Sums of Standard Uniform Random Variables

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This version: June 8, 2019

Abstract

In this paper, we analyze the set of all possible aggregate distributions of the sum of standard uniform random variables, a simply stated yet challenging problem in the literature of distributions with given margins. Our main results are obtained for two distinct cases. In the case of dimension two, we obtain four partial characterization results. For dimension greater or equal to three, we obtain a full characterization of the set of aggregate distributions, which is the first complete characterization result of this type in the literature for any choice of continuous marginal distributions.

Keywords: uniform distribution, aggregation, dependence uncertainty, joint mixability

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1 Introduction

Many questions remain open in the determination of probability measures with given margins and other constraints, since the seminal work of Strassen (1965). One of the challenging and recently active questions is, for n given distributions $F_1, \dots, F_n$ on $\mathbb{R}$, to determine all possible distributions of $S_n = X_1 + \cdots + X_n$ where $X_1, \dots, X_n$ are random variables with respective distributions $F_1, \dots, F_n$ in an atomless probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Formally, denote the set of possible distributions of the sum

$$\mathcal{D}_n = \mathcal{D}_n(F_1, \dots, F_n) = \{\text{cdf of } X_1 + \cdots + X_n : X_i \sim F_i,\ i = 1, \dots, n\}.$$

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where $X \sim F$ means that the cdf of a random variable X is F.

The question of characterizing $\mathcal{D}_n$, although simply stated, is a challenging open question. Generally, it is not easy to determine whether a given distribution G is in $\mathcal{D}_n$, although many moment inequalities can be used as necessary conditions. In the recent literature, some papers partially address the question of $\mathcal{D}_n$ and provide sufficient conditions for $G \in \mathcal{D}_n$; see Bernard et al. (2014), Mao and Wang (2015) and Wang and Wang (2016). A particular question is whether a point-mass belongs to $\mathcal{D}_n$, which is referred to the problem of joint mixability (Wang et al. (2013)), and has found many applications in optimization and risk management.

Yet, there are no known results on the characterization of $\mathcal{D}_n$, except for the trivial case where each of $F_1, \dots, F_n$ is a Bernoulli distribution in a low dimension. Perhaps, the attempt to characterize $\mathcal{D}_n$ for generic $F_1, \dots, F_n$ is too ambitious. In this paper, we focus on the very special case where $F_1, \dots, F_n$ are standard uniform distributions $\mathrm{U}[0,1]$. This problem might look naive at the first glance, but with the technical challenges we shall see in this paper, the characterization of $\mathcal{D}_n$ is highly non-trivial even for uniform distributions.

As a well-known fact, for any random variables $X_1, \dots, X_n$, their sum is dominated in *convex order* by $X_1^c + \cdots + X_n^c$ where X_i^c is identically distributed as X_i, $i = 1, \dots, n$, and $X_1^c, \dots, X_n^c$ are *comonotonic*. Hence, the set $\mathcal{D}_n(F_1, \dots, F_n)$ is contained in the set $\mathcal{C}_n$ of distributions that are dominated in convex order by the distribution of the comonotonic sum (for the precise definitions, see Section 2). These two sets are asymptotically equivalent after normalization as shown by Mao and Wang (2015). One naturally wonders whether they coincide for a finite n.

The main results of this paper can be summarized below in two distinct cases. Recall that $F_1, \dots, F_n$ are standard uniform distributions $\mathrm{U}[0,1]$. For dimension $n = 2$, the sets $\mathcal{D}_n$ and $\mathcal{C}_n$ are not equivalent, and a complete determination of $\mathcal{D}_2$ for uniform margins is still unclear. In Section 3, we provide four results on equivalent conditions for various types of distributions to be in $\mathcal{D}_n$, including unimodal, bi-atomic, and tri-atomic distributions, and distributions dominating a proportion of a uniform one. In Section 4, we are able to analytically characterize the set $\mathcal{D}_n$ for dimension $n \geqslant 3$ by showing that $\mathcal{D}_n = \mathcal{C}_n$. This result came as a pleasant surprise to us, since it is well known that the dependence structure gets much more complicated as the dimension grows. As far as we are aware of, this result is the first full characterization of $\mathcal{D}_n$ for any types of continuous marginal distributions.

For applications of the problem of possible distributions of the sum with specified marginal distributions, we refer to Embrechts et al. (2013), Rüschendorf (2013), McNeil et al. (2015) and

Bernard et al. (2018). A particular problem on the sum of standard uniform random variables is the aggregation of p-values from multiple statistical tests, which are uniform by definition under the null hypothesis. These p-values, as obtained from different tests, typically have an unspecified dependence structure, and hence it is important to understand the possible distributions of the aggregated p-value; see Vovk and Wang (2018).

We remark that the determination of whether $\mathcal{D}_n = \mathcal{C}_n$ for general margins is unclear. Note that the determination of $\mathcal{D}_n = \mathcal{C}_n$ for a given tuple of margins requires more than the determination of joint mixability of the margins, and the latter is known to be an open question in general. We conjecture that for general margins on bounded intervals with sufficient smoothness, there is a dimension n above which $\mathcal{D}_n$ equals $\mathcal{C}_n$, but this is out of reach by current techniques, as our proofs heavily rely on the specific form of uniform distributions.

2 Preliminaries and notation

In this paper, for any (cumulative) distribution function F, we denote by $F^{-1}(t) = \inf\{x : F(x) \geqslant t\}$, $t \in (0, 1]$, the quantile function of F. Denote by $\mathcal{X}$ the set of integrable random variables and $\mathcal{F}$ the set of distributions with finite mean. The terms "distributions" and "distribution functions" are treated as identical in this paper. For $F \in \mathcal{F}$, $\mathrm{Supp}(F)$ is the essential support of the distribution measure induced by F, which will be referred to as the support of F. For any distributions $F, G \in \mathcal{F}$, we denote by $F \oplus G$ the distribution with quantile function $F^{-1}(t) + G^{-1}(t)$, $t \in [0, 1]$. For a distribution F, $\mu(F)$ denotes the expectation of F. Throughout $\lceil x \rceil$ and $\lfloor x \rfloor$ represent for the ceiling and the floor of $x \in \mathbb{R}$, respectively. A density function f is *unimodal* if there exists $a \in \mathbb{R}$ such that f is increasing on $(-\infty, a]$ and decreasing on $[a, \infty)$.

The set $\mathcal{D}_n$ is related to the notion of convex order. A distribution $F \in \mathcal{F}$ is *smaller than* $G \in \mathcal{F}$ *in convex order*, denoted by $F \preccurlyeq_{\mathrm{cx}} G$, if

$$\int_{\mathbb{R}} \phi(x)\,\mathrm{d}F(x) \leqslant \int_{\mathbb{R}} \phi(x)\,\mathrm{d}G(x) \ \text{ for all convex } \phi : \mathbb{R} \to \mathbb{R};$$

provided that both expectations exist (finite or infinite). Standard references for convex order are Müller and Stoyan (2002) and Shaked and Shanthikumar (2007). For a given distribution $F \in \mathcal{F}$, denote by $\mathcal{C}(F)$ the set of all distributions dominated by F in convex order, that is,

$$\mathcal{C}(F) = \{G \in \mathcal{F} : G \preccurlyeq_{\mathrm{cx}} F\}.$$

Checking whether a distribution G is in $\mathcal{C}(F)$ can be conveniently done using an equivalent condition (e.g. Theorem 3.A.1 of Shaked and Shanthikumar (2007)) is

$$\inf_{k\in\mathbb{R}}\left(\int_{\mathbb{R}}(x-k)_+\,\mathrm{d}F(x)-\int_{\mathbb{R}}(x-k)_+\,\mathrm{d}G(x)\right)\geqslant 0. \tag{1}$$

As the focus of this paper is the distribution of the sum of uniform random variables, we use the following simplified notation. For $n\in\mathbb{N}$ and $x\in\mathbb{R}_+$, write

$$\mathcal{D}_n^{\mathrm{U}}=\mathcal{D}_n(\mathrm{U}[0,1],\dots,\mathrm{U}[0,1]) \quad \text{and} \quad \mathcal{C}_x^{\mathrm{U}}=\mathcal{C}(\mathrm{U}[0,x]),$$

where $\mathrm{U}[a,b]$ stands for the uniform distribution over an interval $[a,b]\subset\mathbb{R}$.

Below we list some basic properties of the sets $\mathcal{D}_n(\cdot)$ and $\mathcal{C}(\cdot)$; certainly, they hold also for $\mathcal{D}_n^{\mathrm{U}}$ and $\mathcal{C}_x^{\mathrm{U}}$. First, note that if all distributions $F_1,\dots,F_n$ are shifted by some constants or scaled by the same positive constant, then the elements in $\mathcal{D}_n(F_1,\dots,F_n)$ are also simply shifted or scaled. Moreover, $\mathcal{D}_n(F_1,\dots,F_n)$ is symmetric in the distributions $F_1,\dots,F_n$. These facts allow us to conveniently exchange the order of the distributions $F_1,\dots,F_n$ and normalize these distributions by shifts and a common scale. For given distributions $F_1,\dots,F_n\in\mathcal{F}$, the distribution $F_1\oplus\cdots\oplus F_n$ is the maximum in convex order in the set $\mathcal{D}_n(F_1,\dots,F_n)$. This fact is summarized in the following lemma.

Lemma 1. *For $F_1,\dots,F_n\in\mathcal{F}$, $\mathcal{D}_n(F_1,\dots,F_n)\subset\mathcal{C}(F_1\oplus\cdots\oplus F_n)$.*

Lemma 1 can be equivalently stated as the following. If $X_1\sim F_1,\dots,X_n\sim F_n$ and F is the distribution of $X_1+\cdots+X_n$, then $F\preccurlyeq_{\mathrm{cx}} F_1\oplus\cdots\oplus F_n$. For a history of this result, see, for instance, Puccetti and Wang (2015). In particular, if $F\in\mathcal{D}_n(F_1,\dots,F_n)$, then the mean of F is fixed and is equal to the sum of the means of $F_1,\dots,F_n$.

