Weak comonotonicity

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Abstract. The classical notion of comonotonicity has played a pivotal role when solving diverse problems in economics, finance, and insurance. In various practical problems, however, this notion of extreme positive dependence structure is overly restrictive and sometimes unrealistic. In the present paper, we put forward a notion of weak comonotonicity, which contains the classical notion of comonotonicity as a special case, and gives rise to necessary and sufficient conditions for a number of optimization problems, such as those arising in portfolio diversification, risk aggregation, and premium calculation. In particular, we show that a combination of weak comonotonicity and weak antimonotonicity with respect to some choices of measures is sufficient for the maximization of Value-at-Risk aggregation, and weak comonotonicity is necessary and sufficient for the Expected Shortfall aggregation. Finally, with the help of weak comonotonicity acting as an intermediate notion of dependence between the extreme cases of no dependence and strong comonotonicity, we give a natural solution to a risk-sharing problem.

Key words and phrases: finance; comonotonicity; risk aggregation; conditional beta.

1 Introduction

Two functions are said to be comonotonic if the ups and downs of one function follows those of the other function. Hence, though geometric in nature, comonotonicity is also a kind of dependence notion between functions. It is not surprising, therefore, that comonotonicity has given rise to sufficient conditions when solving a variety of problems in economics, banking, and insurance, and in particular those that deal with portfolio diversification, risk aggregation, and premium calculation principles. Our search for necessary and sufficient conditions has revealed that a certain augmentation of the classical (and inherently point-wise) notion of comonotonicity with appropriately constructed measures achieves more advanced goals than those associated with sufficient conditions. As a by-product, the augmented notion of comonotonicity, which we call weak comonotonicity, provides a natural bridge between a host of concepts in the aforementioned areas of application, and also in statistics, including measures of association. In what follows, we methodically develop the notion of weak comonotonicity from first principles, establish its various properties, and demonstrate manifold uses.

Rigorously speaking, two functions g and h are comonotonic whenever the property

$$\big(g(x) - g(x')\big)\big(h(x) - h(x')\big) \geq 0 \tag{1.1}$$

holds for all $x, x' \in \mathbb{R}$. This notion of comonotonicity (Schmeidler, 1986) has played a pivotal role in sorting out numerous applications and developing new theories (e.g., Yaari, 1987; Denneberg, 1994). Since then, these advances have been in the mainstream of quantitative finance and economics literature (e.g., Dhaene et al., 2002a,b; Föllmer and Schied, 2016). In this paper, we shall focus on dependence concepts between uni-dimensional functions (and random variables); for multivariate extensions and further references on comonotonicity, we refer to Puccetti and Scarsini (2010), Carlier et al. (2012), Ekland et al. (2012), and Rüschendorf (2013). Note that if non-negativity in property (1.1) is replaced by non-positivity, the functions g and h are said to be antimonotonic.

Comonotonicity of (Borel) functions g and h is a sufficient condition for non-negativity of the covariance $\mathrm{Cov}[g(X), h(X)]$, where X is a random variable such that $g(X)$ and $h(X)$ have finite second moments. This is immediately seen from the equations

$$\begin{aligned} 2\,\mathrm{Cov}[g(X), h(X)] &= \mathbb{E}\big[(g(X) - g(X'))(h(X) - h(X'))\big] \\ &= \iint_{\mathbb{R}^2} \big(g(x) - g(x')\big)\big(h(x) - h(x')\big) F_X(\mathrm{d}x) F_X(\mathrm{d}x'), \end{aligned} \tag{1.2}$$

where X' is an independent copy of X, and F_X denotes the cumulative distribution function (cdf) of X. The problem of determining the sign of covariances such as the one above has been of much interest in economics, insurance, banking, reliability engineering, and statistics. Several offshoots have arisen from this type of research, including quadrant dependence (Lehmann, 1966), measures of association (Esary et al., 1967), monotonic (Kimeldorf and Sampson, 1978) and supremum (Gebelein, 1941) correlation coefficients. The following example illustrates the need for such results.

