number of policies. When N itself is random, (1.1) is called a *collective risk model*. For portfolio analysis, individual risk models are a priori the most natural, whereas for ruin theoretic problems, collective risk models are more natural. In the classic treatment of collective risk models, $Y_1, Y_2, \dots$ are iid random variables representing individual claim sizes, and the counting random variable N is assumed to be independent of $(Y_1, Y_2, \dots)$. This classic assumption on the independence of $N, Y_1, Y_2, \dots$ provides great mathematical convenience and elegance, as well as nice interpretations.

In some practical situations, the claims or losses $Y_1, Y_2, \dots$, in individual risk models or collective risk models are dependent, and they may also be dependent on the number of claims N. Think about, for instance, the losses from wind and flood damage in a certain region; see Kousky and Cooke (2009) for related real-life examples. In the context of collective risk models or the closely related setting of compound Poisson processes, certain types of dependence among $N, Y_1, Y_2, \dots$ are studied. For instance, see Cheung et al. (2010), Albrecher et al. (2014) and Landriault et al. (2014) for recent development on dependent Sparre Anderson risk models, and see Denuit et al. (2005) for a comprehensive treatment of dependent losses in actuarial science.

Due to the high dimensionality of the joint model and sometimes limited data, it is often difficult to accurately model or justify a dependence structure. In such situations, the dependence between $Y_1, Y_2, \dots$, is completely or partially unknown, and this setting is nowadays referred to as *dependence* *uncertainty* and extensively developed in the past few years. See Bernard et al. (2014) and Embrechts et al. (2014) for a general discussion on dependence uncertainty. In this paper, we bring in the framework of dependence uncertainty into collective risk models. We assume that $Y_1, Y_2, \ldots$ are identically distributed as in classic collective risk models, but we do not assume a particular model for the dependence structure among random variables in (1.1). Two different practical settings will be considered:

(i) N is independent of $Y_1, Y_2, \ldots$ and the dependence structure of $Y_1, Y_2, \ldots$ is unknown.

(ii) The dependence structure of $N, Y_1, Y_2, \ldots$ is unknown.

From the perspective of risk management, we are particularly interested in quantifying S_N by certain *risk measures* under dependence uncertainty, a crucial concern for risk management in the presence of model uncertainty. The two most popular risk measures in banking and insurance are the Value-at-Risk (VaR) and the Expected Shortfall (ES, also called TVaR in actuarial science). The VaR of a risk X at the confidence level $\alpha \in (0, 1)$ is defined as

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} : F(x) \geqslant \alpha\}, \quad X \in L^0, \tag{1.2}$$

and the ES of a risk X at the confidence level $\alpha \in (0, 1)$ is defined as

$$\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma, \;\; X \in L^1, \tag{1.3}$$

where F is the distribution function of the random variable X. Risk measures for individual and collective risk models are well studied; see for instance Cai and Tan (2007) for optimal stop-loss reinsurance for these models under VaR and ES, and Hürlimann (2003) for ES bound for compound Poisson risks. It is well-known that an analytical calculation of the distribution of S_N, as well as $\mathrm{VaR}_\alpha(S_N)$ and $\mathrm{ES}_\alpha(S_N)$, is often unavailable (see Klugman et al. (2012)). Approximation, simulation or numerical calculation is often needed.

We study the worst-case values of $\mathrm{VaR}_\alpha(S_N)$ and $\mathrm{ES}_\alpha(S_N)$, under the two settings (i) and (ii) above. The recent literature on dependence uncertainty has focused on the individual risk model, in which $N = n$ in (1.1) is non-random. Analytical calculation of the worst-case $\mathrm{VaR}_\alpha(S_n)$ is generally unavailable; some analytical results can be found in Wang et al. (2013) and Jakobsons et al. (2016), and an efficient numerical algorithm is established in Puccetti and Rüschendorf (2012) and Embrechts et al. (2013). In the meanwhile, as a well-known result, the worst-case value of $\mathrm{ES}_\alpha(S_N)$ for a non-random $N = n$ is simply equal to the sum of the individual ES_α values, and this worst-case value is attained by comonotonic $Y_1, \ldots, Y_n$.

Due to a natural connection between collective risk models and individual risk models, using a collective model as in setting (i) can be also seen as one of the several ways to introduce partial dependence information into risk aggregation; see the Appendix for details and a comparison. For more studies of risk aggregation with partial dependence information, see Bernard et al. (2017a,b,c), Bernard and Vanduffel (2015), Bignozzi et al. (2015) and Puccetti et al. (2016, 2017).

The main contributions of this paper are summarized as follows. Based on the classic theory of stochastic orders, we first derive some convex ordering inequalities for collective risk models and thereby obtain analytical formulas for the worst-case values of ES. Using the results on ES for collective risk models, we are able to study the worst-case values of $\mathrm{VaR}_\alpha(S_N)$ and $\mathrm{ES}_\alpha(S_N)$ as $\mathbb{E}[N]$ increases to infinity, that is, a very large insurance portfolio. For simplicity the reader may think of the case where N is Poisson-distributed with parameter $\mathbb{E}[N]$, the most classic choice for the counting random variable N. In both settings (i) and (ii), under some moment and convergence conditions, we show that the worst-case values of $\mathrm{VaR}_\alpha(S_N)$ and $\mathrm{ES}_\alpha(S_N)$ enjoy very nice asymptotic properties. In particular, one can approximate them using the asymptotic equivalent $\mathbb{E}[N]\mathrm{ES}_\alpha(Y_1)$ and the convergence rates are obtained in both settings. The results can be used to approximate VaR and ES of a large insurance portfolio since it is straightforward to calculate $\mathbb{E}[N]\mathrm{ES}_\alpha(Y_1)$. Mathematically, our results generalize the asymptotic equivalence results for homogeneous individual risk models (i.e. N in (1.1) is non-random) in Wang and Wang (2015).

The rest of this paper is organized as follows. In Section 2, we present basic notation and definitions, stochastic orders, properties of ES, and some preliminary results on VaR-ES risk aggregation with dependence uncertainty. In Section 3, we study collective risk models with dependence uncertainty and obtain formulas for the worst-case ES. In Section 4, we establish asymptotic equivalence results under setting (i) and give the convergence rate under this setting. In Section 5, asymptotic equivalence results under setting (ii) are given, albeit stronger regularity conditions are needed compared to the case of setting (i). In Section 6, a brief conclusion is drawn.

2 Preliminaries

2.1 Some notation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space in which all random variables are defined. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough such that for any random variable X that appears in the paper, there exists a random variable independent of X. Let L^p, $p \in \mathbb{R}_+$, be the set of random variables with finite p-th moment, and L^∞ be the set of bounded random variables; in this paper $\mathbb{R}_+ = [0, \infty)$. For two random variables X and Y, we write $X \stackrel{d}{=} Y$ if they have the same distribution. For a distribution F, let $\mathcal{X}_F$ be the set of random variables with distribution F, and for $N \in L^0$, let $\mathcal{X}_F^N$ be the set of random variables in $\mathcal{X}_F$ independent of N. Let $\mathcal{X}_0$ be the set of counting random variables (i.e. taking value in $\{0, 1, \dots, \}$).

For a sequence $\mathbf{Y} = (Y_i,\ i \in \mathbb{N})$, we write (with a slight abuse of notation) $\mathbf{Y} \subset \mathcal{X}_F$ if $Y_i \in \mathcal{X}_F,\ i \in \mathbb{N}$, and similarly for $\mathbf{Y} \subset \mathcal{X}_F^N$. Denote by $\mathcal{Y}_F^N$ the set of random sequences with marginal distribution F and independent of N, that is,

$$\mathcal{Y}_F^N = \{(Y_1, Y_2, \dots) \subset \mathcal{X}_F : (Y_1, Y_2, \dots) \text{ is independent of } N\}.$$

Note that for a sequence $\mathbf{Y} = (Y_i,\ i \in \mathbb{N})$, there is a subtle difference between $\mathbf{Y} \subset \mathcal{X}_F^N$ and $\mathbf{Y} \in \mathcal{Y}_F^N$: the latter requires independence between the sequence $\mathbf{Y}$ and N, whereas the former only requires pair-wise independence between N and Y_i for $i \in \mathbb{N}$.

Throughout, for $N \in \mathcal{X}_0$ and $\mathbf{Y} = (Y_i,\ i \in \mathbb{N}) \subset L^0$, write

$$S_N = \sum_{i=1}^{N} Y_i,$$

where by convention $\sum_{i=1}^{0} Y_i = 0$. In the following, whenever S_N or S_n appears, it implicitly depends on $\mathbf{Y} = (Y_i,\ i \in \mathbb{N})$ which should be clear from the context.

In collective risk models, $Y_i,\ i \in \mathbb{N}$ are always assumed to be identically distributed, since Y_i represents the claim size of the i-th claim from a pool of policies, not the loss from a specific policy. We also assume $Y_i,\ i \in \mathbb{N}$ to be integrable; otherwise $\mathrm{ES}_\alpha(Y_1)$ is infinite for $\alpha \in (0, 1)$. In the case when the claim size Y_1 is not integrable, ES is not a proper risk measure to use in insurance practice; see, for instance, the general discussion on applicability of risk measures in McNeil et al. (2015).

For $p \in (0, 1)$ and any non-decreasing function F, we write

$$F^{-1}(p) = \inf\{x \in \mathbb{R} :\ F(x) \geqslant p\}.$$

It is well known that for any random variable X with distribution F, $F^{-1}(U) \stackrel{d}{=} X$, where U is any U[0, 1]-distributed random variable.