In view of Lemma 1, it would be natural to investigate when the two sets coincide, that is, $\mathcal{D}_n(F_1,\dots,F_n)=\mathcal{C}(F_1\oplus\cdots\oplus F_n)$. Note that for a given $G\in\mathcal{F}$, the determination of $G\in\mathcal{C}(F)$ can be analytically checked with its equivalent condition (1). Hence, if the above two sets coincide, then we have an analytical characterization of $\mathcal{D}_n(F_1,\dots,F_n)$. In the case of uniform distributions, one wonders whether $\mathcal{D}_n^{\mathrm{U}}=\mathcal{C}_n^{\mathrm{U}}$, noting that $\mathcal{D}_n^{\mathrm{U}}\subset\mathcal{C}_n^{\mathrm{U}}$ always holds. Unfortunately, as shown in Mao and Wang (2015) by a counter-example, in the simple case $n=2$, $\mathcal{D}_2^{\mathrm{U}}$ is an essential subset of $\mathcal{C}_2^{\mathrm{U}}$; see Theorem 3 in Section 3 for distributions in $\mathcal{C}_2^{\mathrm{U}}$ but not in $\mathcal{D}_2^{\mathrm{U}}$.

Some basic properties of the set $\mathcal{D}_n(\cdot)$ are given in the following lemma.

Lemma 2. *For any $F_1, \dots, F_n \in \mathcal{F}$, the set $\mathcal{D}_n(F_1, \dots, F_n)$ is non-empty, convex and closed with respect to weak convergence.*

Another simple fact is that a distribution in $\mathcal{D}_n(F_1, \dots, F_n)$ has to have the correct support generated by the marginal distributions. That is, for any $F_1, \dots, F_n \in \mathcal{F}$ and $G \in \mathcal{D}_n(F_1, \dots, F_n)$, we have $\mathrm{Supp}(G) \subset \sum_{i=1}^n \mathrm{Supp}(F_i)$. For more properties on the sets $\mathcal{D}_n(\cdot)$ and $\mathcal{C}(\cdot)$, see Mao and Wang (2015).

A useful concept for our analysis of $\mathcal{D}_n(\cdot)$ is the joint mixability introduced by Wang et al. (2013). An n-tuple of distributions $(F_1, \dots, F_n) \in \mathcal{F}^n$ is said to be *jointly mixable* (JM), if $\mathcal{D}_n(F_1, \dots, F_n)$ contains a point-mass δ_K, $K \in \mathbb{R}$. This point-mass is unique if the distributions $F_1, \dots, F_n$ have bounded supports. If $(F_1, \dots, F_n)$ is JM and $F_1 = \cdots = F_n = F$, then we say that F is *n-completely mixable* (n-CM).

3 Sums of two standard uniform random variables

In this section, we look at the sum of two U$[0,1]$ random variables. Unfortunately, as explained above, a full characterization of $\mathcal{D}_2^{\mathrm{U}}$ appears difficult to obtain. In this section, we provide characterization results for four different types of distributions to be in $\mathcal{D}_2^{\mathrm{U}}$. We first present the main results on $\mathcal{D}_2^{\mathrm{U}}$ in Section 3.1. Their proofs will be given in Section 3.2. Before presenting our main findings, we summarize some existing results on $\mathcal{D}_2^{\mathrm{U}}$. These facts can be derived from existing results on joint mixability in Wang and Wang (2016).

Proposition 1. *We have*

(i) $\mathcal{D}_2^{\mathrm{U}} \subsetneq \mathcal{C}_2^{\mathrm{U}}$.

(ii) Let F be any distribution with a monotone density function on $\mathrm{Supp}(F)$. *Then $F \in \mathcal{D}_2^{\mathrm{U}}$ if and only if* $\mathrm{Supp}(F) \subset [0,2]$ *and F has mean* 1.

(iii) Let F be any distribution with a unimodal and symmetric density function on $\mathrm{Supp}(F)$. *Then $F \in \mathcal{D}_2^{\mathrm{U}}$ if and only if* $\mathrm{Supp}(F) \subset [0,2]$ *and F has mean* 1.

Remark 1. For a uniform distribution on an interval of length a, $\mathrm{U}[1-\frac{a}{2}, 1+\frac{a}{2}] \in \mathcal{D}_2^{\mathrm{U}}$ if and only if $a \in [0,2]$, a special case of (ii) and (iii) of Proposition 1. The case $\mathrm{U}[\frac{1}{2}, \frac{3}{2}] \in \mathcal{D}_2^{\mathrm{U}}$ is shown by Rüschendorf (1982), and the general case $a \in [0,2]$ is shown by Wang and Wang (2016).

3.1 Main results

As a first new result in this paper, we show that the class of distributions with a unimodal density with the correct mean is contained in $\mathcal{D}_2^{\mathrm{U}}$.

Theorem 1. *Let F be a distribution with a unimodal density on $[0,2]$ and mean 1. Then $F \in \mathcal{D}_2^{\mathrm{U}}$.*

Both the two previous results in Proposition 1 (ii) and (iii) are special cases of Theorem 1; thus Theorem 1 generalizes the existing results derived from Wang and Wang (2016).

The second result of the paper concerns the class of distributions which dominate a proportion of a uniform distribution.

Theorem 2. *Let F be a distribution supported in $[a, a+b]$ with mean 1 and density function f. If there exists $h > 0$ such that $f > 3b/(4h)$ on $[1-h, 1+h]$, then $F \in \mathcal{D}_2^{\mathrm{U}}$.*

In Theorem 2, the condition that the density of F dominates $3b/2$ times that of $\mathrm{U}[1-h, 1+h]$ immediately implies the following admissible ranges of a, b and h: $a \geqslant 1/3$, $b \leqslant 2/3$, and $h \leqslant 1/3$. Hence, $\mathrm{Supp}(F) \subset [0,2]$, which is obviously necessary for $F \in \mathcal{D}_2^{\mathrm{U}}(1,1)$. Theorem 2 also immediately implies the following fact: For a distribution F with mean 0 and bounded support, if F has a positive density $f > \varepsilon$ in a neighbourhood of 0 for some $\varepsilon > 0$, then $F \in \mathcal{D}_2(\mathrm{U}[-m,m], \mathrm{U}[-m,m])$ for sufficiently large $m > 0$.

In addition to the two results on continuous distributions, we analyze discrete distributions. We shall obtain two results, one characterizing bi-atomic distributions in $\mathcal{D}_2^{\mathrm{U}}$, and one characterizing equidistant tri-atomic distributions in $\mathcal{D}_2^{\mathrm{U}}$.

Theorem 3. *Let F be a bi-atomic distribution with mean 1 supported on $\{a, a+b\}$ with $b > 0$. Then $F \in \mathcal{D}_2^{\mathrm{U}}$ if and only if $1/b \in \mathbb{N}$.*

For a given $a \in [0,1)$ and $a+b \in (1,2]$, there is a unique distribution on $\{a, a+b\}$ with mean 1. Hence, all the bi-atomic distributions that belong to $\mathcal{D}_2^{\mathrm{U}}$ have the corresponding distribution measures

$$\Big\{\nu : \quad 1 - \nu(\{a\}) = \nu(\{a + 1/k\}) = (1-a)k,\ k \in \mathbb{N},\ a \in [0,1)\Big\}.$$

Note that many bi-atomic distributions supported on $\{a, a+b\}$ are in $\mathcal{C}_2^{\mathrm{U}}$ but not in $\mathcal{D}_2^{\mathrm{U}}$, as long as $1/b \notin \mathbb{N}$. For example, one can choose a bi-atomic distribution F with equal probability on $\{1 - 1/\pi, 1 + 1/\pi\}$, and easily see that $F \in \mathcal{C}_2^{\mathrm{U}}$, whereas from Theorem 3 we find that F is not in $\mathcal{D}_2^{\mathrm{U}}$. Thus, Theorem 3 implies $\mathcal{D}_2^{\mathrm{U}} \subsetneq \mathcal{C}_2^{\mathrm{U}}$, a fact as noted by Mao and Wang (2015).

For a tri-atomic distribution F, write $F = (f_1, f_2, f_3)$ where f_1, f_2, f_3 are the probability masses of F. Note that on given three points, the set of tri-atomic distributions with mean 1 has one degree of freedom. For tractability, we study the case of F having an *equidistant support* in the form of $\{a-b,\ a,\ a+b\}$ for some $b > 0$. We only consider the case $b \leqslant a \leqslant 1$ since the case $a > 1$ is symmetric.

To state our characterization of tri-atomic distribution in $\mathcal{D}_2^{\mathrm{U}}$, we introduce the following notation. For $x > 0$, define a *measure of non-integrity*

$$\lceil x \rfloor = \min\left\{\frac{\lceil x\rceil}{x} - 1, 1 - \frac{\lfloor x \rfloor}{x}\right\} \in [0,1].$$

Obviously $\lceil x \rfloor = 0 \Leftrightarrow x \in \mathbb{N}$.

Theorem 4. *Suppose that $F = (f_1, f_2, f_3)$ is a tri-atomic distribution with mean 1 supported in $\{a-b, a, a+b\}$ and $0 < b \leqslant a \leqslant 1$. Then $F \in \mathcal{D}_2^{\mathrm{U}}$ if and only if it is the following three cases.*

(i) $a = 1$ and $f_2 \geqslant \lceil \frac{1}{2b} \rfloor$.

(ii) $a < 1$ and $\frac{1}{2b} \in \mathbb{N}$.

(iii) $a < 1$, $\frac{1}{2b} - \frac{1}{2} \in \mathbb{N}$ and $f_2 \geqslant a + b - \frac{1}{2}$.

The corresponding distributions in Theorem 4 are summarized below. Write $c = a/b + 1 - 1/(2b)$. $\mathrm{cx}(x, y)$ stands for the convex set generalized by some vectors x, y.

(i) $(f_1, f_2, f_3) \in \mathrm{cx}\{(0,1,0), \frac{1}{2}(1 - \lceil \frac{1}{2b} \rfloor, 2\lceil \frac{1}{2b} \rfloor, 1 - \lceil \frac{1}{2b} \rfloor)\}$.

(ii) $(f_1, f_2, f_3) \in \mathrm{cx}\{(0, c, 1-c), \frac{1}{2}(c, 0, 2-c)\}$.