Example 1.1. Let X be the severity of a risk, which could, for example, be a profit-and-loss variable. Let $g(X)$ be the cost associated with the risk X, and let F_X^h be the so-called (knowledge-based) weighted cdf of the original random variable X (e.g., Rao, 1997, and references therein). That is, F_X^h is defined by the differential equation

$$F_X^h(\mathrm{d}x) = \frac{h(x)}{\mathbb{E}[h(X)]} F_X(\mathrm{d}x), \tag{1.3}$$

where h is a non-negative function such that $\mathbb{E}[h(X)] \in (0, \infty)$. The role of the function h is to modify the probabilities of the original random variable X. For example, in insurance, it is usually designed to lower the left-hand tail of the pdf of X and to lift its right-hand tail, thus making large insurance risks/losses more noticeable and the premiums loaded; we refer to, e.g., Deprez and Gerber (1985) for the Esscher principle of insurance premium calculation, where $h(x) = e^{tx}$ for some constant $t > 0$. Under the weighted cdf F_X^h, the average cost is

$$\mathbb{E}^h[g(X)] = \int g(x) F_X^h(\mathrm{d}x) = \frac{\mathbb{E}[g(X)h(X)]}{\mathbb{E}[h(X)]},$$

which is not smaller than the average cost $\mathbb{E}[g(X)]$ under the true cdf F_X if and only if the covariance $\mathrm{Cov}[g(X), h(X)]$ is non-negative. Several natural questions arise in this context: Under what conditions on the cost function g and the probability weighting function h is the covariance non-negative? Should the functions really be comonotonic, as our earlier arguments would suggest? It is important to note at this point that practical and theoretical considerations may or may not support the latter assumption, due to the complexity of economic agents' behaviour (e.g., Markowitz, 1952; Pennings and Smidts, 2003; Gillen and Markowitz, 2009).

We have organized the rest of the paper as follows. In Section 2, we define, illustrate, and discuss the notion of weak comonotonicity, first for Borel functions and then for random variables (i.e., generic measurable functions). In Section 3 we elucidate the role of weak comonotonicity in risk aggregation. In particular, we show that a combination of weak comonotonicity and weak antimonotonicity with respect to some sets of measures is sufficient for the maximization of Value-at-Risk (VaR) aggregation, and weak comonotonicity is necessary and sufficient for the Expected Shortfall (ES) aggregation. Both the VaR and the ES aggregation problems have been popular in the recent risk management literature (e.g., Rüschendorf, 2013; McNeil et al., 2015; Embrechts et al., 2015). In Section 4, we explore some properties of weak comonotonicity and its relation to other dependence structures and measures of association. As most of this paper deals with weak comonotonicity with respect to product measures, in Section 5 we illuminate the special role of these measures within the general context of joint measures. With the help of the developed theory, in Section 6 we present a detailed solution to a risk-sharing problem by invoking a weak comonotonicity constraint, whose naturalness becomes clear upon noticing that the assumption of arbitrary dependence among admissible allocations might sometimes be too weak, and the assumption of strong comonotonicity might be too strong, and so an intermediate dependence assumption based on weak comonotonicity arises most naturally. Section 7 concludes the paper with a brief overview of main contributions.

2 Weak comonotonicity

Our efforts to tackle problems like those in the previous section, and in particular those related to risk aggregation (Section 3), have naturally led us to a notion of weak comonotonicity (to be defined in a moment) which naturally bridges the arguments around quantities in (1.1) and (1.2) in the following way: First, note the equation

$$\big(g(x)-g(x')\big)\big(h(x)-h(x')\big)=\iint_{\mathbb{R}^2}\big(g(z)-g(z')\big)\big(h(z)-h(z')\big)\delta_x(\mathrm{d}z)\delta_{x'}(\mathrm{d}z'), \tag{2.1}$$

where δ_x and $\delta_{x'}$ are point masses at the points x and x', respectively. It now becomes obvious that by choosing various product measures instead of $\delta_x\times\delta_{x'}$, we can seamlessly move from classical comonotonicity (1.1) to covariance non-negativity (1.2). Formalizing this flexibility gives rise to a general definition of weak comonotonicity, which is the topic of Section 2.1.