2.2 Stochastic orders

Definition 2.1. For $X, Y \in L^1$, X is said to be smaller than Y in *convex order* (resp. *increasing convex order*), denoted by $X \leqslant_{\mathrm{cx}} Y$ (resp. $X \leqslant_{\mathrm{icx}} Y$), if $\mathbb{E}[f(X)] \leqslant \mathbb{E}[f(Y)]$ for all convex functions (resp. increasing convex functions) $f : \mathbb{R} \to \mathbb{R}$, provided that the above expectations exist (can be infinity).

For a general introduction to convex order and increasing convex order, see Müller and Stoyan (2002) and Shaked and Shanthikumar (2007). Convex order is closely associated with the concept of comonotonicity.

Definition 2.2. Two random variables X and Y are *comonotonic* if

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geqslant 0 \text{ for } (\omega, \omega') \in \Omega \times \Omega \ (\mathbb{P} \times \mathbb{P})\text{-a.s.}$$

Comonotonicity of X and Y is equivalent to the existence of a random variable $Z \in L^0$ and two non-decreasing functions f and g, such that $X = f(Z)$ and $Y = g(Z)$ almost surely. We say that several random variables $X_1, \dots, X_n$ are comonotonic if X_i and X_j are comonotonic for each pair of $i, j = 1, \dots, n$[1]. See Dhaene et al. (2002) and Rüschendorf (2013) for an overview on comonotonicity.

Given random variables $X_1, X_2, \dots, X_n$, the following lemma presents an upper bound for sums $S_n = X_1 + X_2 + \cdots + X_n$ in the sense of convex order; see Theorem 7 of Dhaene et al. (2002) and Theorem 3.5 of Rüschendorf (2013). In particular, Rüschendorf (2013, Chapter 3) contains two different proofs and a brief history of this celebrated result.

Lemma 2.1. *For any random vector $(X_1, \dots, X_n) \in (L^1)^n$ we have*

$$X_1 + \cdots + X_n \leqslant_{\text{cx}} X_1^c + \cdots + X_n^c,$$

where $X_i^c \overset{d}{=} X_i$, $i = 1, \dots, n$, and $X_1^c, \dots, X_n^c \in L^1$ are comonotonic.

Another property about increasing convex order and comonotonicity is given in the following lemma, which is Corollary 3.28 (c) of Rüschendorf (2013).

Lemma 2.2. *For $X, Y, X^c, Y^c \in L^1$ such that X^c, Y^c are comonotonic, $X \overset{d}{=} X^c$, $Y \overset{d}{=} Y^c$ and $X^cY^c \in L^1$, we have*

$$XY \leqslant_{\text{icx}} X^cY^c.$$

The stochastic inequality in the above lemma holds for every monotonic supermodular function of X and Y; see Theorem 2 of Tchen (1980) and Theorem 2.1 of Puccetti and Wang (2015).

[1] Generally, this definition is stronger than assuming that X_1 and X_j are comonotonic for $j = 2, \dots, n$; see for instance, Example 4 of Cheung et al. (2014).

2.3 Some properties of ES

In this paper, we will frequently use some well-known properties of the Expected Shortfall defined in (1.3); see e.g. McNeil et al. (2015) for details on properties of ES.

Lemma 2.3. *For $\alpha \in (0,1)$, $\mathrm{ES}_\alpha : L^1 \to \mathbb{R}$ satisfies: for any $X, Y \in L^1$,*

(i) Monotonicity: $\mathrm{ES}_\alpha(X) \leqslant \mathrm{ES}_\alpha(Y)$ *if* $X \leqslant Y$ $\mathbb{P}$*-a.s;*

(ii) Cash-invariance: $\mathrm{ES}_\alpha(X - m) = \mathrm{ES}_\alpha(X) - m$ *for any* $m \in \mathbb{R}$*;*

(iii) Subadditivity: $\mathrm{ES}_\alpha(X + Y) \leqslant \mathrm{ES}_\alpha(X) + \mathrm{ES}_\alpha(Y)$*;*

(iv) Positive homogeneity: $\mathrm{ES}_\alpha(\lambda X) = \lambda \mathrm{ES}_\alpha(X)$ *for any* $\lambda > 0$*;*

(v) Law-invariance: $\mathrm{ES}_\alpha(X) = \mathrm{ES}_\alpha(Y)$ *if* $X \stackrel{d}{=} Y$*;*

(vi) Comonotonic additivity: $\mathrm{ES}_\alpha(X + Y) = \mathrm{ES}_\alpha(X) + \mathrm{ES}_\alpha(Y)$ *if X and Y are comonotonic.*

The reader is referred to Föllmer and Schied (2011, Chapter 4) and Delbaen (2012) for interpretations of these standard properties in the literature of risk measures. The following lemma is also well known in the literature of convex order (see Theorem 4.A.3 of Shaked and Shanthikumar (2007)).

Lemma 2.4. *For $X, Y \in L^1$, $X \leqslant_{\mathrm{icx}} Y$ if and only if $\mathrm{ES}_\alpha(X) \leqslant \mathrm{ES}_\alpha(Y)$ for all $\alpha \in (0,1)$.*

As a consequence of Lemma 2.4, for $\alpha \in (0,1)$, ES_α preserves increasing convex order (and hence convex order). Another property that will be used later is the L^1-continuity of ES below; for a proof of this property, see, for instance, Svindland (2008).

Lemma 2.5. *For $\alpha \in (0,1)$, $\mathrm{ES}_\alpha : L^1 \to \mathbb{R}$ is continuous with respect to the L^1-norm.*

Recalling the definition of the L^1-continuity, the above lemma means that for a sequence of random variables $X_1, X_2, \dots$ and $X \in L^1$, as $n \to \infty$, $\mathbb{E}[|X_n - X|] \to 0$ implies that $\mathrm{ES}_\alpha(X_n) \to \mathrm{ES}_\alpha(X)$.

2.4 VaR-ES asymptotic equivalence in risk aggregation

We give some preliminary results on the VaR-ES *asymptotic equivalence* in risk aggregation, which will be useful to the main results in this paper.

Lemma 2.6 (Corollary 3.7 of Wang and Wang (2015))**.** *For any distribution F and $Y \in \mathcal{X}_F$,*

$$\lim_{n\to\infty} \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha(S_n)}{n} = \mathrm{ES}_\alpha(Y), \qquad \alpha \in (0,1). \tag{2.1}$$

The result in (2.1) can be rewritten as

$$\lim_{n\to\infty} \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha(S_n)}{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha(S_n)} = 1, \qquad \alpha \in (0,1), \tag{2.2}$$

provided that $0 < \mathrm{ES}_\alpha(Y) < \infty$, $Y \in \mathcal{X}_F$. Result of type (2.2) is called an *asymptotic equivalence* between VaR and ES.

The equivalence (2.2) was shown in Puccetti and Rüschendorf (2014) and Puccetti et al. (2013) under different conditions, and with generality in Wang and Wang (2015). For equivalence of type (2.2) under the setting of inhomogeneous marginal distributions and for risk measures other than VaR and ES, see Embrechts et al. (2015), Wang et al. (2015) and Cai et al. (2017). The convergence rate of (2.2) is given in the following lemma.

Lemma 2.7 (Corollary 3.8 of Wang and Wang (2015))**.** *Suppose that the distribution F has finite p-th moment, $p \geqslant 1$, and* ES *at level $\alpha \in (0,1)$ is non-zero. Then as $n \to \infty$,*

$$\frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha(S_n)}{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha(S_n)} = 1 - o\left(n^{1/p-1}\right).$$

3 Collective risk models with dependence uncertainty

3.1 Setup and a motivating example

In this section, we study the worst-case values of VaR and ES for collective risk models. As mentioned in the introduction, we consider two different settings of dependence uncertainty:

(i) the number of claims N is independent of the claim sizes $Y_1, Y_2, \dots$ and the dependence structure of $Y_1, Y_2, \dots$ is unknown;

(ii) the dependence structure of $N, Y_1, Y_2, \dots$ is unknown.

We refer to the setting (i) as the *classic* collective risk model with dependence uncertainty and to the setting (ii) as the *generalized* collective risk model with dependence uncertainty. Using the notation introduced in Section 2, for some distribution F on $\mathbb{R}_+$ (i.e. non-negative claim sizes), setting (i) reads as $\mathbf{Y} \in \mathcal{Y}_F^N$ and setting (ii) reads as $\mathbf{Y} \subset \mathcal{X}_F$. The quantities of interest in setting (i) are

$$\sup_{\mathbf{Y}\in\mathcal{Y}_F^N} \mathrm{VaR}_\alpha(S_N) \quad \text{and} \quad \sup_{\mathbf{Y}\in\mathcal{Y}_F^N} \mathrm{ES}_\alpha(S_N), \tag{3.1}$$

and the quantities of interest in setting (ii) are

$$\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha(S_N) \quad \text{and} \quad \sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha(S_N). \tag{3.2}$$

It turns out that under both settings (i) and (ii), the worst-case value of ES is straightforward to calculate, whereas an analytical formula for the worst-case value of VaR is not available. This is similar to the well-studied case of individual risk models; see Embrechts et al. (2014) for a review on worst-case VaR aggregation when N is non-random.