(iii) $(f_1, f_2, f_3) \in \mathrm{cx}\{(0, c, 1-c), \frac{1}{2}(c(1-b), 2bc, 2 - c(1-b))\}$.

3.2 Proofs of the main results

To prove Theorem 1, we need the following lemma. To state it, we introduce the notion of special simple unimodal functions. A function h is called a *special simple unimodal (SSU) function on* $[a, a+n)$, $a \in \mathbb{R}$, $n \in \mathbb{N}$, if it is unimodal, and $h(x)$ is a constant on $[a+k-1, a+k)$ for each $k = 1, \dots, n$.

Lemma 3. *Let F be a distribution function with density function f and support $[a, a+n)$, $a \in \mathbb{R}_-$, $n \in \mathbb{N}$. Suppose that F has mean 0 and f is a SSU function on $[a, a+n)$. Then for $c \geqslant \max\{a+n, -a\}/2$, we have $F \in \mathcal{D}_2(\mathrm{U}[-c,c], \mathrm{U}[-c,c])$.*

Proof. We show the result by induction. For $n = 1$, then we have F is the uniform distribution on $[a, a+1)$ with mean 0. This means $a = -0.5$, that is, F is the distribution of $\mathrm{U}[-0.5, 0.5]$, and $c \geqslant 0.5$. By Theorem 3.1 of Wang and Wang (2016), we know that $\mathrm{U}[-0.5, 0.5]$, $\mathrm{U}[-c, c]$ and $\mathrm{U}[-c, c]$ are jointly mixable as the mean inequality $-0.5 - 2c + 2c \leqslant 0 \leqslant 0.5 + 2c - 2c$ is satisfied. This means $F \in \mathcal{D}_2(\mathrm{U}[-c,c], \mathrm{U}[-c,c])$.

Next, we assume the result holds for $n \leqslant k$ and show it holds for $n = k + 1$. Without loss of generality, we assume $a + n > -a$. Otherwise consider the distribution of $X^* = -X$ with $X \sim F$. Define a distribution H with density function $h : [a, a + n) \to \mathbb{R}_+$ defined as

$$h(x) = \frac{n + 1 + 2a}{n} =: \alpha, \ \ x \in [a, a + 1), \quad h(x) = \frac{-1 - 2a}{n(n-1)} =: \beta, \quad x \in [a + 1, a + n).$$

It can be easily verified that it has mean 0 and $\alpha \geqslant 1/n \geqslant \beta > 0$ as $2a + n > 0$. That is, h is a decreasing density function on $[a, a + n)$ with mean 0. Then by Theorem 3.2 of Wang and Wang (2016), we have H, $\mathrm{U}[-c, c]$ and $\mathrm{U}[-c, c]$) are jointly mixable. Hence, we have $H \in \mathcal{D}_2(\mathrm{U}[-c, c], \mathrm{U}[-c, c])$.

Denote $a_i = f(a + i)$, $i = 0, \ldots, n - 1$. Since f is unimodal, without loss of generality, assume that $a_1 \leqslant \ldots \leqslant a_k \geqslant \ldots \geqslant a_n$ for some $k \in \{0, \ldots, n - 1\}$. Let

$$\lambda := \min\left\{\frac{a_1}{\alpha}, \frac{a_2}{\beta}, \ldots, \frac{a_n}{\beta}\right\}.$$

By contradiction, it immediately follows from $\sum_{i=1}^{n} a_i = 1$ and $\alpha + (n - 1)\beta = 1$ that $\lambda \leqslant 1$. If $\lambda = 1$, then $a_1 = \alpha$ and $\alpha_i = \beta$, $i = 2, \ldots, n$, that is, $F = H \in \mathcal{D}_2(\mathrm{U}[-c, c], \mathrm{U}[-c, c])$, which shows the statement in the lemma. Next, we consider the case $\lambda < 1$. We first assert that the following sequence

$$b_1 = a_1 - \lambda\alpha, \quad b_i = a_i - \lambda\beta, \quad i = 2, \ldots, n.$$

is unimodal such that either b_1 or b_n is 0. To see it, we only need to show it is unimodal by the definition of λ. We consider the following two cases.

(i) If $k \geqslant 2$, then $a_1 \leqslant a_2$ and hence, $b_1 = a_1 - \lambda\alpha \leqslant a_2 - \lambda\beta = b_2$ as $\alpha \geqslant \beta$. Also, note that $b_2 \leqslant \ldots \leqslant b_k \geqslant \ldots \geqslant b_n$. Hence, the sequence $\{b_i, \, i = 1, \ldots, n\}$ is unimodal.

(ii) If $k = 1$, then $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n$, and hence, $b_2 \geqslant \ldots \geqslant b_n$. No matter $b_1 \geqslant b_2$ or $b_1 \leqslant b_2$, we have that $\{b_i, \, i = 1, \ldots, n\}$ is unimodal.

Define $F_0 = \frac{F - \lambda H}{1 - \lambda}$. It is easy to check that F_0 is a distribution function with a density function f_0 taking value $b_i/(1 - \lambda)$ on the interval $[a, a + i)$, $i = 1, \ldots, n$. By the above observations on the sequence $\{b_i, \, i = 1, \ldots, n\}$, we know that F_0 is a distribution with mean 0 and with support on a subset of $[a, a + n - 1)$ or $[a + 1, a + n)$. Then by induction, we have $F_0 \in \mathcal{D}_2(\mathrm{U}[-c, c], \mathrm{U}[-c, c])$. Since $F = (1 - \lambda)F_0 + \lambda H$, and $\mathcal{D}_2$ is a convex set, we have $F \in \mathcal{D}_2(\mathrm{U}[-c, c], \mathrm{U}[-c, c])$. $\square$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Denote by f the density function of F, which is unimodal. Then there exists $x_0 \in [0,2]$ such that $f(x_0) = \max\{f(x), x \in [0,2]\}$. Let X and Y be two independent random variables such that $X \sim F$ and $Y \sim \mathrm{U}[0,1]$, and define $X_m = \lfloor m(X-1) \rfloor + Y$, $m \in \mathbb{N}$. Then the density function of X_m, denoted by h_m, is given by

$$h_m(x) = \int_{j/m}^{(j+1)/m} f(x)\,\mathrm{d}x =: p_j, \quad x \in [j-m, j-m+1),\ j = 0, \dots, 2m-1.$$

Let $k = \lfloor m x_0 \rfloor$. Then we have

$$p_0 \leqslant p_1 \leqslant \dots \leqslant p_{k-1}, \quad p_{k+1} \geqslant p_{k+2} \geqslant \dots \geqslant p_{2m},$$

and by $f(x)$ is unimodal on $[k/m, (k+1)/m]$,

$$p_k = \int_{k/m}^{(k+1)/m} f(x)\,\mathrm{d}x \geqslant \frac{1}{m} \min\{f(k/m),\ f((k+1)/m)\} \geqslant \min\{p_{k-1},\ p_{k+1}\}.$$

That is, the sequence $\{p_k,\ k = 1, \dots, 2m\}$ is unimodal. It can be verified that

$$\mathbb{E}[X_m] = \mathbb{E}[\lfloor m(X-1) \rfloor - m(X-1)] + \frac{1}{2},$$

which lies in $[-0.5, 0.5]$. Then the random variable $Y_m := X_m - \mu(X_m)$ has mean 0 and takes value in a subset of $[-m-0.5, m+0.5]$. By Lemma 3, we have the distribution of Y_m belongs to $\mathcal{D}_2(\mathrm{U}[-c,c], \mathrm{U}[-c,c])$ with $c = m/2 + 0.25$. Then obviously, we have the distribution of $Y_m/(2m+0.5)+1$ belongs to $\mathcal{D}_2^{\mathrm{U}}$. Note that $Y_m/(2m+0.5)+1$ converges to X in L^∞-norm as $m \to \infty$. Hence, by the closure of $\mathcal{D}_2^{\mathrm{U}}$ with respect to L^∞-norm, we have $F \in \mathcal{D}_2^{\mathrm{U}}$. This completes the proof. $\square$

To prove Theorems 2 - 4, we need the following lemma.

Lemma 4. *Let Z be a random variable with distribution F supported in $\{b-k, b-k+1, \dots, b\}$, $k \in \mathbb{N}$, $b \in \mathbb{R}$, satisfying*

$$\mathbb{P}(Z = b - i) = p_i \geqslant 0, \quad i = 0, \dots, k, \quad \text{and} \quad \sum_{i=0}^{k} p_i = 1.$$

Then we have the following statements hold.

(i) If $F \in \mathcal{D}_2(\mathrm{U}[0,T], \mathrm{U}[-T,0])$, then at least one of b and T is an integer.

(ii) If $k=1$, then $F \in \mathcal{D}_2(\mathrm{U}[0,T], \mathrm{U}[-T,0])$ if and only if $b \in (0,1)$ and $T \in \mathbb{N}$.

(iii) If $k=2$, then $F \in \mathcal{D}_2(\mathrm{U}[0,T], \mathrm{U}[-T,0])$ if and only if one of the following three cases holds:

(a) $b=1$, $p_2 = p_0 = (1-p_1)/2$, $p_1 \geqslant r/T$ with $T = 2m \pm r$ with $m \in \mathbb{N}$ and $r \in (0,1)$.

(b) $b \in (0,1) \cup (1,2)$, T is even, $p_1 \in (0, \min\{b, 2-b\})$, $p_2 = (b-p_1)/2$ and $p_0 = 1-(b+p_1)/2$.

(c) $b \in (0,1) \cup (1,2)$, T is odd, $p_1 \in (b^*/T, b^*)$ with $b^* = \min\{b, 2-b\}$, $p_2 = (b-p_1)/2$ and $p_0 = 1-(b+p_1)/2$.