2.1 Weak comonotonicity of Borel functions

In what follows, we use $(\mathbb{R},\mathcal{B})$ to denote the Borel measurable space, where $\mathcal{B}:=\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and we also work with the measurable space $(\mathbb{R}^2,\mathcal{B}^2)$, where $\mathcal{B}^2:=\mathcal{B}\otimes\mathcal{B}$.

Definition 2.1. Let $\mathcal{R}$ be any subset of product measures $\varrho_1\times\varrho_2$ on $(\mathbb{R}^2,\mathcal{B}^2)$. We say that two functions g and h are *weakly comonotonic with respect to* $\mathcal{R}$ whenever

$$\iint_{\mathbb{R}^2}\big(g(x)-g(x')\big)\big(h(x)-h(x')\big)\varrho_1(\mathrm{d}x)\varrho_2(\mathrm{d}x')\ge 0 \tag{2.2}$$

for every $\varrho_1\times\varrho_2\in\mathcal{R}$. In case $\mathcal{R}$ is a singleton, we also say that g and h are weakly comonotonic with respect to $\rho_1\times\rho_2$ if (2.2) holds.

We also speak of *weak antimonotonicity* if non-negativity in (2.2) is replaced by non-positivity. Property (2.2) gives rise to a whole spectrum of comonotonicity notions, at one end of which is the classical notion of comonotonicity (i.e., property (1.1)), which can be viewed as g and h being weakly comonotonic with respect to $\mathcal{R} = \{\delta_x \times \delta_{x'} : x, x' \in \mathbb{R}\}$. In other words, the classical notion of comonotonicity can be thought of as the point-wise or strong comonotonicity. On the other hand, Definition 2.1 and equation (1.2) imply that the covariance $\mathrm{Cov}[g(X), h(X)]$ is non-negative if and only if the functions g and h are weakly comonotonic with respect to $\{F_X \times F_X\}$, where F_X is the cdf of X. By choosing various product measures, we thus arrive at a large array of comonotonicity notions. The following example is designed to illustrate, and in particular enhance our intuitive understanding of, the notion of weak comonotonicity.

Example 2.1. Let $g(x) = \sin(x)$ and $h(x) = \cos(x)$. In the classical sense, the two functions are neither comonotonic nor antimonotonic on the interval $[0, \pi]$, but they are antimonotonic on $[0, \pi/2]$ and comonotonic on $[\pi/2, \pi]$. As to their weak comonotonicity, consider the integral

$$\Delta(a) := \iint_{\mathbb{R}^2} \big(g(x) - g(x')\big)\big(h(x) - h(x')\big) F(\mathrm{d}x) F(\mathrm{d}x')$$

with respect to the following three uniform distributions $F = F_{[0,a]}$, $F_{[(\pi-a)/2,(\pi+a)/2]}$, and $F_{[\pi-a,\pi]}$ on the noted intervals, where $a \in [0, \pi]$ in every case. We have

$$\Delta(a) = \begin{cases} \dfrac{\sin^2(a)}{a} - \dfrac{2\sin(a)(1-\cos(a))}{a^2} & \text{when} \quad F = F_{[0,a]}, \\ 0 & \text{when} \quad F = F_{[(\pi-a)/2,(\pi+a)/2]}, \\ \dfrac{2\sin(a)(1-\cos(a))}{a^2} - \dfrac{\sin^2(a)}{a} & \text{when} \quad F = F_{[\pi-a,\pi]}. \end{cases}$$