Before we carry out a theoretical treatment, we illustrate with a simple example in the theory of loss models by comparing an individual risk model and a corresponding collective risk model formulation. Assume both models admit dependence uncertainty, and we evaluate worst-case ES for both models as in (3.2). We shall see that the ES bound is largely reduced by knowing the distribution of the claim frequency, as opposed to an *uncertain distribution of the claim frequency* implied by the individual risk model with dependence uncertainty.

Example 3.1. Let $n = 40000$. Consider an individual risk model

$$S = \sum_{i=1}^{n} X_i,$$

where for $i = 1, \dots, n$, X_i follows a distribution F such that $\mathbb{P}(X_i > x) = \frac{1}{1000}e^{-x}$, $x \geqslant 0$. If we assume that $X_1, \dots, X_n$ are independent, then the collective reformulation of S is given by

$$S_N = \sum_{i=1}^{N} Y_i,$$

where N follows the Poisson distribution with parameter $\lambda = 40$ (denoted by Pois(40)), Y_i follows an Exponential distribution with mean 1 (denoted by Expo(1)), $i \in \mathbb{N}$, and $N, Y_1, Y_2, \dots$ are independent. Below we assume that only N and $(Y_i,\ i \in \mathbb{N})$ are independent, but the dependence among $X_1, \dots, X_n$ and the dependence among $Y_1, Y_2, \dots,$ are uncertain. Take $\alpha = 0.95$. To evaluate the corresponding worst-case ES_α values, we have[2]

$$\sup_{\mathbf{Y}\in\mathcal{Y}_F^N} \mathrm{ES}_\alpha\left(\sum_{i=1}^N Y_i\right) = 164.09,$$
$$\sup_{X_i\in\mathcal{X}_F, i\leqslant n} \mathrm{ES}_\alpha\left(\sum_{i=1}^n X_i\right) = 800.$$

As we can see from the numerical results, the knowledge of $N \sim \mathrm{Pois}(40)$ greatly reduces the worst-case ES value, as compared to the individual risk model. In the sequel, we shall investigate the VaR and ES bounds for collective risk models under dependence uncertainty.

3.2 VaR and ES bounds for collective risk models

In this section we establish some explicit formulas for VaR and ES bounds in (3.1) and (3.2). We first provide a simple result on convex order for collective risk models with unknown dependence.

Lemma 3.1. *Suppose that $(Y_i, N) \in L^1 \times \mathcal{X}_0$, $i \in \mathbb{N}$, have identical joint distributions and $NY_1 \in L^1$. We have*

$$\sum_{i=1}^N Y_i \leqslant_{\mathrm{cx}} NY_1. \tag{3.3}$$

Proof. First, one can easily verify $\mathbb{E}[\sum_{i=1}^N Y_i] = \mathbb{E}[NY_1]$ and hence both sides of (3.3) are in L^1. Let $D = \{n \in \{0, 1, \dots\} : \mathbb{P}(N = n) > 0\}$ be the range of N. Denote by F_n the conditional distribution of Y_1 given $N = n$ for $n \in D$. Let f be a convex function such that both $\mathbb{E}[f(\sum_{i=1}^N Y_i)]$ and $\mathbb{E}[f(NY_1)]$ are properly defined. For $n \in D$, there exist some U[0, 1]-distributed random variables $U_1^n, \dots, U_n^n$ such that

$$\mathbb{E}[f(Y_1 + \cdots + Y_n)|N = n] = \mathbb{E}[f(F_n^{-1}(U_1^n) + \cdots + F_n^{-1}(U_n^n))].$$

It follows from Lemma 2.1 that

$$\mathbb{E}[f(Y_1 + \cdots + Y_n)|N = n] \leqslant \mathbb{E}[f(nF_n^{-1}(U_1^n))] = \mathbb{E}[f(nY_1)|N = n].$$

Summing up over $n \in D$ yields

$$\mathbb{E}[f(Y_1 + \cdots + Y_N)] \leqslant \mathbb{E}[f(NY_1)],$$

and hence by definition, (3.3) holds. □

[2]the first value is calculated via Theorem 3.3 (see below) and the average of 100 repetitions of simulation with a sample of size 100,000, and the second value is calculated analytically.

As a special case of Lemma 3.1, if N is in L^1 and independent of the identically distributed random variables $Y_1, Y_2, \ldots \in L^1$, then (3.3) holds. This particular result will be used later.

To deal with setting (ii) in which the dependence structure between N and $Y_1, Y_2, \ldots$ is unspecified, we give a result in the following lemma on increasing convex order instead of convex order. Note that for $X, Y \in L^1$, $X \leqslant_{\mathrm{cx}} Y$ implies that $\mathbb{E}[X] = \mathbb{E}[Y]$. Since $\mathbb{E}[S_N]$ depends on the dependence structure between N and $Y_1, Y_2, \ldots$, convex order between collective risk models under different dependence structures cannot be expected.

Lemma 3.2. *Suppose that the distribution F on $\mathbb{R}_+$ has finite second moment, and $N \in \mathcal{X}_0 \cap L^2$. For $Y_1, Y_2, \ldots \in \mathcal{X}_F$, we have*

$$\sum_{i=1}^{N} Y_i \leqslant_{\mathrm{icx}} NY,$$

where $Y \in \mathcal{X}_F$ and N, Y are comonotonic.

Proof. Note that $NY \in L^1$ by Hölder's inequality. Define $X_n = \sum_{i=1}^{n} Y_i \mathrm{I}_{\{N \geqslant i\}}$, $n \in \mathbb{N}$ and $X_\infty = \sum_{i=1}^{\infty} Y_i \mathrm{I}_{\{N \geqslant i\}}$. Note that $\mathbb{P}(X_\infty > X_n) \to 0$ as $n \to \infty$, and hence $\mathbb{P}(X_\infty < \infty) = 1$. Thus X_∞ is a finite random variable. Then we have $X_n \to X_\infty$ almost surely and hence $X_n \to X_\infty$ in distribution. Since $F \to F^{-1}(\gamma)$ is weakly continuous at each F_0 for which $s \to F_0^{-1}(s)$ is continuous at $s = \gamma$ (see e.g. Cont et al. (2010)), we have

$$\mathrm{VaR}_\gamma(X_n) \to \mathrm{VaR}_\gamma(X_\infty) \quad \text{almost everywhere in } \gamma \in [0,1]. \tag{3.4}$$

For any $\mathbf{Y} \subset \mathcal{X}_F$ and any $\alpha \in (0,1)$, we have

$$\begin{aligned}
\mathrm{ES}_\alpha\left(\sum_{i=1}^{N} Y_i\right) = \mathrm{ES}_\alpha\left(\sum_{i=1}^{\infty} Y_i \mathrm{I}_{\{N \geqslant i\}}\right) &= \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_\gamma(X_\infty)\mathrm{d}\gamma \\
\text{(by (3.4))} \quad &= \frac{1}{1-\alpha}\int_\alpha^1 \lim_{n\to\infty} \mathrm{VaR}_\gamma(X_n)\mathrm{d}\gamma \\
\text{(Fatou's Lemma)} \quad &\leqslant \liminf_{n\to\infty} \mathrm{ES}_\alpha(X_n) \\
\text{(subadditivity of ES)} \quad &\leqslant \liminf_{n\to\infty} \sum_{i=1}^{n} \mathrm{ES}_\alpha\left(Y_i \mathrm{I}_{\{N \geqslant i\}}\right) \\
\text{(by Lemmas 2.2 and 2.4)} \quad &\leqslant \liminf_{n\to\infty} \sum_{i=1}^{n} \mathrm{ES}_\alpha\left(Y \mathrm{I}_{\{N \geqslant i\}}\right) \\
\text{(comonotonic additivity of ES)} \quad &= \liminf_{n\to\infty} \mathrm{ES}_\alpha\left(\sum_{i=1}^{n} Y \mathrm{I}_{\{N \geqslant i\}}\right) \\
\text{(L^1-continuity of ES)} \quad &= \mathrm{ES}_\alpha(NY).
\end{aligned}$$

Since $\mathrm{ES}_\alpha\left(\sum_{i=1}^{N} Y_i\right) \leqslant \mathrm{ES}_\alpha(NY)$ for all $\alpha \in (0,1)$, by Lemma 2.4, we have $\sum_{i=1}^{N} Y_i \leqslant_{\mathrm{icx}} NY$. ◻

Remark 3.1. Using the same proof, the stochastic inequality in Lemma 3.2 can be generalized to the random sum of non-identically distributed random variables as follows. Suppose that $N \in \mathcal{X}_0$, $Y_i \geqslant 0$, $i \in \mathbb{N}$, and $\sum_{i=1}^N Y_i^c \in L^1$, where $Y_i^c \stackrel{d}{=} Y_i$, $i \in \mathbb{N}$, and $Y_1^c, Y_2^c, \dots$ and N are comonotonic. Then we have

$$\sum_{i=1}^N Y_i \leqslant_{\mathrm{icx}} \sum_{i=1}^N Y_i^c.$$

With the help of Lemmas 3.1 and 3.2, we arrive at the worst-case values of ES for collective risk models under dependence uncertainty.