Proof. First we introduce the notation: for any random variable X and any set $L \subset \mathbb{R}$, define random events

$$A_X(L) := \{X - n \in L \text{ for some } n \in \mathbb{N}\} = \{X \mod 1 \in L\},$$

and a function

$$g_X(x) = \lim_{\delta \downarrow 0} \frac{\mathbb{P}(A_X([x-\delta, x+\delta]))}{2\delta}, \quad x \in \mathbb{R}, \quad \text{if the limit exists.}$$

Note that $Z \equiv b \mod 1$, $\mathbb{E}[Z] = 0$ and hence, $b - k \leqslant 0 \leqslant b$. If $F \in \mathcal{D}_2(\mathrm{U}[0,T], \mathrm{U}[-T,0])$, then there exist two random variables $X \sim \mathrm{U}[0,T]$ and $Y \sim \mathrm{U}[-T,0]$ such that $Z = X+Y$ a.s. Since $Z = X+Y \equiv b \mod 1$ a.s., we have $A_X(L) = A_Y(b-L)$ a.s. for any $L \subset \mathbb{R}$, where $b - L = \{b - x : x \in L\}$. We first show that at least one of b and T is an integer. To this end, assume that T is not an integer. Then there exists $\ell \in \mathbb{N}$ such that $T = \ell + t$ with $t \in (0,1)$. Note that $X \sim \mathrm{U}[0,T]$. We have

$$\begin{aligned}
g_X(x) &= \lim_{\delta \downarrow 0} \frac{\mathbb{P}(X \mod 1 \in [x-\delta, x+\delta])}{2\delta} \\
&= \frac{1}{2T}\left(\#\{n \in \mathbb{N} : n + x \in (0,T)\} + \#\{n \in \mathbb{N} : n + x \in [0,T]\}\right) \\
&= \begin{cases} \frac{\ell+1}{T}, & \text{if } x \mod 1 \in (0,t), \\ \frac{\ell+1/2}{T}, & \text{if } x \mod 1 = 0 \text{ or } t, \\ \frac{\ell}{T}, & \text{if } x \mod 1 \in (t,1). \end{cases}
\end{aligned}$$

By the definition of $A_X(L)$ and $Y \overset{\mathrm{d}}{=} -X$, we have $A_Y([x-\delta, x+\delta]) = A_{-X}([x-\delta, x+\delta]) = A_X([-x-\delta, -x+\delta])$ a.s., and thus, $g_Y(x) = g_X(-x)$ for $x \in \mathbb{R}$. Also, note that $A_X(L) = A_Y(b-L)$

a.s. which implies $g_X(x) = g_Y(b-x)$. Therefore, we have

$$g_X(x) = g_Y(b-x) = g_X(x-b) \text{ for any } x \in \mathbb{R}.$$

Note that there exists $x \in \mathbb{R}$ such that $x \mod 1 \in (0,t)$ and $x - b \mod 1 \in (t,1)$ which contradicts with the formula of g_X. Hence, b must be an integer.

Next we consider the two cases that $k=1$ and $k=2$.

(i) For $k=1$, note that for any $b \in \mathbb{N}$, $\mu(F)$ could not be 0. Hence, we only need to show $F \in \mathcal{D}_2(\mathrm{U}[0,T], \mathrm{U}[-T,0])$ when $b \in (0,1)$ and $T \in \mathbb{N}$. By $\mu(F) = 0$, we have $p_1 = b$ and $p_0 = 1-b$. Let $X \sim \mathrm{U}[0,T]$ and define random variable Y such that

$$[Y|X \in [k, k+b]] = b - 1 - X, \text{ a.s. for } k = 0, \dots, T-1$$

and

$$[Y|X \in (k+b, k+1]] = b - X, \text{ a.s. for } k = 0, \dots, T-1.$$

Then it is easy to see that $Y \sim \mathrm{U}[-T,0]$ and $X+Y$ has the distribution F. Thus, we have $F \in \mathcal{D}_2(\mathrm{U}[0,T], \mathrm{U}[-T,0])$.

(ii) If $k=2$, by the necessity condition that at least one of b and T is an integer, we consider the following three cases. Without loss of generality, assume $p_i = \mathbb{P}(Z = b-i) > 0$, $i = 0,1,2$.

(a) If b is an integer, for the mean-constraint to be satisfied, we have $b=1$ and

$$\mathbb{P}(Z=-1) = \mathbb{P}(Z=1) = p_0 > 0 \text{ and } \mathbb{P}(Z=0) = p_1 = 1 - 2p_0.$$

Let $T = 2m \pm r$ with $r \in [0,1]$ and $m \in \mathbb{N}$. We only need to find the smallest value of p_1 such that $F \in \mathcal{D}_2(\mathrm{U}[0,T], \mathrm{U}[-T,0])$ as $\delta_0 \in \mathcal{D}_2(\mathrm{U}[0,T], \mathrm{U}[-T,0])$ and $\mathcal{D}_2(\mathrm{U}[0,T], \mathrm{U}[-T,0])$ is closed under mixture, where δ_0 is the point-mass at 0. Assume that there exist $X \sim \mathrm{U}[0,T]$ and $Y \sim \mathrm{U}[-T,0]$ such that $Z = X+Y$ a.s. If $T = 2m+r$, then since b is an integer, we have

$$\begin{aligned} A_X((r,1)) &= \left\{X \in \cup_{k=1}^{2m} (r+k-1, k)\right\} \\ &= A_Y(\{(-1,-r)\}) = \left\{Y \in \cup_{k=1}^{2m} (-k, -r-k+1)\right\} \text{ a.s.} \end{aligned}$$

We let

$$[Y|X \in A_1] = -X - 1 \;\text{ and }\; [Y|X \in A_2] = -X + 1 \;\text{ a.s.}$$

where $A_1 = \cup_{k=1}^{m}(r + 2k - 3, 2k - 1)$ and $A_2 = \cup_{k=1}^{m}(r + 2k - 1, 2k)$. Then $[X + Y|X \in A_1 \cup A_2] = [X + Y|A_X((r, 1))]$ takes values on $\{-1, 1\}$. On the other hand, note that

$$A_X((0, r)) = \left\{X \in \cup_{k=0}^{2m}(k, k + r)\right\} = A_Y((-r, 0)) = \left\{Y \in \cup_{k=0}^{2m}(-k - r, -k)\right\}.$$

Then on the set $A_X((0, r))$, $X + Y$ could not only take values on $\{-1, 1\}$. There is at least one $k \in \{0, \ldots, 2m\}$ such that $X + Y = 0$ on $\{X \in (k, k + r)\}$. Hence, $p_1 \geqslant (2m + 1)r/(2m + 1)T = r/T$. We next show that r/T can be attained by p_1 for $F \in \mathcal{D}_2(\mathrm{U}[0, T], \mathrm{U}[-T, 0])$. It suffices to let $[Y|X \in (2m, 2m + r)] = -X$ a.s. and for $k = 0, \ldots, m - 1$

$$[Y|X \in (2k, 2k + r)] = -X - 1, \quad [Y|X \in (2k + 1, 2k + 1 + r)] = -X + 1 \text{ a.s.}$$

In this case, we have $p_1 = r/T$. By symmetry, we can also get the same result if $T = 2m - r$. To see this, note that almost surely

$$\begin{aligned} A_X((0, 1 - r)) &= \left\{X \in \cup_{k=0}^{2m-1}(k, k + 1 - r)\right\} \\ &= A_Y((r - 1, 0)) = \left\{Y \in \cup_{k=0}^{2m-1}(-k - 1 + r, -k)\right\}. \end{aligned}$$

We let

$$[Y|X \in A_1] = -X - 1 \;\text{ and }\; [Y|X \in A_2] = -X + 1 \;\text{ a.s.}$$

where $A_1 = \cup_{k=0}^{m-1}(2k, 2k + 1 - r)$ and $A_2 = \cup_{k=0}^{m-1}(2k + 1, 2k + 2 - r)$. Then $[X + Y|X \in A_1 \cup A_2] = [X + Y|A_X((0, 1 - r))]$ takes values on $\{-1, 1\}$. On the other hand, note that almost surely

$$\begin{aligned} A_X((1 - r, 1)) &= \left\{X \in \cup_{k=1}^{2m-1}(k - r, k)\right\} \\ &= A_Y(\{(-1, r - 1)\}) = \left\{Y \in \cup_{k=1}^{2m-1}(-k, r - k)\right\}. \end{aligned}$$

Then on the set $A_X((1-r,1))$, $X+Y$ could not only take values on $\{-1,1\}$. There is at least one $k \in \{1,\dots,2m-1\}$ such that $X+Y=0$ on $\{X \in (k-r,k)\}$. Hence, $p_1 \geqslant (2m-1)r/(2m-1)T = r/T$. We next show that r/T can be attained by p_1 for $F \in \mathcal{D}_2(\mathrm{U}[0,T],\mathrm{U}[-T,0])$. It suffices to let $[Y|X \in (2m-1-r, 2m-1)] = -X$ a.s. and for $k=1,\dots,m-1$

$$[Y|X \in (2k-r,2k)] = -X-1, \quad [Y|X \in (2k+1-r, 2k+1)] = -X+1 \text{ a.s.}$$

In this case, we have $p_1 = r/T$. Hence, we have when b is an integer, $Z \in \mathcal{D}_2(\mathrm{U}[0,T],\mathrm{U}[-T,0])$ if and only if $b=1$ and $\mathbb{P}(Z=-1)=\mathbb{P}(Z=1) \leqslant 1/2 - r/(2T)$.

(b) If b is not an integer and T is a even integer, without loss of generality, assume $b \in (0,1)$ as the case $b \in (1,2)$ can be discussed similarly by considering the symmetric distribution on $\{-b, 1-b, 2-b\}$ and noting that $\mathcal{D}_2(\mathrm{U}[0,T],\mathrm{U}[-T,0])$ is closed under symmetric transform. To make sure $\mathbb{E}[Z]=0$, we have $p_0 - p_2 = 1-b$. By Part (i), we know the distribution, denoted by F_1, on $\{b-1,b\}$ with mean 0 belongs to $\mathcal{D}_2(\mathrm{U}[0,T],\mathrm{U}[-T,0])$ which is also closed under mixture. Hence, we have $F \in \mathcal{D}_2(\mathrm{U}[0,T],\mathrm{U}[-T,0])$ implies

$$\begin{aligned}\lambda F + (1-\lambda)F_0 &= \lambda p_2 \delta_{b-2} + (\lambda p_1 + (1-\lambda)b)\delta_{b-1} + (\lambda p_0 + (1-\lambda)(1-b))\delta_b \\ &= \lambda p_2 \delta_{b-2} + (\lambda p_1 + (1-\lambda)b)\delta_{b-1} + (\lambda p_2 + 1 - b)\delta_b \\ &\in \mathcal{D}_2(\mathrm{U}[0,T],\mathrm{U}[-T,0]) \quad \text{ for any } \lambda \in [0,1],\end{aligned}$$

where the second equality follows from $p_0 - p_2 = 1-b$. Therefore, we only need to find the smallest and the largest values of p_1 such that $p_0+p_1+p_2=1$, $p_0-p_2 = 1-b$ and $F \in \mathcal{D}_2(\mathrm{U}[0,T],\mathrm{U}[-T,0])$.