When $F = F_{[\pi-a,\pi]}$, we depict $\Delta(a)$ as a function of $a \in [0, \pi]$ in Figure 2.1. It is non-negative for every $a \in [0, \pi]$, thus implying that the functions $\sin(x)$ and $\cos(x)$, which are neither comonotonic nor antimonotonic on $[0, \pi]$ in the classical sense, are nevertheless *weakly comonotonic* with respect to $\{F_{[\pi-a,\pi]} \times F_{[\pi-a,\pi]} : a \in [0, \pi]\}$. On the other hand, when $F = F_{[0,a]}$, the function $\Delta(a)$ is non-positive for every $a \in [0, \pi]$, and thus $\sin(x)$ and $\cos(x)$ are *weakly antimonotonic* with respect to $\{F_{[0,a]} \times F_{[0,a]} : a \in [0, \pi]\}$. Finally, under the distribution $F_{[(\pi-a)/2,(\pi+a)/2]}$, the two functions are both weakly comonotonic and weakly antimonotonic. This concludes Example 2.1.

It is useful to reflect upon Example 2.1 from a general perspective, for which we employ Bayesian terminology. Namely, we first impose the (improper) uniform prior $\pi(x) \propto 1$ on the entire real line. Then we weight the prior using the indicator function $\mathbb{I}_{[x_0,x_1]}(x)$, where $[x_0, x_1]$ can be any compact

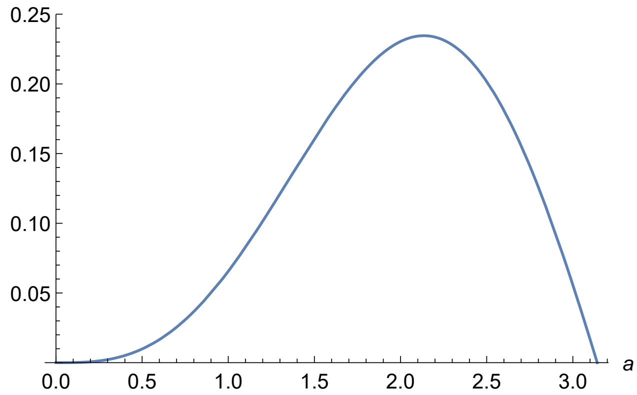

Figure 2.1: Weak comonotonicity of $\sin(x)$ and $\cos(x)$ when F is the uniform on $[\pi - a, \pi]$ distribution, depicted as the function $\Delta(a)$ for all $a \in [0, \pi]$.

interval. This gives rise to the uniform distribution $F_{[x_0,x_1]}$ defined by the differential equation

$$F_{[x_0,x_1]}(\mathrm{d}x) = \frac{\mathbb{I}_{[x_0,x_1]}(x)}{\mathbb{E}^\pi[\mathbb{I}_{[x_0,x_1]}]}\pi(\mathrm{d}x) \tag{2.3}$$

(compare it with equation (1.3)). This uniform distribution, whose density (pdf) takes the form $f_{[x_0,x_1]}(x) = \mathbb{I}_{[x_0,x_1]}(x)/(x_1 - x_0)$, can be thought of as a magnifying glass over the window $[x_0, x_1]$: by sliding it over the domain of definition of functions, we explore weak comonotonicity of the functions, as we have done in Example 2.1.

2.2 Weak comonotonicity of random variables

Note that the moment we had shifted our focus from non-decreasing functions to comonotonic ones, we lost the need for having order relationship in the underlying measurable space. Hence, we can work with abstract measurable space $(\Omega, \mathcal{F})$, in which case $\mathcal{F}$-measurable functions like $X, Y : \Omega \to \mathbb{R}$ are called random variables, and this is the general framework within which we work next. Namely, X and Y are said to be comonotonic whenever

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0$$

for all $\omega, \omega' \in \Omega$. The definition is independent of any choice of measure.

Definition 2.2. Let $\mathcal{P}$ be any subset of probability product measures $\pi_1 \times \pi_2$ on $(\Omega^2, \mathcal{F}^2)$. We say that two random variables X and Y are *weakly comonotonic with respect to* $\mathcal{P}$ whenever

$$\iint_{\Omega^2} (X(\omega) - X(\omega'))(Y(\omega) - Y(\omega'))\pi_1(\mathrm{d}\omega)\pi_2(\mathrm{d}\omega') \geq 0 \tag{2.4}$$

for every $\pi_1 \times \pi_2 \in \mathcal{P}$.