Theorem 3.3. *Suppose that F is a distribution on $\mathbb{R}_+$, $N \in \mathcal{X}_0$ and $Y, Y^* \in \mathcal{X}_F$ such that N, Y are independent and N, Y^* are comonotonic.*

(i) If $Y, N \in L^1$, then

$$\sup_{\mathbf{Y} \in \mathcal{Y}_F^N} \mathrm{ES}_\alpha(S_N) = \mathrm{ES}_\alpha(NY), \quad \alpha \in (0,1). \tag{3.5}$$

(ii) If $Y, N \in L^2$, then

$$\sup_{\mathbf{Y} \subset \mathcal{X}_F} \mathrm{ES}_\alpha(S_N) = \mathrm{ES}_\alpha(NY^*), \quad \alpha \in (0,1). \tag{3.6}$$

Proof. Note that $NY \in L^1$ since N, Y are independent. Since $\mathbf{Y} \subset \mathcal{X}_F^N$ for any $\mathbf{Y} \in \mathcal{Y}_F^N$, we have

$$\sup_{\mathbf{Y} \in \mathcal{Y}_F^N} \mathrm{ES}_\alpha(S_N) \leqslant \sup_{\mathbf{Y} \subset \mathcal{X}_F^N} \mathrm{ES}_\alpha(S_N).$$

By Lemma 2.4, ES preserves increasing convex order. Further, by Lemmas 3.1 and 3.2, we have

$$\sup_{\mathbf{Y} \in \mathcal{Y}_F^N} \mathrm{ES}_\alpha(S_N) \leqslant \sup_{\mathbf{Y} \subset \mathcal{X}_F^N} \mathrm{ES}_\alpha(S_N) \leqslant \mathrm{ES}_\alpha(NY) \quad \text{and} \quad \sup_{\mathbf{Y} \subset \mathcal{X}_F} \mathrm{ES}_\alpha(S_N) \leqslant \mathrm{ES}_\alpha(NY^*).$$

It suffices to take $Y_1, Y_2, \dots$ to be identical to $Y \in \mathcal{X}_F^N$ to show that $\sup_{\mathbf{Y} \in \mathcal{Y}_F^N} \mathrm{ES}_\alpha(S_N) \geqslant \mathrm{ES}_\alpha(NY)$ in (i) and to take $Y_1, Y_2, \dots$ to be identical to $Y^* \in \mathcal{X}_F$ to show $\sup_{\mathbf{Y} \subset \mathcal{X}_F} \mathrm{ES}_\alpha(S_N) \geqslant \mathrm{ES}_\alpha(NY^*)$ in (ii). □

The results in Theorem 3.3 are consistent with simple intuition. Assume that the riskiness of an insurance portfolio is measured by an ES. If the number of claims and the claim sizes are independent, then, in the worst-case dependence scenario, all claims are comonotonic. If the number of claims and the claim sizes are also dependent, then in the worst-case dependence scenario, all claims are comonotonic and they are further comonotonic with the number of claims. This could for instance be close to reality in the case of insurance losses from flood damage in an area, where the claim sizes and the number of claims are largely determined by the magnitude of the flood, and hence they are all positively correlated. Thus, the portfolio of insurance policies with heavy positive dependence has the most dangerous dependence structure, if an ES is the risk measure in use. Note that such an intuition is not valid for the risk measure VaR.

The values of $\mathrm{ES}_\alpha(NY)$ and $\mathrm{ES}_\alpha(NY^*)$ in (3.5) and (3.6) are straightforward to calculate. For (3.5), one needs to calculate the distribution of NY, which is the product of two independent random variables. This involves a one-step convolution after a logarithm transformation. For (3.6), note that $NY^* \overset{d}{=} G^{-1}(U)F^{-1}(U)$, where U is U[0, 1]-distributed and G is the distribution of N. In that case, its ES is simply

$$\mathrm{ES}_\alpha(NY^*) = \frac{1}{1-\alpha}\int_\alpha^1 G^{-1}(u)F^{-1}(u)\mathrm{d}u,$$

which is as simple as calculating the ES of any known distribution.

The following corollary gives an ES ordering for an individual risk model with dependence uncertainty, a collective risk model under setting (i), and a collective risk model under setting (ii).

Corollary 3.4. *Suppose that F is a distribution on $\mathbb{R}_+$ with finite first moment, $N \in X_0$ and $\mathbb{E}[N] \in \mathbb{N}$. We have the following orders*

$$\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha\left(S_{\mathbb{E}[N]}\right) \leqslant \sup_{\mathbf{Y}\in\mathcal{Y}_F^N} \mathrm{ES}_\alpha(S_N) \leqslant \sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha(S_N), \quad \alpha \in (0,1). \tag{3.7}$$

Proof. Since $\mathbf{Y} \in \mathcal{Y}_F^N$ implies $\mathbf{Y} \subset \mathcal{X}_F$, the second inequality follows immediately. To show the first inequality, take $Y \in \mathcal{X}_F^N$. Note that from the properties of ES,

$$\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha\left(S_{\mathbb{E}[N]}\right) = \mathbb{E}[N]\mathrm{ES}_\alpha(Y) = \mathrm{ES}_\alpha(\mathbb{E}[N]Y),$$

and from Theorem 3.3,

$$\sup_{\mathbf{Y}\in\mathcal{Y}_F^N} \mathrm{ES}_\alpha\left(S_N\right) = \mathrm{ES}_\alpha(NY).$$

By Theorem 3.A.33 of Shaked and Shanthikumar (2007), $\mathbb{E}[N]Y \leqslant_{\mathrm{cx}} NY$. The rest of the proof follows since ES preserves convex order as in Lemma 2.4. □

In the case of $N, Y \in L^2$, the order in (3.7) can be formulated as follows. For $N \in \mathcal{X}_0$, $Y \overset{d}{=} Y^*$ such that N, Y are independent and N, Y^* are comonotonic, we have

$$\mathbb{E}[N]\mathrm{ES}_\alpha(Y) \leqslant \mathrm{ES}_\alpha(NY) \leqslant \mathrm{ES}_\alpha(NY^*), \quad \alpha \in (0,1). \tag{3.8}$$

As for the problem of the worst-case value of VaR for collective risk models, there is no simple analytical formula, as expected from classic results on dependence uncertainty. Note that VaR_α is dominated by ES_α, for $\alpha \in (0,1)$; thus $\mathrm{VaR}_\alpha(S_N) \leqslant \mathrm{ES}_\alpha(S_N)$ for all model settings. From Theorem 3.3, we have

$$\sup_{\mathbf{Y}\in\mathcal{Y}_F^N} \mathrm{VaR}_\alpha(S_N) \leqslant \mathrm{ES}_\alpha(NY) \quad \text{and} \quad \sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha(S_N) \leqslant \mathrm{ES}_\alpha(NY^*), \tag{3.9}$$

where F, N, Y and Y^* are as in Theorem 3.3. In the next two sections, we will see that

$$\sup_{\mathbf{Y}\in\mathcal{Y}_F^N} \mathrm{VaR}_\alpha(S_N) \approx \mathrm{ES}_\alpha(NY) \quad \text{and} \quad \sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha(S_N) \approx \mathrm{ES}_\alpha(NY^*),$$

if N is large (in some sense). That is, the inequalities in (3.9) are almost sharp and can be used to approximate VaR.

Remark 3.2. In Lemma 3.2 and Theorem 3.3 (ii), we require $Y, N \in L^2$ so that $NY^* \in L^1$; recall that L^1 is the domain of ES_α. One may also use the slightly more general assumption that $N \in L^p$ and $Y \in L^q$ for some $p, q > 1$ such that $1/p + 1/q = 1$.

4 Asymptotic results for classic collective risk models

4.1 Setup and objectives

The rest of the paper is dedicated to the study of an analog of the asymptotic equivalence in (2.2) for collective risk models. Recall that throughout we write

$$S_{N(v)} = \sum_{i=1}^{N(v)} Y_i, \quad \mathbf{Y} = (Y_1, Y_2, \dots).$$

For some distribution F on $\mathbb{R}_+$, and a counting random variable $N(v)$ with parameter v, the analog of (2.2) in setting (i) is

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{ES}_\alpha\left(S_{N(v)}\right)} = 1, \tag{4.1}$$

and the analog of (2.2) in setting (ii) is

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha\left(S_{N(v)}\right)} = 1. \tag{4.2}$$

Here, $v \to \infty$ indicates that the expected number of claims goes to infinity. The parameter v is interpreted as the *volume* of the insurance portfolio, and it can be chosen as, for instance, $\mathbb{E}[N(v)]$.

One of the key assumptions we propose is $N(v)/v \to 1$ in L^1. This assumption naturally holds if $N(v)$ is a Poisson random variable with parameter $v > 0$, or $N(v)$ is the partial sum of a short-range dependent stationary sequence (so that a law of large numbers holds). Indeed, the problem we study in this paper first appeared as a question of measuring large insurance portfolios under dependence uncertainty, where $N(v)$ is a Poisson random variable with a large parameter. Moreover, an insurance company can analyze effects from potential extension of business by measuring the insurance portfolio as v increases.

Results under setting (i) are presented in this section and results under setting (ii) are given in Section 5 below. Since $v\mathrm{ES}_\alpha(Y)$ is straightforward to calculate and thus serves as a basis for approximation of the two worst-case values of interest, we present our results in terms of the two ratios

$$\frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha(Y)} \quad \text{and} \quad \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{ES}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha(Y)}.$$

We also establish convergence rates in both cases.