Note that $F \in \mathcal{D}_2(\mathrm{U}[0,T],\mathrm{U}[-T,0])$ is equivalent to there exist $X \sim \mathrm{U}[0,T]$ and $Y \sim \mathrm{U}[-T,0]$ such that $Z = X+Y$ a.s. Then by $A_X(L) = A_Y(b-L)$ a.s. for any $L \subset \mathbb{R}$, we have

$$\begin{aligned}A_X((b,1)) &= \left\{X \in \cup_{k=1}^T (b+k-1,k)\right\} \\ &= A_Y(\{(b-1,0)\}) = \left\{Y \in \cup_{k=1}^T (b-k, 1-k)\right\}.\end{aligned}$$

It is easy to verify that $\mathbb{E}[X+Y|A_X((b,1))] = b$. Note that the $X+Y$ takes values on $\{b-2, b-1, b\}$. Then $[X+Y|A_X((b,1))] = b$ a.s. This implies on $A_X((b,1))$, $X+Y$ is not equal to $b-1$ a.s., which in turn implies b is an upper bound of p_1. On the other hand, note that

$$A_X((0,b)) = \{X \in \cup_{k=1}^T (k-1, b+k-1)\}$$
$$= A_Y(\{(0,b)\}) = \{Y \in \cup_{k=1}^T (-k, b-k)\}.$$

Letting $[Y|A_X((0,b))] = b - X - 1$ a.s. and $[Y|A_X((b,1))] = b - X$ a.s. yield that b is the largest value of p_1.

To find the smallest value of p_1, we consider two cases that T is even and odd. If T is even, then $T = 2m$ for some $m \in \mathbb{N}$. On the set $A_X(0,b)$, for $k = 0, 1, \ldots, m-1$, we let

$$[Y|X \in (2k, b+2k)] = b - X - 2 \quad \text{and} \quad [Y|X \in (2k+1, b+2k+1)] = b - X, \text{ a.s.}$$

In this case, p_1 is zero which is the smallest possible value of p_1.

(c) If b is not an integer and T is an odd integer, then similar to Part (b), we only need to find the largest and the smallest value of p_1 for the case $b \in (0,1)$. The largest value of p_1 is b. To find the smallest value, note that $T = 2m+1$ for some $m \in \mathbb{N}$. On the set $A_X(0,b)$, we let $[Y|X \in (2m, b+2m)] = b - X$ a.s. and for $k = 0, 1, \ldots, m-1$,

$$[Y|X \in (2k, b+2k)] = b - X - 2, \quad [Y|X \in (2k+1, b+2k+1)] = b - X, \text{ a.s.}$$

In this case, $p_1 = b/T$ which is the smallest possible value of p_1. This is due to $[X+Y|X \in (2m, b+2m)] > -1 > b-2$ as $\{X \in (2m, b+2m)\} \subset A_Y(0,b)$ and $[Y|A_Y((0,b))] \geqslant -2m-1$ a.s.

Combining the above three cases, we complete the proof for the case of $k = 2$. □

Proof of Theorem 2. Note that $3b/4h \times 2h = 3b/2 \leqslant 1$, hence $b \leqslant 2/3$. Therefore, $[a, a+b] \subset [0,2]$. Denote $\lambda = 3b/2 \geqslant 3h$ and let H denote the distribution function of $\mathrm{U}[1-h, 1+h]$. Define $G = (F - \lambda H)/(1-\lambda)$. By the assumption on F, G is also a distribution with positive density and mean 1. There exists a sequence of distribution functions $\{G_n\}_{n \geqslant 2}$ with finite support on $[a, a+b]$ and mean 1 which converges to G in distribution as $n \to \infty$. Define $F_n = \lambda H + (1-\lambda) G_n$, $n \geqslant 2$. Then F_n converges to F is distribution as $n \to \infty$. Note that $\mathcal{D}_2^{\mathrm{U}}$ is closed with respect to weak convergence by Lemma 2. We only need to show $F_n \in \mathcal{D}_2^{\mathrm{U}}$ for $n \geqslant 2$. Without loss of generality, assume

$$G_n(\{x_i\}) = p_i, \quad i = 1, \dots, n \text{ with } \; p_1 + \cdots + p_n = 1, \; a \leqslant x_1 < \cdots < x_n \leqslant a + b.$$

We show it by induction on n. Note that $a \leqslant 1 - h \leqslant 1 + h \leqslant b$. For $n = 2$, without loss of generality, let $c, d > 0$ be such that $x_1 = 1 - c$ and $x_2 = 1 + d > 0$, $3(c + d)/2 \leqslant \lambda$ and then

$$G_2(\{1 - c\}) = \frac{d}{c + d} \quad \text{and} \quad G_2(\{1 + d\}) = \frac{c}{c + d}. \tag{2}$$

Since $1 \geqslant \lambda \geqslant c + d + h$, we have

$$\frac{1 - h}{c + d} - \frac{1 - \lambda}{c + d} \geqslant 1 \quad \text{and} \quad \frac{1 - h}{c + d} \geqslant 1.$$

Then there exists some integer $k \geqslant 1$ such that

$$\frac{1 - h}{c + d} \geqslant k \geqslant \frac{1 - \lambda}{c + d}.$$

That is, $1 - k(c + d) \geqslant h$ and $k(c + d) \geqslant 1 - \lambda$. Denote

$$\theta := \frac{k(c + d) + \lambda - 1}{k(c + d)} \in [0, \lambda] \quad \text{as} \quad \lambda - \theta = \frac{(1 - \lambda)(1 - k(c + d))}{k(c + d)} \geqslant 0.$$

Note that the length of the support of $\mathrm{U}[0, k(c+d)]$ ($\mathrm{U}[1-k(c+d), 1]$) is a multiple of $(1+d)-(1-c)$. By Lemma 4 (i), we have $G_2 \in \mathcal{D}_2(\mathrm{U}[0, k(c + d)], \mathrm{U}[1 - k(c + d), 1])$. Similarly, we have $H \in \mathcal{D}_2(\mathrm{U}[k(c + d), 1], \mathrm{U}[0, 1 - k(c + d)])$. Also, by Theorem 1, we know $H \in \mathcal{D}_2^{\mathrm{U}}$. It follows that

$$F_2 = (1 - \lambda)G_2 + (\lambda - \theta)H + \theta H \in \mathcal{D}_2^{\mathrm{U}}.$$

Suppose that $F_n = (1-\lambda)G_n + \lambda H \in \mathcal{D}_2^{\mathrm{U}}$ for $n \leqslant k$ and we aim to show $F_{k+1} = (1-\lambda)G_{k+1} + \lambda H \in \mathcal{D}_2^{\mathrm{U}}$. Let $G_{k+1,1}$ be defined by (2) with $c = 1 - x_1 > 0$ and $d = x_{k+1} - 1 > 0$. That is,

$$G_{k+1,1}(\{x_1\}) = \frac{x_{k+1} - 1}{x_{k+1} - x_1} \quad \text{and} \quad G_{k+1,1}(\{x_{k+1}\}) = \frac{1 - x_1}{x_{k+1} - x_1}.$$

Let $\alpha := \min\{p_1(c + d)/d, p_n(c + d)/c\}$. Then $G_{2:k+1} := (G_{k+1} - \alpha G_{k+1,1})/(1 - \alpha)$ is a distribution function with support on $\{x_1, \dots, x_k\}$ or $\{x_2, \dots, x_{k+1}\}$ and mean 1. Then we have

$$\begin{aligned} F_{k+1} &= (1-\lambda)(\alpha G_{k+1,1} + (1-\alpha)G_{2:k+1}) + \lambda H \\ &= \alpha((1-\lambda)G_{k+1,1} + \lambda H) + (1-\alpha)((1-\lambda)G_{2:k+1} + \lambda H)). \end{aligned}$$

By induction, we have $(1-\lambda)G_{k+1,1} + \lambda H \in \mathcal{D}_2^{\mathrm{U}}$ and $(1-\lambda)G_{2:k+1} + \lambda H \in \mathcal{D}_2^{\mathrm{U}}$. Then by the convexity of $\mathcal{D}_2^{\mathrm{U}}$ from Lemma 2, we have $F_{k+1} \in \mathcal{D}_2^{\mathrm{U}}$. Thus, we complete the proof. $\square$

Proof of Theorems 3 and 4. We only give the proof of Theorem 4 as Theorem 3 can be proved similarly based on Lemma 4 (ii). Note that the cumulative distribution function of a random variable Z belongs to $\mathcal{D}_2^{\mathrm{U}}$ if and only if the cumulative distribution function of $T(Z-1)$ belongs to $\mathcal{D}_2(\mathrm{U}[0,T], \mathrm{U}[-T,0])$ for $T \geqslant 0$. Let Z be a random variable with distribution F and let $T := 1/b$ and $c := (a+b-1)/b$, that is, $b = 1/T$ and $a = c/T + 1 - 1/T$. Then $Z_T := T(Z-1)$ is a random variable satisfying

$$\mathbb{P}(Z_T = c - i) = f_{i+1}, \quad i = 0, 1, 2.$$

By Lemma 4 (iii), we know the cumulative distribution function of Z_T belongs to $\mathcal{D}_2^{\mathrm{U}}$ if and only if one of the three cases of Lemma 4 (iii) holds by replacing b by c and T by $1/b$ respectively. That is,

(a) $a + b - 1/b = 1$, $f_2 \geqslant rb$ with $1/b = 2m \pm r$ with $m \in \mathbb{N}$ and $r \in (-1, 1)$.