Again, we also speak of *weak antimonotonicity* if non-negativity in (2.4) is replaced by non-positivity. This definition not only generalizes Definition 2.1 but also paves a path toward the notion of conditional correlation, and thus, in turn, toward conditional beta that has prominently featured in problems such as dynamic asset pricing and risk estimation with non-synchronous prices (Engle, 2016, see also references therein). The next example elucidates the connection.

Example 2.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space of financial scenarios $\omega \in \Omega$, and let $X, Y : \Omega \to \mathbb{R}$ be, for example, risk severities of two financial instruments. Quite often, it is of interest to measure association between the two instruments over certain events $A \in \mathcal{F}$ of positive probabilities. In this case, the original probability $\mathbb{P}$ is re-weighted

$$\mathbb{P}(\mathrm{d}\omega|A) = \frac{\mathbb{I}_A(\omega)}{\mathbb{P}(A)}\mathbb{P}(\mathrm{d}\omega),$$

thus reducing property (2.4) via $\pi_1(\mathrm{d}\omega) = \pi_2(\mathrm{d}\omega) = \mathbb{P}(\mathrm{d}\omega|A)$ to

$$\int_A \int_A (X(\omega) - X(\omega'))(Y(\omega) - Y(\omega'))\mathbb{P}(\mathrm{d}\omega)\mathbb{P}(\mathrm{d}\omega') \geq 0. \tag{2.5}$$

Property (2.5) can in turn be rewritten as $\mathrm{Corr}[X, Y|A] \geq 0$, which can equivalently be interpreted as the non-negativity requirement on the conditional beta (Engle, 2016) over the events $A \in \mathcal{F}$ of interest, which could, for example, make up the σ-field of historical events (see, e.g., Box et al. (2015) for a time series context; and Pflug and Römisch (2007), Föllmer and Schied (2016) for risk measurement and management contexts).

Coming now back to Definition 2.2, we check that the following four statements are equivalent:

(i) X and Y are (strongly, or point-wise) comonotonic;

(ii) X and Y are weakly comonotonic with respect to every probability product measures $\pi_1 \times \pi_2$ on $(\Omega^2, \mathcal{F}^2)$;

(iii) X and Y are weakly comonotonic with respect to $\mathcal{P} = \{\delta_\omega \times \delta_{\omega'} : \omega, \omega' \in \Omega\}$;

(iv) there exist non-decreasing functions f_1 and f_2 and a random variable Z such that $X = f_1(Z)$ and $Y = f_2(Z)$; according to Denneberg's Lemma (Denneberg, 1994, Proposition 4.5), we can set $Z := X + Y$.

We are now ready to elucidate the fundamental role of weak comonotonicity in problems associated with risk aggregation.

3 Risk aggregation and weak comonotonicity

Two of the most popular classes of risk measures used in banking and insurance practice are the Value-at-Risk (VaR) and the Expected Shortfall (ES, also known as TVaR, CTE, CVaR, AVaR). We fix an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a random variable X, the VaR at level $p \in (0,1)$ is defined as

$$\mathrm{VaR}_p(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) > p\},$$

and the ES at level $p \in (0,1)$ is defined as

$$\mathrm{ES}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q(X)\mathrm{d}q.$$

A classic problem in the field of risk management is risk aggregation with given marginal distributions (e.g., McNeil et al., 2015, Section 8.4). Let X and Y be two integrable random variables. For $p \in (0,1)$, we say that (X,Y) maximizes the VaR_p aggregation, if

$$\mathrm{VaR}_p(X+Y) = \max\{\mathrm{VaR}_p(X'+Y') : X' \overset{\mathrm{d}}{=} X,\ Y' \overset{\mathrm{d}}{=} Y\},$$

and similarly for the ES aggregation, where "$\overset{\mathrm{d}}{=}$" stands for equality in distribution.