4.2 VaR-ES asymptotic equivalence

Theorem 4.1. *Suppose that the distribution F on $\mathbb{R}_+$ has finite first moment, $Y \in \mathcal{X}_F$, and $\{N(v),\ v \geqslant 0\} \subset \mathcal{X}_0$ such that $N(v)/v \to 1$ in L^1 as $v \to \infty$. Then for $\alpha \in (0,1)$,*

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v} = \lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{ES}_\alpha\left(S_{N(v)}\right)}{v} = \mathrm{ES}_\alpha\left(Y\right). \tag{4.3}$$

Proof. By the independence of $N(v)$ and Y, and $\frac{N(v)}{v} \xrightarrow{L^1} 1$, we have

$$\mathbb{E}\left|\frac{N(v)Y}{v} - Y\right| \leqslant \mathbb{E}\left|\frac{N(v)}{v} - 1\right| \cdot \mathbb{E}\left[Y\right] \to 0, \qquad \text{as } v \to \infty.$$

Hence, $\frac{N(v)Y}{v} \xrightarrow{L^1} Y$. Continuity of ES with respect to the L^1-norm implies

$$\lim_{v\to\infty} \mathrm{ES}_\alpha\left(\frac{N(v)Y}{v}\right) = \mathrm{ES}_\alpha\left(Y\right). \tag{4.4}$$

From Theorem 3.3 (i), we have

$$\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{ES}_\alpha(S_{N(v)}) = \mathrm{ES}_\alpha\left(N(v)Y\right). \tag{4.5}$$

By (4.4) and the positive homogeneity of ES, we have

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{ES}_\alpha(S_{N(v)})}{v} = \mathrm{ES}_\alpha\left(Y\right).$$

Thus, we obtain the second equality in (4.3).

Since L^1-convergence implies convergence in probability, $\frac{N(v)}{v} \xrightarrow{L^1} 1$ yields that for any $\varepsilon > 0$ and $\delta > 0$, there exists an $M_1 > 0$ such that for all $v \geqslant M_1$,

$$\mathbb{P}\left(\left|\frac{N(v)}{v} - 1\right| > \delta\right) < \varepsilon.$$

Write $S_{N(v)} = Y_1 + \cdots + Y_{N(v)}$. Define

$$S^*_{N(v)} = \begin{cases} S_{N(v)} & \text{if } N(v)/v \geqslant 1 - \delta, \\ 0 & \text{if } N(v)/v < 1 - \delta. \end{cases}$$

Since $S_{N(v)} \geqslant S^*_{N(v)}$, we have

$$\begin{aligned} \mathrm{VaR}_{\alpha+\varepsilon}(S_{N(v)}) \geqslant \mathrm{VaR}_{\alpha+\varepsilon}\left(S^*_{N(v)}\right) &= \inf\left\{t \in \mathbb{R} : \mathbb{P}\left(S^*_{N(v)} \leqslant t\right) \geqslant \alpha + \varepsilon\right\} \\ &= \inf\{t \in \mathbb{R} : \mathbb{P}(S_{N(v)} \leqslant t, N(v)/v \geqslant 1 - \delta) + \mathbb{P}(0 \leqslant t, N(v)/v < 1 - \delta) \geqslant \alpha + \varepsilon\} \\ &\geqslant \inf\{t \in \mathbb{R} : \mathbb{P}(S_{N(v)} \leqslant t, N(v)/v \geqslant 1 - \delta) \geqslant \alpha\} \\ &\geqslant \inf\{t \in \mathbb{R} : \mathbb{P}(S_{\lfloor(1-\delta)v\rfloor} \leqslant t) \geqslant \alpha\} = \mathrm{VaR}_\alpha(S_{\lfloor(1-\delta)v\rfloor}). \end{aligned} \tag{4.6}$$

By Lemma 2.6, for any $\varepsilon_2 > 0$, there exists an $M_2 > 1/\varepsilon$ such that for all $v > M_2$,

$$\frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_{\alpha-\varepsilon}(S_{\lfloor(1-\delta)v\rfloor})}{\lfloor(1-\delta)v\rfloor} > \mathrm{ES}_{\alpha-\varepsilon}(Y) - \varepsilon_2.$$

Thus, for the above $\varepsilon > 0$ and $v > \max\{M_1, M_2\}$,

$$\begin{aligned} \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha(S_{N(v)})}{v} &\geqslant \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_{\alpha-\varepsilon}(S_{\lfloor(1-\delta)v\rfloor})}{v} \\ &= \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_{\alpha-\varepsilon}(S_{\lfloor(1-\delta)v\rfloor})}{\lfloor(1-\delta)v\rfloor} \cdot \frac{\lfloor(1-\delta)v\rfloor}{v} \\ &> [\mathrm{ES}_{\alpha-\varepsilon}(Y) - \varepsilon_2] \cdot \frac{(1-\delta)v - 1}{v} \\ &> [\mathrm{ES}_{\alpha-\varepsilon}(Y) - \varepsilon_2](1 - \delta - \varepsilon), \end{aligned}$$

which implies

$$\liminf_{v\to\infty} \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha(S_{N(v)})}{v} \geqslant \mathrm{ES}_\alpha(Y).$$

On the other hand,

$$\limsup_{v\to\infty} \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha(S_{N(v)})}{v} \leqslant \lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{ES}_\alpha(S_{N(v)})}{v} = \mathrm{ES}_\alpha(Y).$$

Therefore,

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha(S_{N(v)})}{v} = \mathrm{ES}_\alpha(Y).$$

Thus, we obtain the first equality in (4.3). $\square$

Theorem 4.1, together with Lemma 2.6, suggests that for $\alpha \in (0,1)$ and $Y \in \mathcal{X}_F$, the following five quantities are all asymptotically equivalent as $v \to \infty$:

(i) $\sup_{\mathbf{Y} \in \mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha(S_{N(v)})$; (iii) $\sup_{\mathbf{Y} \subset \mathcal{X}_F} \mathrm{VaR}_\alpha(S_{\lfloor v \rfloor})$; (v) $v\mathrm{ES}_\alpha(Y)$.

(ii) $\sup_{\mathbf{Y} \in \mathcal{Y}_F^{N(v)}} \mathrm{ES}_\alpha(S_{N(v)})$; (iv) $\sup_{\mathbf{Y} \subset \mathcal{X}_F} \mathrm{ES}_\alpha(S_{\lfloor v \rfloor})$;

Hence, one may use (v) above (straightforward to calculate) to approximate the other four quantities. The approximation error, that is, the convergence rate in Theorem 4.1, is studied in the following section.

Remark 4.1. Since $\mathrm{VaR}_\alpha \leqslant \mathrm{ES}_\alpha$, the quantity in (i) is smaller than or equal to the quantity in (ii), and similarly for (iii) and (iv). Another observation is that $\sup_{\mathbf{Y} \subset \mathcal{X}_F} \mathrm{ES}_\alpha(S_{\lfloor v \rfloor}) = \lfloor v \rfloor \mathrm{ES}_\alpha(Y) \leqslant v\mathrm{ES}_\alpha(Y)$. From Corollary 3.4, the quantity in (iv) is smaller than or equal to the quantity in (ii), provided that $\mathbb{E}[N(v)] = \lfloor v \rfloor$. However, there is no general order between (i) and (v) (or (iv)); when we approximate $\sup_{\mathbf{Y} \in \mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha(S_{N(v)})$ with $v\mathrm{ES}_\alpha(Y)$, it is not clear which one is larger. See Theorem 4.2 below for more detailed analysis on their relationship.

4.3 Rate of convergence

Theorem 4.2. *Suppose that the distribution F on $\mathbb{R}_+$ has finite p-th moment, $p \geqslant 1$, $Y \in \mathcal{X}_F$, $\mathbb{E}[Y] > 0$, and $\limsup_{v\to\infty} v^q \mathbb{E}\left|\frac{N(v)}{v} - 1\right| \leqslant c$ for some $q > 0,\ c > 0$. Then for $\alpha \in (0,1)$,*

$$-2C^{1/2}v^{-q/2} + o\left(v^{1/p-1}\right) + o\left(v^{-q/2}\right) \leqslant \frac{\sup_{\mathbf{Y} \in \mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha(S_{N(v)})}{v\mathrm{ES}_\alpha(Y)} - 1 \leqslant Cv^{-q} + o\left(v^{-q}\right), \tag{4.7}$$

and

$$\left|\frac{\sup_{\mathbf{Y} \in \mathcal{Y}_F^{N(v)}} \mathrm{ES}_\alpha(S_{N(v)})}{v\mathrm{ES}_\alpha(Y)} - 1\right| \leqslant Cv^{-q} + o\left(v^{-q}\right), \tag{4.8}$$

where $C = \frac{c}{1-\alpha}$.