(b) $(a+b-1)/b < 1$, $1/b$ is even, $f_2 \in (0, (a+b-1)/b)$, $f_3 = (c - f_2)/2$ and $f_1 = 1 - ((a+b-1)/b + f_2)/2$.

(c) $(a+b-1)/b < 1$, $1/b$ is odd, $f_2 \in (a+b-1, (a+b-1)/b)$, $f_3 = (c-f_2)/2$ and $f_1 = 1-(c+f_2)/2$.

Note that under the constraint $0 < b \leqslant a \leqslant 1$, we have $c \leqslant 1$; $c = 1$ is equivalent to $a = 1$; $rb = \lceil 1/(2b) \rfloor$; $1/b$ is even if and only if $1/(2b) \in \mathbb{N}$; $1/b$ is odd if and only if $1/(2b) - 1/2 \in \mathbb{N}$. Also, by $\mathbb{E}[Z] = 1$, we have $f_3 = f_1 + (1-a)/b$ and thus, $f_2 = 1 - 2f_1 - (1-a)/b$ which implies that $f_2 \in (0, (a+b-1)/b)$ always holds. Hence, the statement in Theorem 4 holds. $\square$

4 Sums of three or more standard uniform random variables

In this section, we aim to show that for $n \geqslant 3$, the two sets $\mathcal{D}_n^{\mathrm{U}}$ and $\mathcal{C}_n^{\mathrm{U}}$ are identical, in sharp contrast to the case of $n = 2$ analyzed in Section 3. We start with the bi-atomic distribution. Let F be the distribution function of a random variable X such that $\mathbb{P}(X = a) = p$ and $\mathbb{P}(X = b) = 1 - p$

with $\mathbb{E}[X] = 1/2$, $a < b$ and $0 < p < 1$ and $T_n(F)$ be the distribution function of nX. That is,

$$F = p\delta_a + (1-p)\delta_b \quad \text{and} \quad T_n(F) = p\delta_{na} + (1-p)\delta_{nb},$$

where δ_x denotes the point-mass at $x \in \mathbb{R}$.

Lemma 5. *Let F be a bi-atomic distribution on $\{a, b\}$ with $a < b$. Then the following statements are equivalent.*

(i) $F \preccurlyeq_{\mathrm{cx}} \mathrm{U}[0,1]$.

(ii) $b - a \leqslant 1/2$ *and* $\mu(F) = 1/2$.

(iii) $T_n(F) \in \mathcal{D}_n^{\mathrm{U}}(1, \ldots, 1)$ *for* $n \geqslant 3$.

Proof. It is easy to verify (iii) $\Rightarrow$ (i). It suffices to show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Let X be a random variable having distribution F and $\mathbb{P}(X = a) = p = 1 - \mathbb{P}(X = b)$. Note that under the constraint of (i), we must have $a, b \in (0,1)$ and $\mu(F) = 1/2$ which implies $0 < a < 1/2 < b < 1$; under the constraint of (ii), by $\mu(F) = 1/2$, we have $a < 1/2 < b$. Then by $b - a \leqslant 1/2$, we also have $a > 0$ and $b < 1$. Thus, in the following proof, we always assume $0 < a < 1/2 < b < 1$.

Without loss of generality, we also assume $0 < a \leqslant 1 - b \leqslant 1/2$ by symmetry. Otherwise, consider another random variable $X^* = 1 - X$ with distribution F^* and it suffices to note that $\mu(F^*) = 1/2$, $b^* - a^* = b - a.$, and $F \preccurlyeq_{\mathrm{cx}} \mathrm{U}[0,1]$ is equivalent to $F^* \preccurlyeq_{\mathrm{cx}} \mathrm{U}[0,1]$.

(i) $\Rightarrow$ (ii): By $F \preccurlyeq_{\mathrm{cx}} \mathrm{U}[0,1]$, we have $\mathbb{E}[X] = 1/2$ and $\mathbb{E}[(X-t)_+] \leqslant (1-t)^2/2$, for $t \in [0,1]$, that is,

$$(b-t)(1-p) \leqslant \frac{(1-t)^2}{2}, \quad a \leqslant t \leqslant b. \tag{3}$$

To show (ii), we only need to show $b - a \leqslant 1/2$, that is, the largest possible value of $b - a$ is $1/2$. To do this, we fix the value of p and denote $\ell_1(t) := (b-t)(1-p)$, $t \in [a,b]$ and $\ell_2(t) := (1-t)^2/2$, $t \in [0,1]$. Note that $\ell_1(t)$ is a linear function with fixed and constant derivative equaling to $p - 1$ and lies below the quadratic curve $\ell_2(t)$ for $t \in [a,b]$. Also note that as a decreases or b increases, ℓ_1 moves upwards. Hence, when $b - a$ attains its largest possible value, we have the line ℓ_1 and the quadratic curve ℓ_2 are tangent and the tangent point satisfies the equation $\ell_2'(t) = t - 1 = p - 1$, that is, the tangent point is $t = p$. Then by $\ell_1(p) = \ell_2(p)$, we get $b = 1 + p/2$ which is the largest possible value of b. Then by $\mathbb{E}[X] = 1/2$, we can get $a = p/2$, which is the smallest possible value of a. Therefore, we have the largest possible value of $b - a$ is $1/2$.

(ii) $\Rightarrow$ (iii): Let $G_{x,y}$ be the distribution of $\mathrm{U}[x,y]$. It suffices to show that $G_{0,1} = (1-p)H_1 + pH_2$ such that both H_1 and H_2 are n-CM, $\mu(H_1) = a$ and $\mu(H_2) = b$. We first define two distributions H_1^* and H_2^* for the following three cases with $1-b \geqslant a$ in mind.

(a) If $a \geqslant 1/n$, define $H_1^* = H_2^* = G_{0,1}$, that is, the distribution function of $\mathrm{U}[0,1]$.

(b) If $1-b \geqslant 1/n \geqslant a$, define

$$H_1^* = G_{0,na} \quad \text{and} \quad H_2^* = \frac{na-p}{1-p}G_{0,na} + \frac{1-na}{1-p}G_{na,1}.$$

By $\mathbb{E}[X] = 1/2$, we have

$$p = \frac{b-1/2}{b-a} = 1 - \frac{1/2-a}{b-a} \leqslant 1 - 2(\frac{1}{2} - a) = 2a < na,$$

where the inequality follows from $b-a \leqslant 1/2$. Hence, H_2^* is a distribution with positive density function. It is obvious that $\mu(H_1^*) = na/2 \geqslant a$ and thus $\mu(H_2^*) \leqslant b$ as $p\,\mu(H_1^*)+(1-p)\mu(H_2^*) = 1/2$ and $pa + (1-p)b = 1/2$.

(c) If $1-b < 1/n$, then we have $n = 3$ as $b - a \leqslant 1/2$ and $a \leqslant 1-b$. Define

$$H_1^* = \frac{3b-2}{p}G_{0,3b-2} + \frac{p+2-3b}{p}G_{3b-2,3a} \;\text{ and }\; H_2^* = \frac{3a-p}{1-p}G_{3b-2,3a} + \frac{1-3a}{1-p}G_{3a,1}.$$

Note that $p = \frac{b-1/2}{b-a} \geqslant 2(b-1/2) > 3b-2$ and $p < 3a$. Hence, we have both H_1^* and H_2^* are distribution functions with positive densities. It is easy to calculate that

$$\mu(H_1^*) = \frac{3a+3b}{2} - 1 + \frac{3a}{p} - \frac{9ab}{2p}.$$

Note that

$$\begin{aligned}
2p(\mu(H_1^*) - a) &= p(3b + a - 2) + 3a(2 - 3b) \\
&\geqslant (3b-2)(3b+a-2) - 2(b-a)3a(3b-2) \\
&= (3b-2)(3b+a-2+2(b-a)3a) \\
&\overset{\mathrm{sgn}}{=} 3b - 2 + a + 2(b-a)3a \geqslant 0,
\end{aligned}$$

where the first inequality follows from $p > 3b-2$ and $2(b-a) \leqslant 1$, $A \overset{\mathrm{sgn}}{=} B$ represents that A and B have the same sign, and the $\overset{\mathrm{sgn}}{=}$ is due to $1-b < 1/n$ and $n = 3$ by the observations at the beginning of (c). Hence, we have $\mu(H_1^*) \geqslant a$ and similarly $\mu(H_2^*) \leqslant b$.

In each of the above three cases, we have $G_{0,1} = pH_1^* + (1-p)H_2^*$, H_1^* has a decreasing density on $[0, na]$ with $\mu(H_1^*) \geqslant a$, and H_2^* has an increasing density on $[1-n(1-b), 1]$ with $\mu(H_2^*) \leqslant b$.

On the other hand, let $H_1^0 = G_{0,p}$ and $H_2^0 = G_{p,1}$. Then it is obvious that $G_{0,1} = pH_1^0 + (1-p)H_2^0$, $\mu(H_1^0) \leqslant a$ and $\mu(H_2^0) \geqslant b$. Hence, we can find some $\alpha \in [0,1]$ such that $\alpha\mu(H_1^*) + (1-\alpha)\mu(H_1^0) = a$. Then the distribution $H_1 := \alpha H_1^* + (1-\alpha)H_1^0$ is supported in $[0, na]$ with decreasing density and $\mu(H_1) = a$. By Theorem 3.2 of Wang and Wang (2016), G is n-CM, that is, $\delta_{na} \in \mathcal{D}_n(G, \dots, G)$. Similarly, we have $H_2 = \alpha H_2^* + (1-\alpha)H_2^0$ is supported in $[1-n(1-b), 1]$ with an increasing density and $\mu(H_2) = b$, which implies that H_2 is also n-CM. That is, $\delta_{nb} \in \mathcal{D}_n(H_2, \dots, H_2)$. By Lemma 2, we have

$$T_n(F) = p\delta_{na} + (1-p)\delta_{nb} \in \mathcal{D}_n(pH_1 + (1-p)H_2, \dots, pH_1 + (1-p)H_2) = \mathcal{D}_n(\mathrm{U}[0,1], \dots, \mathrm{U}[0,1]).$$

Thus, we complete the proof. $\square$

Now we are ready to show our main result in dimension $n \geqslant 3$. It turns out that for standard uniform distributions, the two sets $\mathcal{D}_n(F_1, \dots, F_n)$ and $\mathcal{C}(F_1 \oplus \dots \oplus F_n)$ in Lemma 1 coincide. The following theorem is, to the best of our knowledge, the first analytical characterization of $\mathcal{D}_n$ for continuous marginal distributions.