It is well-known (e.g., McNeil et al., 2015, Section 8.4.4) that the maximization of ES aggregation is achieved by (strong) comonotonicity, that is, (X,Y) maximizes the ES_p aggregation if they are strongly comonotonic. A similar statement holds for all convex-order consistent risk measures, or variability measures, such as the variance, the standard deviation, convex and coherent risk measures, and the Gini Shortfall (Furman et al., 2017), and this is because of the well-known fact (e.g., Puccetti and Wang, 2015) that comonotonicity maximizes convex order of the sum. Note that for a specific $p \in (0,1)$, (strong) comonotonicity is a sufficient condition for (X,Y) to maximize the ES_p aggregation, but it is not necessary.

Another well-known phenomenon (e.g., McNeil et al., 2015, Proposition 8.31), which is in sharp contrast to the above situation, is that the maximization of VaR aggregation is not achieved by comonotonicity. This is due to the fact that VaR_p is generally not subadditive. The calculation of the worst-case VaR aggregation is technically very challenging and the corresponding dependence structure is quite complicated. For recent analytical and numerical results, we refer to Wang et al. (2013) and Embrechts et al. (2013, 2014, 2015). Fortunately, the case of $n = 2$ admits an analytical solution, which is originally due to Makarov (1981) and Rüschendorf (1982).

To summarize, strong comonotonicity is sufficient but not necessary for the maximization of ES aggregation, and it is neither sufficient nor necessary for the maximization of VaR aggregation.

This calls for weaker and alternative dependence notions compared to strong comonotonicity. We shall see later in Theorem 3.1 that the notion of weak comonotonicity serves this purpose very well, as it gives a sufficient condition for the maximum VaR_p aggregation, as well as a necessary and sufficient condition for the maximum ES_p aggregation.

To prepare for Theorem 3.1, we need some notation and a lemma. For a random variable X and for any $p \in (0,1)$, we write

$$A_p^X = \{\omega \in \Omega : X(\omega) > \mathrm{VaR}_p(X)\}.$$

Note that $\mathbb{P}(A_p^X) = 1-p$ if X is continuously distributed. In this case, A_p^X is the event of probability p on which X takes its largest possible values. Further, let

$$\mathcal{P}_p^X = \{\delta_\omega \times \delta_{\omega'} : \omega \in A_p^X,\ \omega' \in (A_p^X)^c\},$$

where A^c stands for the complement of a subset A of Ω, and let

$$\mathcal{Q}_p^X = \{\delta_\omega \times \delta_{\omega'} : \omega, \omega' \in A_p^X\}.$$

In what follows, we treat $\mathbb{P}$-a.s. equal random variables as identical, and thus statements like "X and Y are weakly comonotonic with respect to $\mathcal{P}_p^X$" should be interpreted as they hold for a representative pair of the random variables X and Y.

Lemma 3.1. *Let X and Y be two continuously distributed random variables, and let $p \in (0,1)$. The following three statements are equivalent:*

(i) *X and Y are weakly comonotonic with respect to $\mathcal{P}_p^X$;*

(ii) *X and Y are weakly comonotonic with respect to $\mathcal{P}_p^Y$;*

(iii) *$A_p^X = A_p^Y$ a.s. with respect to $\mathbb{P}$.*

Proof. We only show (i)⇔(iii) since (ii)⇔(iii) holds by symmetry. First, we assume that statement (i) holds. For $\omega \in A_p^X$ and $\omega' \in (A_p^X)^c$, we have $X(\omega) - X(\omega') > 0$. By definition of weak comonotonicity, this implies $Y(\omega) - Y(\omega') \geq 0$. Therefore, Y takes its largest values on A_p^X. Since $\mathbb{P}(A_p^X) = p$, we have $A_p^Y = \{Y > \mathrm{VaR}_p(Y)\} = A_p^X$ a.s. Next, we assume that statement (iii) holds. Then, for a.s. $\omega \in A_p^Y$ and $\omega' \in (A_p^Y)^c$, we have $X(\omega) - X(\omega') > 0$ and $Y(\omega) - Y(\omega') > 0$. This gives the weak comonotonicity of X and Y; more precisely, of a representative version of (X, Y). □

We are now ready to state our main result on the relationship between risk aggregation and weak comonotonicity.