Proof. Let $\delta = \sqrt{C}v^{-q/2}$, $\eta = \left(v^q\mathbb{E}\left|\frac{N(v)}{v} - 1\right| - c\right)_+$, and $\varepsilon = \frac{c+\eta}{\delta}v^{-q}$. Clearly $\varepsilon = \sqrt{c(1-\alpha)}v^{-q/2} + o(v^{-q/2})$. Note that

$$v^{-q}(c+\eta) \geqslant \mathbb{E}\left|\frac{N(v)}{v} - 1\right| \geqslant \int_{|N(v)/v-1|>\delta} \left|\frac{N(v)}{v} - 1\right| \mathrm{d}\mathbb{P} > \delta\,\mathbb{P}\left(\left|\frac{N(v)}{v} - 1\right| > \delta\right).$$

Hence,

$$\mathbb{P}\left(\left|\frac{N(v)}{v} - 1\right| > \delta\right) < \frac{c+\eta}{\delta}v^{-q} = \varepsilon.$$

This implies $\mathrm{VaR}_\alpha\left(S_{N(v)}\right) \geqslant \mathrm{VaR}_{\alpha-\varepsilon}\left(S_{\lfloor(1-\delta)v\rfloor}\right)$ as shown in (4.6). By Lemma 2.7, we have

$$\begin{aligned}\frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha(Y)} &\geqslant \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_{\alpha-\varepsilon}\left(S_{\lfloor(1-\delta)v\rfloor}\right)}{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_{\alpha-\varepsilon}\left(S_{\lfloor(1-\delta)v\rfloor}\right)} \cdot \frac{\lfloor(1-\delta)v\rfloor\mathrm{ES}_{\alpha-\varepsilon}(Y)}{v\mathrm{ES}_\alpha(Y)} \\ &\geqslant \left[1-o\left(v^{1/p-1}\right)\right]\cdot\left(1-\delta-v^{-1}\right)\cdot\frac{\mathrm{ES}_{\alpha-\varepsilon}(Y)}{\mathrm{ES}_\alpha(Y)}.\end{aligned} \tag{4.9}$$

Note that

$$\left|1-\frac{\mathrm{ES}_{\alpha-\varepsilon}(Y)}{\mathrm{ES}_\alpha(Y)}\right| = \frac{\left(\frac{1}{1-\alpha}-\frac{1}{1-\alpha+\varepsilon}\right)\int_\alpha^1 \mathrm{VaR}_\gamma(Y)\,\mathrm{d}\gamma - \frac{1}{1-\alpha+\varepsilon}\int_{\alpha-\varepsilon}^{\alpha}\mathrm{VaR}_\gamma(Y)\,\mathrm{d}\gamma}{\mathrm{ES}_\alpha(Y)} \leqslant \frac{\varepsilon}{1-\alpha}.$$

Therefore,

$$\frac{\mathrm{ES}_{\alpha-\varepsilon}(Y)}{\mathrm{ES}_\alpha(Y)} \geqslant 1-\frac{\varepsilon}{1-\alpha}.$$

Plugging the above inequality into (4.9), one has

$$\begin{aligned}\frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha(Y)} &\geqslant \left[1-o\left(v^{1/p-1}\right)\right]\cdot\left(1-\delta-v^{-1}\right)\cdot\left(1-\frac{\varepsilon}{1-\alpha}\right) \\ &= 1-2\sqrt{C}v^{-q/2}-o\left(v^{1/p-1}\right)-o\left(v^{-q/2}\right).\end{aligned}$$

Thus, we obtain the first inequality in (4.7).

In the next step we show (4.8). From Theorem 3.3 (i),

$$\frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{ES}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha(Y)} = \frac{\mathrm{ES}_\alpha(N(v)Y)}{v\mathrm{ES}_\alpha(Y)}.$$

By the subadditivity of ES, we have

$$\mathrm{ES}_\alpha(Y) = \mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y+Y-\frac{N(v)}{v}Y\right) \leqslant \mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y\right)+\mathrm{ES}_\alpha\left(Y-\frac{N(v)}{v}Y\right).$$

Similarly, $\mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y\right) \leqslant \mathrm{ES}_\alpha(Y)+\mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y-Y\right)$. It follows that

$$\mathrm{ES}_\alpha(Y)-\mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y\right) \leqslant \mathrm{ES}_\alpha\left(Y-\frac{N(v)}{v}Y\right) \leqslant \mathrm{ES}_\alpha\left(\left|Y-\frac{N(v)}{v}Y\right|\right),$$

and

$$\mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y\right)-\mathrm{ES}_\alpha(Y) \leqslant \mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y-Y\right) \leqslant \mathrm{ES}_\alpha\left(\left|Y-\frac{N(v)}{v}Y\right|\right).$$

Therefore,

$$\begin{aligned}\left|\mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y\right)-\mathrm{ES}_\alpha(Y)\right| &\leqslant \mathrm{ES}_\alpha\left(\left|Y-\frac{N(v)}{v}Y\right|\right) \\ &\leqslant \frac{1}{1-\alpha}\mathbb{E}\left|\frac{N(v)}{v}-1\right|\cdot\mathbb{E}[Y],\end{aligned} \tag{4.10}$$

which implies

$$\left|\frac{\mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y\right)}{\mathrm{ES}_\alpha(Y)}-1\right|\leqslant Cv^{-q}+o(v^{-q}).$$

Thus we obtain (4.8) as

$$\left|\frac{\sup_{\mathbf{Y}\in\mathcal{X}_F^{N(v)}}\mathrm{ES}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha\left(Y\right)}-1\right|\leqslant Cv^{-q}+o(v^{-q}),$$

and the second inequality in (4.7) is automatically implied since VaR_α is dominated by ES_α. □

In Example 4.1 of the next section, we will see that $q = 1/2$ for Poisson(v)-distributed $N(v)$. In this case, assuming $p \geqslant 4/3$ (typically true), the convergence rate in the left-hand side of (4.7) is led by $O(v^{-1/4})$ and the one in (4.8) is led by $O(v^{-1/2})$. Admittedly, the convergence rate $O(v^{-1/4})$ is not very fast in general, and its applicability for approximation depends on the models and the magnitude of v. However, for risk management purpose, one should be on the conservative side; as such, the faster rate $O(v^{-q})$ in the right-hand side of (4.7) and in (4.8) is more important in practice. In Example 4.4 below, we will see that the term $O(v^{-q})$ for the upper bounds in Theorem 4.2 is sharp.

4.4 Some examples

Example 4.1 (Poisson number of claims)**.** As the primary example, suppose that $N(v)$ follows a Poisson distribution with parameter v. We check the conditions and parameters in Theorems 4.1 and 4.2. Clearly, $N(v)/v \to 1$ in L^1 as $v \to \infty$ by the L^1-Law of Large Numbers. Indeed, note that $\mathbb{E}\,|N(v)-v| = 2e^{-v}\frac{v^{\lfloor v\rfloor+1}}{\lfloor v\rfloor!}$, and further by Stirling's formula and some elementary analysis, one has

$$2e^{-v}\frac{v^{\lfloor v\rfloor+1}}{\lfloor v\rfloor!}v^{-1/2}\to\sqrt{\frac{2}{\pi}},$$

which means

$$\lim_{v\to\infty}v^{1/2}\mathbb{E}\left|\frac{N(v)}{v}-1\right|=\sqrt{\frac{2}{\pi}}.$$

Therefore in Theorem 4.2, $c = \sqrt{\frac{2}{\pi}}$ and $q = 1/2$.

Example 4.2 (Non-random number of claims)**.** Suppose that $N(v)$ equals $\lfloor v\rfloor$. Then $q = \infty$ in the conditions of Theorem 4.2, and the lower bound on VaR convergence rate given in (4.7) is equivalent to Lemma 2.7.

Example 4.3 (Non-random claim sizes)**.** Suppose that Y is not random and $\limsup_{v\to\infty} v^q\mathbb{E}\left|\frac{N(v)}{v}-1\right| \leqslant c$ for some $q > 0$, $c > 0$. In this case, we have a convergence rate that is slightly stronger than the one given in (4.7),

$$1 - 2C^{1/2}v^{-q/2} - o\left(v^{-q/2}\right) \leqslant \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha(Y)}$$

$$\leqslant \frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{ES}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha(Y)} \leqslant 1 + Cv^{-q} + o(v^{-q}), \tag{4.11}$$

where $C = \frac{c}{1-\alpha}$. Compared with Theorem 4.2, the term $o(v^{1/p-1})$ in the lower bound for VaR convergence disappears. This is quite natural since the term $o(v^{1/p-1})$ is due to the randomness of Y as suggested by Lemma 2.7. To see the first inequality in (4.11), let $\varepsilon = \alpha$ and $\delta = \sqrt{C}v^{-q/2}$. For v large enough,

$$\mathbb{P}\left(\left|\frac{N(v)}{v} - 1\right| > \delta\right) < \varepsilon \quad \text{and} \quad \mathbb{P}\left(\frac{N(v)}{v} < 1 - \delta\right) < \alpha,$$

which imply

$$\mathrm{VaR}_\alpha\left(\frac{N(v)}{v}\right) \geqslant 1 - \delta.$$

Therefore,

$$\frac{\sup_{\mathbf{Y}\in\mathcal{Y}_F^{N(v)}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha(Y)} = \frac{\mathrm{VaR}_\alpha\left(N(v)\right)}{v} \geqslant 1 - \delta \geqslant 1 - 2C^{1/2}v^{-q/2} - o\left(v^{-q/2}\right).$$

The rest of (4.11) comes from Theorem 4.2.

Example 4.4 (Sharpness of the rate in the right-hand side of (4.7) and in (4.8))**.** For some $q > 0$, take $N(v) = \lfloor v + v^{1-q} \rfloor$ and let F be a degenerate distribution of a constant, say 1. In this case, $S_{N(v)} = N(v)$ is not random, and obviously

$$\frac{\mathrm{VaR}_\alpha(S_{N(v)})}{v} = \frac{\mathrm{ES}_\alpha(S_{N(v)})}{v} = O(v^{-q}).$$

This shows that the leading term v^{-q} in the right-hand side of (4.7) and in (4.8) is sharp up to a constant scale, even in the case when $Y_1, Y_2, \ldots$ and $N(v)$ are deterministic.

5 Asymptotic results for generalized collective risk models

In this section, we study the more complicated setting (ii) in which N and $Y_1, Y_2, \ldots$ are not necessarily independent, and their joint distribution is also uncertain. We have similar results as in Theorem 4.1 and Theorem 4.2 under stronger regularity conditions.