Theorem 5. *For $n \geqslant 3$, we have $\mathcal{D}_n^{\mathrm{U}} = \mathcal{C}_n^{\mathrm{U}}$.*

Proof. As $\mathcal{D}_n^{\mathrm{U}} \subseteq \mathcal{C}_n^{\mathrm{U}}$, it suffices to show $\mathcal{C}_n^{\mathrm{U}} \subseteq \mathcal{D}_n^{\mathrm{U}}$, that is, for any distribution function F, $F \preccurlyeq_{\mathrm{cx}} \mathrm{U}[0,1]$ implies $\mathcal{D}_n^{\mathrm{U}}(1/n)$. Here and throughout the proof, we use the notation $\mathcal{D}_n^{\mathrm{U}}(x) = \mathcal{D}_n(\mathrm{U}[0,x], \dots, \mathrm{U}[0,x])$ for $x \in \mathbb{R}$. For a distribution F, denote

$$W_F(t) = \mathbb{E}[(X-t)_+] - \frac{(1-t)^2}{2}, \quad t \in [0,1],$$

where X is a random variable having the distribution function F. We first consider the special case that F is a distribution function of a discrete random variable $(a_1, p_1; \dots; a_m, p_m)$ with $0 \leqslant a_1 < \dots < a_m \leqslant 1$ and $G_{x,y}$ is the distribution function of $\mathrm{U}[x,y]$, $x < y$. By Lemma 5, we know the result holds for $m = 2$. Next, we show it holds for general $m \geqslant 2$ by induction.

For general $m \geqslant 2$, by $F \preccurlyeq_{\mathrm{cx}} G_{0,1}$, we have for $W_F(t) \leqslant 0$ for $t \in [0,1]$, that is, for $k = 2, \dots, m$, we have

$$(a_k - t)p_k + \dots + (a_m - t)p_m \leqslant \frac{(1-t)^2}{2}, \quad a_{k-1} \leqslant t \leqslant a_k. \tag{4}$$

Next, we consider two cases.

(a) If there exists $t \in [a_{k-1}, a_k)$ such that the equality of (4) holds, then $p_1 + \cdots + p_{k-1} = t$. To see this, denote $\ell_1(t) := (a_k - t)p_k + \cdots + (a_m - t)p_m$, and $\ell_2(t) = (1-t)^2/2$, $t \in [a_{k-1}, a_k]$. Note that ℓ_1 is a linear function which lies below the decreasing quadratic curveℓ_2. Thus, ℓ_1 and the quadratic curve ℓ_2 are tangent at the point t and the tangent point t satisfies the equation $t - 1 = -p_k - \cdots - p_m$, that is, $p_1 + \cdots + p_{k-1} = t$. Let X_1 and X_2 be two random variables satisfying $\mathbb{P}(X_1 = a_i) = p_i/t$, $i = 1, \ldots, k-1$, and $\mathbb{P}(X_2 = a_i) = p_i/(1-t)$, $i = k, \ldots, m$. Denote by F_1 and F_2 the distributions of X_1 and X_2, respectively. It is easy to verify that $F = tF_1 + (1-t)F_2$, $F_1 \preccurlyeq_{\rm cx} G_{0,t}$ and $F_2 \preccurlyeq_{\rm cx} G_{t,1}$. Then by induction, we have $F_1 \in \mathcal{D}_n^{\rm U}(t/n)$ and $F_2 \in \mathcal{D}_n(\mathrm{U}[t/n, 1/n], \ldots, \mathrm{U}[t/n, 1/n])$. That is, there exist $X_{11}, \ldots, X_{1n} \sim \mathrm{U}[0, t/n]$ and $X_{21}, \ldots, X_{2n} \sim \mathrm{U}[t/n, 1/n]$ such that

$$X_{11} + \cdots + X_{1n} \sim F_1 \quad \text{and} \quad X_{21} + \cdots + X_{2n} \sim F_2.$$

Without loss of generality, assume that $X_{11}, \ldots, X_{1n}$ are independent of $X_{21}, \ldots, X_{2n}$. Let A be a random event independent of $X_{11}, \ldots, X_{1n}, X_{21}, \ldots, X_{2n}$ such that $\mathbb{P}(A) = p_1 + \cdots + p_{k-1} = t$. Define

$$Y_i = X_{1i} 1_A + X_{2i} 1_{A^c}, \qquad i = 1, \ldots, n.$$

It is obvious that $Y_i \sim \mathrm{U}[0,1]$, $i = 1, \ldots, n$, and $Y_1 + \cdots + Y_n \sim F$. This means $F \in \mathcal{D}_n^{\rm U}(1/n, \ldots, 1/n)$.

(b) If the inequality of (4) is strict for every $t \in (0,1)$, define two functions G and H as

$$G = \lambda \delta_{a_1} + (1-\lambda)\delta_{a_m} \quad \text{and} \quad H = \frac{F - \theta G}{1 - \theta},$$

where δ_a denote the degenerated distribution at point a,

$$\lambda = \frac{a_m - 1/2}{a_m - a_1} \quad \text{and} \quad \theta = \min\left\{\frac{p_1}{\lambda}, \frac{p_m}{1-\lambda}\right\} < 1.$$

It is easy to verify that G and H are two distribution functions satisfying $F = \theta G + (1-\theta)H$. We also assert that $H \preccurlyeq_{\rm cx} F$. To see it, let $X \sim F$, $Y \sim G$, and $Z \sim H$. Then for any convex function ϕ, we have

$$\phi(X) \leqslant \frac{a_m - X}{a_m - a_1}\phi(a_1) + \frac{X - a_1}{a_m - a_1}\phi(a_m), \quad a.s.,$$

It then follows that

$$\mathbb{E}[\phi(X)] \leqslant \frac{a_m - \mathbb{E}[X]}{a_m - a_1}\phi(a_1) + \frac{\mathbb{E}[X] - a_1}{a_m - a_1}\phi(a_m) = \lambda\phi(a_1) + (1-\lambda)\phi(a_m) = \mathbb{E}[\phi(Y)],$$

Also, note that $F = \theta G + (1-\theta)H$ which implies $\mathbb{E}[\phi(X)] = \theta\mathbb{E}[\phi(Y)] + (1-\theta)\mathbb{E}[\phi(Z)]$. Combined with $\mathbb{E}[\phi(X)] \leqslant \mathbb{E}[\phi(Y)]$, we have $\mathbb{E}[\phi(Z)] \leqslant \mathbb{E}[\phi(X)]$, that is, $H \preccurlyeq_{\mathrm{cx}} F$. This implies $H \preccurlyeq_{\mathrm{cx}} \mathrm{U}[0,1]$ and note that the support of distribution H has at most $m-1$ points. Using inductive hypothesis, we have $H \in \mathcal{D}_n^{\mathrm{U}}(1/n)$.

On the other hand, if $a_m - a_1 \leqslant \frac{1}{2}$, we know $G \preccurlyeq_{\mathrm{cx}} \mathrm{U}[0,1]$, $G \in \mathcal{D}_n^{\mathrm{U}}(1/n, \dots, 1/n)$. Then $F = \theta G + (1-\theta)H \in \mathcal{D}_n^{\mathrm{U}}(1/n)$. Otherwise, if $a_m - a_1 > \frac{1}{2}$, then $G \not\preccurlyeq_{\mathrm{cx}} \mathrm{U}[0,1]$, that is, $W_G(t_0) > 0$ for some $t_0 \in (0,1)$. Define a function

$$\alpha(t) = -\frac{W_G(t)}{W_F(t)}, \quad t \in (0,1)$$

which is continuous function satisfying $\alpha(0+) = \alpha(1-) = -1$ as $W_F(t) = W_G(t) = -t^2/2$ for $t \in [0, a_1]$ and $W_F(t) = W_G(t) = -(1-t)^2/2$ for $t \in [a_m, 1]$. Hence, we have $\alpha(t)$ takes its maximum value at some $t \in (0,1)$ and the maximum value is positive. Without loss of generality, assume $\alpha_0 = \alpha(t_1) > 0$ is its maximum value. Then we have

$$W_G(t) + \alpha_0 W_F(t) \leqslant 0 \ \text{ for all } t \in [0,1] \ \text{ and } \ W_G(t_1) + \alpha_0 W_F(t_1) = 0. \tag{5}$$

Define a distribution $F_0 := (G + \alpha_0 F)/(1+\alpha_0)$. Then by (5), we have $F_0 \preccurlyeq_{\mathrm{cx}} \mathrm{U}[0,1]$ and $W_{F_0}(t_1) = 0$. By Case (a), we have $F_0 \in \mathcal{D}_n^{\mathrm{U}}(1/n)$.

Note that $G = (1+\alpha_0)F_0 - \alpha_0 F$ and $F = \theta G + (1-\theta)H$, which implies $F = (1+\alpha_0)\theta F_0 - \alpha_0\theta F + (1-\theta)H$, that is,

$$F = \frac{(\theta + \alpha_0\theta)F_0 + (1-\theta)H}{1 + \alpha_0\theta}.$$

Then by Lemma 2, we have $F \in \mathcal{D}_n^{\mathrm{U}}(1/n)$.