Theorem 3.1. *Let X and Y be two continuously distributed and integrable random variables, and let $p \in (0,1)$. We have the following two statements:*

(i) *If X and Y are weakly comonotonic with respect to $\mathcal{P}_p^X$, and X and Y are weakly antimonotonic with respect to $\mathcal{Q}_p^X$, then (X,Y) maximizes the VaR_p aggregation;*

(ii) *X and Y are weakly comonotonic with respect to $\mathcal{P}_p^X$ if and only if (X,Y) maximizes the ES_p aggregation.*

Proof. First, we prove statement (i). By Lemma 3.1, $A_p^X = A_p^Y$ a.s. Also note that X and Y are (strongly) antimonotonic on the set A_p^X. Let $U = F_X(X)$, which is uniformly distributed on $[0,1]$, and we know that X and U are strongly comonotonic. As a consequence, $X = \mathrm{VaR}_U(X)$ a.s., and the sets A_p^X, A_p^Y and $\{U > p\}$ are a.s. equal. Because Y and U are antimonotonic on the set $\{U > p\}$, if U takes value $u \in (p,1)$, then Y takes the value $\mathrm{VaR}_{1+p-u}(Y)$ a.s., and hence $Y = \mathrm{VaR}_{1+p-U}(Y)$ a.s. on $\{U > p\}$. Further, note that if $U \le p$, then $X+Y \le \mathrm{VaR}_p(X)+\mathrm{VaR}_p(Y)$ a.s. and if $U > p$, then $X + Y \ge \mathrm{VaR}_p(X) + \mathrm{VaR}_p(Y)$ a.s. As a consequence, by definition of the p-quantile (VaR_p), $\mathrm{VaR}_p(X + Y)$ is the smallest value ($\mathbb{P}$-a.s.) $X + Y$ takes on the set $\{U > p\}$, which is the smallest value of $\mathrm{VaR}_U(X) + \mathrm{VaR}_{1+p-U}(Y)$ for $U \in (p,1)$. Therefore,

$$\mathrm{VaR}_p(X+Y) = \inf\{\mathrm{VaR}_{p+t}(X) + \mathrm{VaR}_{1-t}(Y) : t \in (0, 1-p)\}.$$

This gives the maximum value of the VaR_p aggregation according to Makarov (1981, equation (2)) or McNeil et al. (2015, Proposition 8.31), thus concluding the proof of statement (i).

To prove statement (ii), we need some preliminaries. Namely, we use the dual representation of ES_p in the form

$$\mathrm{ES}_p(Z) = \max\{\mathbb{E}[Z|B] : B \in \mathcal{F},\ \mathbb{P}(B) = 1-p\} \tag{3.1}$$

for any random variable Z, and $B = A_p^Z$ attains the maximum in (3.1) if Z is continuously distributed (e.g., Embrechts and Wang, 2015, Lemma 3.1). Because of subadditivity of ES_p, we have

$$\mathrm{ES}_p(X) + \mathrm{ES}_p(Y) = \max\{\mathrm{VaR}_p(X'+Y') : X' \overset{\mathrm{d}}{=} X,\ Y' \overset{\mathrm{d}}{=} Y\}.$$

Hence, (X,Y) maximizes the ES_p aggregation if and only if $\mathrm{ES}_p(X+Y) = \mathrm{ES}_p(X)+\mathrm{ES}_p(Y)$. Note that $\mathrm{ES}_p(X+Y) \le \mathrm{ES}_p(X) + \mathrm{ES}_p(Y)$ always holds. Now we are able to establish the "if and only if" statement (ii).

($\Rightarrow$) Suppose that X and Y are weakly comonotonic with respect to $\mathcal{P}_p^X$. This implies $A_p^X = A_p^Y$ a.s. by Lemma 3.1. Therefore, by equation (3.1),

$$\mathrm{ES}_p(X+Y) \geq \mathbb{E}\left[X+Y|A_p^X\right] = \mathbb{E}\left[X|A_p^X\right] + \mathbb{E}\left[Y|A_p^Y\right] = \mathrm{ES}_p(X) + \mathrm{ES}_p(Y).$$

Hence, (X,Y) maximizes the ES_p aggregation.