5.1 VaR-ES asymptotic equivalence

Theorem 5.1. *Suppose that the distribution F on $\mathbb{R}_+$ has finite second moment, $Y \in \mathcal{X}_F$, and $\{N(v),\ v \geqslant 0\} \subset \mathcal{X}_0$ such that $N(v)/v \to 1$ in L^2 as $v \to \infty$. Then for $\alpha \in (0,1)$,*

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v} = \lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha\left(S_{N(v)}\right)}{v} = \mathrm{ES}_\alpha\left(Y\right). \tag{5.1}$$

Proof. From Theorem 3.3, for fixed $v > 0$, we have

$$\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha\left(S_{N(v)}\right) = \mathrm{ES}_\alpha\left(N(v)Y^*\right), \tag{5.2}$$

where $Y^* \in \mathcal{X}_F$ is comonotonic with $N(v)$. Hölder's inequality implies

$$\mathbb{E}\left|\frac{N(v)Y^*}{v} - Y^*\right| \leqslant \sqrt{\mathbb{E}\left|\frac{N(v)}{v} - 1\right|^2 \cdot \mathbb{E}\left[(Y^*)^2\right]} \to 0, \qquad \text{as } v \to \infty.$$

Hence, $\frac{N(v)Y^*}{v} \xrightarrow{L^1} Y^*$. As a consequence, continuity of ES with respect to the L^1-norm implies

$$\lim_{v\to\infty} \mathrm{ES}_\alpha\left(N(v)Y^*/v\right) = \mathrm{ES}_\alpha\left(Y^*\right) = \mathrm{ES}_\alpha(Y).$$

Therefore,

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha(S_{N(v)})}{v} = \lim_{v\to\infty} \frac{\mathrm{ES}_\alpha\left(N(v)Y^*\right)}{v} = \mathrm{ES}_\alpha\left(Y\right).$$

Thus we obtain the second equality in (5.1).

For the first equality in (5.1), $\frac{N(v)}{v} \xrightarrow{L^2} 1$ implies that for any $\varepsilon > 0$ and $\delta > 0$, for v large enough, one has

$$\mathbb{P}\left(\left|\frac{N\left(v\right)}{v} - 1\right| > \delta\right) < \varepsilon.$$

Similarly to the proof of Theorem 4.2, we have

$$\frac{\mathrm{ES}_{\alpha-\varepsilon}\left(Y\right)}{\mathrm{ES}_\alpha\left(Y\right)} \geqslant 1 - \frac{\varepsilon}{1-\alpha},$$

and

$$\frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha(S_{N(v)})}{v\mathrm{ES}_\alpha\left(Y\right)} \geqslant \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha\left(Y\right)} \geqslant \left[1 - o\left(v^{1/p-1}\right)\right] \cdot \left(1 - \delta - v^{-1}\right) \cdot \frac{\mathrm{ES}_{\alpha-\varepsilon}\left(Y\right)}{\mathrm{ES}_\alpha\left(Y\right)}.$$

Thus,

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v} = \mathrm{ES}_\alpha\left(Y\right),$$

and we obtain the first equality in (5.1). □

5.2 Rate of convergence

In this section we provide the convergence rate in generalized collective risk models. Similarly to Theorem 5.1, stronger regularity conditions are required as compared to results in Section 4.

Theorem 5.2. *Suppose that the distribution F on $\mathbb{R}_+$ has finite p-th moment, $p \geqslant 2$, $Y \in \mathcal{X}_F$, $\mathbb{E}[Y] > 0$, and $\limsup_{v\to\infty} v^r \mathbb{E}\left|\frac{N(v)}{v} - 1\right|^2 \leqslant c$ for some $r > 0$ and $c > 0$. Then we have*

$$-2\left(\frac{c}{1-\alpha}\right)^{1/3} v^{-r/3} + o\left(v^{1/p-1}\right) + o\left(v^{-r/3}\right) \leqslant \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha\left(Y\right)} - 1 \tag{5.3}$$

$$\leqslant \left|\frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha\left(Y\right)} - 1\right| \leqslant \frac{\sqrt{\mathbb{E}[Y^2]}}{\mathrm{ES}_\alpha(Y)}\frac{\sqrt{c}}{1-\alpha} v^{-r/2} + o\left(v^{-r/2}\right). \tag{5.4}$$

Proof. Let $\delta = \left(\frac{c}{1-\alpha}v^{-r}\right)^{1/3}$, $\eta = \left(v^r\mathbb{E}\left|\frac{N(v)}{v} - 1\right|^2 - c\right)_+$, and $\varepsilon = \frac{c+\eta}{\delta^2}v^{-r}$. Clearly $\varepsilon = \left(c(1-\alpha)^2 v^{-r}\right)^{1/3} + o(v^{-r/3})$. Similar to the proof of Theorem 4.2, we have

$$\mathbb{P}\left(\left|\frac{N\left(v\right)}{v} - 1\right| > \delta\right) < \varepsilon \quad \text{and} \quad \frac{\mathrm{ES}_{\alpha-\varepsilon}\left(Y\right)}{\mathrm{ES}_\alpha\left(Y\right)} \geqslant 1 - \frac{\varepsilon}{1-\alpha}.$$

Moreover,

$$\begin{aligned}\frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha\left(Y\right)} &\geqslant \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_{\alpha-\varepsilon}\left(S_{\lfloor(1-\delta)v\rfloor}\right)}{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_{\alpha-\varepsilon}\left(S_{\lfloor(1-\delta)v\rfloor}\right)} \cdot \frac{\lfloor(1-\delta)v\rfloor \mathrm{ES}_{\alpha-\varepsilon}\left(Y\right)}{v\mathrm{ES}_\alpha\left(Y\right)} \\ &\geqslant \left[1 - o\left(v^{1/p-1}\right)\right]\cdot\left(1 - \delta - v^{-1}\right)\left(1 - \frac{\varepsilon}{1-\alpha}\right) \\ &\geqslant 1 - 2\left(\frac{c}{1-\alpha}\right)^{1/3} v^{-r/3} - o\left(v^{1/p-1}\right) - o\left(v^{-r/3}\right).\end{aligned}$$

Thus we obtain (5.3). The first inequality in (5.4) comes from the fact that ES_α dominates VaR_α.

By (4.10) and Hölder's inequality, we have

$$\left|\mathrm{ES}_\alpha\left(\frac{N(v)}{v}Y\right) - \mathrm{ES}_\alpha\left(Y\right)\right| \leqslant \mathrm{ES}_\alpha\left(\left|Y - \frac{N(v)}{v}Y\right|\right) \leqslant \frac{1}{1-\alpha}\sqrt{\mathbb{E}\left|\frac{N(v)}{v} - 1\right|^2 \cdot \mathbb{E}\left[Y^2\right]}.$$

As a consequence,

$$\left|\frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{ES}_\alpha\left(S_{N(v)}\right)}{v\mathrm{ES}_\alpha\left(Y\right)} - 1\right| \leqslant \frac{\sqrt{\mathbb{E}[Y^2]}}{\mathrm{ES}_\alpha(Y)}\frac{\sqrt{c}}{1-\alpha} v^{-r/2} + o\left(v^{-r/2}\right).$$

Thus we obtain the second inequality in (5.4). □

Example 5.1 (Poisson number of claims, revisited)**.** Suppose that $N(v)$ follows a Poisson distribution with parameter v. We can check the parameters in Theorem 5.2. Since $\mathbb{E}[|N(v)/v - 1|^2] = \mathrm{Var}(N(v))/v^2 = 1/v$, we have $r = 1$ and $c = 1$. Therefore, the leading term in the left-hand side of (5.3) is $O(v^{-1/3})$, which converges to zero faster than $O(v^{-1/4})$ as in Example 4.1 under setting (i). This is intuitive as $\sup_{\mathbf{Y}\subset\mathcal{X}_F} \mathrm{VaR}_\alpha\left(S_{N(v)}\right) \geqslant \sup_{\mathbf{Y}\subset\mathcal{X}_F^{N(v)}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)$. The right-hand side of (5.4) remains the same order $O(v^{-1/2})$.

5.3 A remark on the dependence between the claim frequency and the claim sizes

In Sections 4 and 5, we studied the asymptotic equivalence of VaR and ES in two settings. A natural question that follows would be whether an asymptotic equivalence holds also for specified dependence structures between $N(v)$ and $Y_1, Y_2, \ldots$ other than independence. That is, whether the following limit

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\subset\hat{\mathcal{X}}_F^{N(v)}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{\sup_{\mathbf{Y}\subset\hat{\mathcal{X}}_F^{N(v)}} \mathrm{ES}_\alpha\left(S_{N(v)}\right)} = 1 \tag{5.5}$$

holds, where $\hat{\mathcal{X}}_F^{N(v)} \subset \mathcal{X}_F$ is the set of random variables with distribution F and a pre-specified dependence structure (copula) with $N(v)$. Note that from Lemma 3.1, the worst-case ES can be calculated as $\mathrm{ES}_\alpha(N(v)Y)$, where $Y \in \hat{\mathcal{X}}_F^{N(v)}$.