If F is a general distribution function such that $F \preccurlyeq_{\mathrm{cx}} \mathrm{U}[0,1]$, then $\mathrm{Supp}(F) \subset [0,1]$ and it has no mass on 0 and 1. For any $n \in \mathbb{N}$, define F_n as the distribution function of X_n

$$X_n = \sum_{k=1}^{n} \mathbb{E}\left[X \,\middle|\, \frac{k-1}{n} \leqslant X < \frac{k}{n}\right] 1_{\{\frac{k-1}{n} \leqslant X < \frac{k}{n}\}}$$

where X is a random variable having distribution function F. Then F_n converges to F in weak convergence as $n \to \infty$, and $F_n \preccurlyeq_{\mathrm{cx}} \mathrm{U}[0,1]$. By the above proof for discrete disitributions with finite support, we have $F_n \in \mathcal{D}_n^{\mathrm{U}}(1/n)$ for each $n \in \mathbb{N}$. Then by Lemma 2, we have $F \in \mathcal{D}_n^{\mathrm{U}}(1/n)$. Thus, we complete the proof. $\square$

For any random variable $X \sim F$ with mean 0, we have $F_a \preccurlyeq_{\mathrm{cx}} F$ for any $a \in [0,1]$, where F_a is the distribution of aX. Hence, we immediately get the following corollary. Note that this corollary, although looks simple, does not seem to allow for an elementary proof without using Theorem 5.

Corollary 1. *For $n \geqslant 3$, if $F \in \mathcal{D}_n(\mathrm{U}[-1,1], \dots, \mathrm{U}[-1,1])$, then so is F_a for all $a \in [0,1]$.*

5 An application

In risk management, often one needs to optimize a statistical functional, mapping $\mathcal{F}$ to $\mathbb{R}$ (such as a risk measure), over the set of $\mathcal{D}_n(F_1, \dots, F_n)$, and this type of problem is called *risk aggregation with dependence uncertainty* (see e.g. Embrechts et al. (2013) and Bernard et al. (2014)). These problems are typically quite difficult to solve in general, as the set $\mathcal{D}_n(F_1, \dots, F_n)$ is a complicated object. For uniform marginal distributions, using results in this paper (in particular, Theorem 5), we are able to translate many optimization problems on $\mathcal{D}_n^{\mathrm{U}}$ to $\mathcal{C}_n^{\mathrm{U}}$ for $n \geqslant 3$, which is a convenient object to work with.

We study an application of the problem of minimizing or maximizing for a given interval A, the value of $\mathbb{P}(S \in A)$ where S is the sum of n standard uniform random variables. A special case of the problem concerns bounds on $\mathbb{P}(S \leqslant x)$ for $x \in \mathbb{R}$, i.e., bounds on $F(x)$ for $F \in \mathcal{D}_n^{\mathrm{U}}$, is studied by Rüschendorf (1982). Using Theorem 5, we are able to solve the problem of $\mathbb{P}(S \in A)$ completely.

Proposition 2. *For $n \geqslant 3$, and $0 \leqslant a \leqslant a+b \leqslant n$, we have*

$$\min_{F_S \in \mathcal{D}_n^{\mathrm{U}}} \mathbb{P}(S \in (a, a+b)) = \left(\frac{2b}{n} - 1\right)_+, \tag{6}$$

and

$$\max_{F_S \in \mathcal{D}_n^{\mathrm{U}}} \mathbb{P}(S \in [a, a+b]) = \min\left\{\frac{2(a+b)}{n}, \frac{2(n-a)}{n}, 1\right\}, \tag{7}$$

where F_S stands for the cdf of S.

Proof. As Theorem 5 gives $\mathcal{D}_n^{\mathrm{U}} = \mathcal{C}_n^{\mathrm{U}}$, it suffices to look at the optimization problems for $\mathcal{C}_n^{\mathrm{U}}$. For $0 \leqslant u \leqslant v \leqslant n$, let $\mathcal{A}_{u,v}$ be the sigma field generated by $\{U \leqslant u\}$ and $\{U \leqslant v\}$. $S = \mathbb{E}[U|\mathcal{A}_{u,v}]$ is tri-atomically distributed with distribution measure

$$\frac{u}{n}\delta_{u/2} + \frac{v-u}{n}\delta_{(u+v)/2} + \frac{n-v}{n}\delta_{(n+v)/2}. \tag{8}$$

Note that $F_S \preccurlyeq_{\mathrm{cx}} F_U$ because S is a conditional expectation of U, and thus $F_S \in \mathcal{C}_n^{\mathrm{U}} = \mathcal{D}_n^{\mathrm{U}}$.

We first verify the following inequality (indeed, it is Theorem 1 of Rüschendorf (1982)). For any S which is the sum of n standard uniform random variables and $x \in \mathbb{R}$, we have

$$\mathbb{P}(S \leqslant x) \leqslant 2x/n \quad \text{and} \quad \mathbb{P}(S \geqslant x) \leqslant 2(n-x)/n. \tag{9}$$

Equation (9) can be shown using the equivalent condition of convex order (see Theorem 3.A.5 of Shaked and Shanthikumar (2007)). Using $F_S \preccurlyeq_{\mathrm{cx}} F_U$,

$$F_S^{-1}(\alpha)\alpha \geqslant \int_0^\alpha F_S^{-1}(t)\,\mathrm{d}t \geqslant \int_0^\alpha F_U^{-1}(t)\,\mathrm{d}t = \frac{n\alpha^2}{2}, \quad \alpha \in (0,1).$$

Therefore, $F_S^{-1}(\alpha) \geqslant \frac{n\alpha}{2}$, $\alpha \in (0,1)$, and equivalently $F_S(x) \leqslant 2x/n$, $x \in \mathbb{R}$. The other inequality in (9) is symmetric by noting that $n - S$ is still the sum of n standard uniform random variables and $\mathbb{P}(S \geqslant x) = \mathbb{P}(n - S \leqslant n - x)$.

We now analyze the problem of the minimum in (6).

(i) Suppose $b \leqslant n/2$. Since $(n+u)/2 - u/2 = n/2 > b$, we can find $u \in [0, n]$ such that $u/2 \leqslant a$ and $(n+u)/2 \geqslant b + a$. By letting $S = \mathbb{E}[U|\mathcal{A}_{u,u}]$ and using (8), we have $\mathbb{P}(S = u/2) = u/n$ and $\mathbb{P}(S = (n+u)/2) = (n-u)/n$. In this case, $\mathbb{P}(S \in (a, a+b)) = 0$; thus (6) holds.

(ii) Suppose $b > n/2$, which implies $a < n/2$ and $a + b > n/2$. Let S be given by $\mathbb{E}[U|\mathcal{A}_{u,v}]$ where $u = 2a$ and $v = 2(a+b) - n$. Note that $\mathbb{P}(S \in (a, a+b)) = \frac{v-u}{n} = \frac{2b}{n} - 1$. This shows the "$\leqslant$" direction of (6). On the other hand, by (9), for any S which is the sum of n standard uniform random variables, $\mathbb{P}(S \leqslant a) \leqslant 2a/n$ and $\mathbb{P}(S \geqslant a+b) \leqslant 2(n-a-b)/n$. Thus,

$$\mathbb{P}(S \in (a, a+b)) \geqslant 1 - \frac{2a}{n} - \frac{2(n-a-b)}{n} = \frac{2b}{n} - 1.$$

This shows the "$\geqslant$" direction of (6).

Next, we analyze the problem of the maximum in (7).

(i) If $a + b > n/2$ and $a < n/2$, then $n/2 \in [a, a+b]$. Taking $S = \mathbb{E}[U] = n/2$ gives $\mathbb{P}(S \in [a, a+b]) = 1$; thus (7) holds.

(ii) Suppose $a+b \leqslant n/2$. By (9), for any S which is the sum of n standard uniform random variables, $\mathbb{P}(S \in [a, a+b]) \leqslant \mathbb{P}(S \leqslant a+b) \leqslant 2(a+b)/n$. To see that such a bound is attainable, take $S = \mathbb{E}[U|\mathcal{A}_{u,u}]$ where $u = 2(a+b)$. Then, by (8), we have

$$\mathbb{P}(S \in [a, a+b]) \geqslant \mathbb{P}(S = a+b) = \mathbb{P}\left(S = \frac{u}{2}\right) = \frac{u}{n} = \frac{2(a+b)}{n}.$$

Therefore, (7) holds.

(iii) Suppose $a \geqslant n/2$. Similar to the above case, by (9), for any S which is the sum of n standard uniform random variables, $\mathbb{P}(S \in [a, a+b]) \leqslant \mathbb{P}(S \geqslant a) \leqslant 2(n-a)/n$. To see that such a bound is attainable, take $S = \mathbb{E}[U|\mathcal{A}_{u,u}]$ where $u = 2a - n$. Then, by (8), we have

$$\pi(S \in [a, a+b]) \geqslant \mathbb{P}(S = a) = \mathbb{P}\left(S = \frac{n+u}{2}\right) = \frac{n-u}{n} = \frac{2(n-a)}{n}.$$

Therefore, (7) holds. $\square$

Remark 2. Based on the proof of Proposition 2, we can identify some minimizing distributions for (6) and some maximizing distributions for (7). For $0 \leqslant u \leqslant v \leqslant n$, write the distribution

$$F_{u,v} = \frac{u}{n}\delta_{u/2} + \frac{v-u}{n}\delta_{(u+v)/2} + \frac{n-v}{n}\delta_{(n+v)/2}. \tag{10}$$

There are a few cases. For the minimum in (6):

1. If $b \leqslant n/2$, then $F_{u,u}$ attains (6) for $u \in [2a+2b-2n, 2a]$.
2. If $b > n/2$, then $F_{u,v}$ attains (6) where $u = 2a$ and $v = 2(a+b) - n$.

For the maximum in (7):

1. If $a+b > n/2$ and $a < n/2$, then $\delta_{n/2}$ attains (7).
2. If $a+b \leqslant n/2$, then $F_{u,u}$ attains (7) where $u = 2(a+b)$.
3. If $a \geqslant n/2$, then $F_{u,u}$ attains (7) where $u = 2a - n$.

Acknowledgements

The authors thank the Editor, an Associate Editor, and an anonymous referee for careful reading of our paper and very helpful comments which improve the quality of the paper. T. Mao was supported by the NNSF of China (grant numbers: 71671176, 71871208, 11501575). B. Wang acknowledges financial support from the NNSF of China (grant number: 11501017). R. Wang acknowledges financial support from the Natural Sciences and Engineering Research Council of Canada (RGPIN-2018-03823, RGPAS-2018-522590), and the Center of Actuarial Excellence Research Grant from the Society of Actuaries.

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