In general, the knowledge on the dependence structure of $(N(v), Y_i)$, $i = 1, 2, \ldots,$ would put some restrictions on the dependence structure of $(Y_1, Y_2, \ldots)$; the latter was assumed to be arbitrary in our settings (i) and (ii), as well as in the classic setup of dependence uncertainty. With the "effect of dependence uncertainty" demolished, (5.5) may no longer hold true. This is evidenced by the following (rather extreme) example where $N(v), Y_i$ are comonotonic for $i = 1, 2, \ldots$ (note that this does not necessarily imply that $Y_1, Y_2, \ldots$ are comonotonic since $N(v)$ is discrete). For other pre-specified dependence structures between $(N(v), Y_i)$, $i = 1, 2, \ldots,$ the question of (5.5) requires a case-by-case study.

Assume that the distribution F has finite second moment, $\{N(v),\ v \geqslant 0\} \subset \mathcal{X}_0$ such that $N(v)/v \to 1$ in L^2 as $v \to \infty$, and $Y \in \mathcal{X}_F^{c,v}$. Denote by $\mathcal{X}_F^{c,v} \subset \mathcal{X}_F$ the set of random variables with distribution F and comonotonic with $N(v)$. In this case one still has the ES convergence as in Theorem 4.1,

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F^{c,v}} \mathrm{ES}_\alpha\left(S_{N(v)}\right)}{v} = \mathrm{ES}_\alpha\left(Y\right), \quad \alpha \in (0,1), \tag{5.6}$$

whereas the VaR convergence

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F^{c,v}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v} = \mathrm{ES}_\alpha\left(Y\right), \quad \alpha \in (0,1), \tag{5.7}$$

may fail to hold.

To see (5.6), by Hölder's inequality, we have

$$\mathbb{E}\left|\frac{N(v)Y}{v} - Y\right| \leqslant \sqrt{\mathbb{E}\left|\frac{N(v)}{v} - 1\right|^2 \cdot \mathbb{E}\left[Y^2\right]} \to 0, \qquad \text{as } v \to \infty.$$

Hence, $\frac{N(v)Y}{v} \xrightarrow{L^1} Y$. From Lemma 3.1, we have $\sup_{\mathbf{Y}\subset\mathcal{X}_F^{c,v}} \mathrm{ES}_\alpha\left(S_{N(v)}\right) = \mathrm{ES}_\alpha\left(N(v)Y\right)$, and (5.6) follows from the continuity of ES with respect to the L^1-norm.

To see that (5.7) may not hold true, we simply give a counter-example. Take any $\alpha \in (0,1)$. Let F be a Bernoulli distribution with parameter $(1-\alpha)/2$, and assume that for each $v > 0$, there exists a positive integer f_v such that $\mathbb{P}(N(v) > f_v) = (1-\alpha)/2$. For fixed v and any $Y_1, Y_2, \dots \in \mathcal{X}_F^{c,v}$, we have $\{Y_i = 1\} = \{N(v) > f_v\}$ almost surely for each $i = 1, 2, \dots$, and hence $Y_1, Y_2, \dots$ are almost surely equal. As a consequence, there is indeed no dependence uncertainty: $S_{N(v)} = N(v)Y_1$ almost surely. Since $\mathbb{P}(N(v)Y_1 > 0) \leqslant \mathbb{P}(Y_1 > 0) = (1-\alpha)/2$, we have

$$\sup_{\mathbf{Y}\subset\mathcal{X}_F^{c,v}} \mathrm{VaR}_\alpha(S_{N(v)}) = \mathrm{VaR}_\alpha(N(v)Y_1) = 0.$$

Therefore,

$$\lim_{v\to\infty} \frac{\sup_{\mathbf{Y}\subset\mathcal{X}_F^{c,v}} \mathrm{VaR}_\alpha\left(S_{N(v)}\right)}{v} = 0.$$

Thus (5.7) does not hold noting that $\mathrm{ES}_\alpha(Y) > 0$.

6 Conclusion

In this paper, we study the worst-case values of VaR and ES of the aggregate loss in collective risk models under two settings of dependence uncertainty. Analytical formulas for the worst-case values of ES are obtained. For both settings, an asymptotic equivalence of the VaR and ES for a random sum of risks is established under some general moment and regularity conditions. The conditions in our main results are easily satisfied by common models, including the classic compound Poisson collective risk models. Our main results suggest that under dependence uncertainty, we can use $v\mathrm{ES}_\alpha(Y)$ to approximate the worst-case risk aggregation when the risk measure is VaR_α or ES_α and v is large enough; the approximation error is also obtained in terms of some moment and convergence rate of the claim sizes and the claim frequency.

There are various methods developed to incorporate dependence information into risk aggregation with model uncertainty. Practically, one should choose a formulation depending on information available to the user; collective risk models become relevant should the claim frequency distribution be available. For individual risk models, various types of dependence information were considered, such as moment information and variance information by Bernard et al. (2017a,b), positive dependence or independence information by Bignozzi et al. (2015) and Puccetti et al. (2017), and factor relation by Bernard et al. (2017c). Information similar to the above types can be naturally incorporated into the collective risk model. This leads to promising future research directions; we anticipate great mathematical and practically relevant implications.

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A Appendix

Below we discuss the difference between a collective risk model and a corresponding individual risk model. Let N be the counting random variable which is bounded by some $n \in \mathbb{N}$, i.e. $N \leqslant n$, and $Y_i \sim F$, $i \in \mathbb{N}$ (in fact, only $Y_1, \ldots, Y_n$ are used). A random sum S_N may be written in two ways: a collective risk model

$$S_N = \sum_{i=1}^{N} Y_i \tag{A.1}$$

and an individual risk model

$$S_N = \sum_{i=1}^{n} Y_i \mathrm{I}_{\{N \geqslant i\}} = \sum_{i=1}^{n} Z_i, \tag{A.2}$$

where $Z_i = Y_i \mathrm{I}_{\{N \geqslant i\}}$, $i = 1, \ldots, n$. Note that this setup is different from the collective reformulation in Example 3.1, where one starts with a homogeneous individual risk model with small probability of loss from each individual risk, and arrives at a Poisson collective risk model.

In the recent literature of dependence uncertainty for an individual risk model, $Z_1, \ldots, Z_n$ in (A.2) are assumed to have an arbitrary dependence. In our collective risk model, although S_N may be written as in (A.2), the dependence among $Z_1, \ldots, Z_n$ is not arbitrary anymore, as it is driven by a common random variable N. There are further essential differences, if we look at the two formulations more closely under the two settings of dependence uncertainty studied in this paper.

(i) **N and the sequence $Y_1, Y_2, \ldots$ are independent.** In this case, the distribution of Z_i can be determined by that of Y_i and N. Denote this distribution by F_i. We can consider the worst-case risk measure (take an ES for instance) in our model

$$\sup_{\mathbf{Y}\in\mathcal{Y}_F^N} \mathrm{ES}_\alpha\left(\sum_{i=1}^{N} Y_i\right) \tag{A.3}$$

and in the classic model

$$\sup_{Z_i\in\mathcal{X}_{F_i}, i\leqslant n} \mathrm{ES}_\alpha\left(\sum_{i=1}^{n} Z_i\right). \tag{A.4}$$

Clearly, through (A.1) and (A.2), the collective risk model formulation $\mathbf{Y} \in \mathcal{Y}_F^N$ in (A.3) is a submodel of the individual risk model formulation $Z_i \in \mathcal{X}_{F_i},\ i \leqslant n$ in (A.4), and hence the worst-case value in (A.3) should be smaller than or equal to the one in (A.4). We shall illustrate this difference with a numerical example where one has

$$\sup_{\mathbf{Y}\in\mathcal{Y}_F^N} \mathrm{ES}_\alpha\left(\sum_{i=1}^{N} Y_i\right) < \sup_{Z_i\in\mathcal{X}_{F_i}, i\leqslant n} \mathrm{ES}_\alpha\left(\sum_{i=1}^{n} Z_i\right).$$

See Example A.1 below.

(ii) **The dependence between N and the sequence $Y_1, Y_2, \ldots$ is also unknown.** In this case, the distribution of Z_i, and the conditional distribution of Z_i given N are both unknown. Hence, no existing result in the literature of dependence uncertainty that we are aware of can be applied to this setting.

Example A.1. Let $n = 10$. Suppose that for $i = 1, \ldots, n$, Y_i follows Expo(1), and N follows the binomial distribution with parameters n and $1/3$ (denoted by $\mathrm{Bin}(n, 1/3)$), independent of $\{Y_i,\ i \in \mathbb{N}\}$. For $i = 1, \ldots, n$, denote the distribution of $Y_i \mathrm{I}_{\{N\geqslant i\}}$ by F_i. Take $\alpha = 0.95$. By Theorem 3.3, we can calculate

$$\sup_{\mathbf{Y}\in\mathcal{Y}_F^N} \mathrm{ES}_\alpha\left(\sum_{i=1}^{N} Y_i\right) = \mathrm{ES}_\alpha(NY_1) = 15.813,$$

$$\sup_{Z_i\in\mathcal{X}_{F_i}, i\leqslant n} \mathrm{ES}_\alpha\left(\sum_{i=1}^{n} Z_i\right) = \sum_{i=1}^{n} \mathrm{ES}_\alpha(Z_i) = 19.026,$$

where the first value is the average of 100 repetitions of simulation with a sample of size 100,000, and the second value is calculated analytically.

The above illustration shows that, under the above setting (i), the collective risk model imposes a special type of dependence through the counting random variable N, and has a smaller worst-case ES value of the aggregate risk as compared to the corresponding individual risk model with dependence uncertainty. Thus, using a collective risk model is one of the many ways of introducing partial dependence information into risk aggregation.

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