($\Leftarrow$) Suppose that (X,Y) maximizes the ES_p aggregation. Then, using equation (3.1), we have, for some $B \in \mathcal{F}$,

$$\begin{aligned}\mathbb{E}\left[X+Y|B\right] = \mathrm{ES}_p(X+Y) = \mathrm{ES}_p(X) + \mathrm{ES}_p(Y) &= \mathbb{E}\left[X|A_p^X\right] + \mathbb{E}\left[Y|A_p^Y\right] \\ &\geq \mathbb{E}\left[X|B\right] + \mathbb{E}\left[Y|B\right].\end{aligned}$$

Therefore, $\mathbb{E}[X|A_p^X] = \mathbb{E}[X|B]$. Since X is continuously distributed and takes its largest values on A_p^X, and $\mathbb{P}(A_p^X) = 1-p = \mathbb{P}(B)$, we conclude that $A_p^X = B$ a.s. Similarly, we conclude that $A_p^Y = B$ a.s. Using Lemma 3.1 again, we obtain that X and Y are weakly comonotonic with respect to $\mathcal{P}_p^X$

This finishes the proof of Theorem 3.1. ☐

Note that the weak comonotonicity condition on $\mathcal{P}_p^X$ in Theorem 3.1 is truly weaker than strong comonotonicity, as it does not specify the copula of X and Y. As discussed by Embrechts et al. (2014, Section 3), the typical worst-case scenario of VaR aggregation is a combination of positive dependence and negative dependence in some non-rigorous sense. Theorem 3.1(i) answers precisely what these non-rigorous positive and negative dependence structures mean: weak comonotonicity with respect to $\mathcal{P}_p^X$ and weak antimonotonicity with respect to $\mathcal{Q}_p^X$. Furthermore, Theorem 3.1(ii) gives a necessary and sufficient condition for the dependence structure maximizing the ES_p aggregation.

As a direct consequence of Theorem 3.1, there exists a dependence structure that maximizes the VaR_p and ES_p aggregations simultaneously, as specified in Theorem 3.1(i). Note that the weak comonotonicity of X and Y with respect to $\mathcal{P}_p^X$ can be interpreted as a positive dependence in which the large values of X and Y appear simultaneously; but they are not perfectly aligned as in strong comonotonicity. It is straightforward to see, however, that this dependence structure, although necessary and sufficient for the ES_p aggregation, is not necessary for the VaR_p aggregation. For instance, if Y is positive and $X(\omega)$ is large enough, say $X(\omega) > \mathrm{VaR}_p(X+Y)$, then it does not matter what value $Y(\omega)$ takes because it does not affect the calculation of $\mathrm{VaR}_p(X+Y)$.

Remark 3.1. Theorem 3.1(ii) is formulated for a specific $p \in (0,1)$. If one likes (X,Y) to maximize ES_p aggregation for all $p \in (0,1)$ or, equivalently, maximize the convex order of the sum, then strong comonotonicity is the only dependence structure (e.g., Cheung, 2010, Theorem 3). This, in particular, highlights the lack of practical attractiveness of the classical notion of comonotonicity, as it is unnecessarily too strong, at least from the perspective of ES_p aggregation. Indeed, practical considerations place emphasis on special values of p, usually specified by regulators, and they are, for example, close to 1 in banking and insurance (e.g., Basel IV and Solvency II; see McNeil et al. (2015)). More generally, we can think of examples when we would be concerned with p's in certain subinterval of $(0,1)$, but not in the entire interval $(0,1)$. This serves yet another justification for the introduction and explorations of the notion of weak comonotonicity.

Remark 3.2. The VaR aggregation problem is equivalent to the problem of maximizing or minimizing $\mathbb{P}(X+Y>x)$ for a given $x \in \mathbb{R}$ and given marginal distributions of X and Y. Indeed, this is the problem originally studied by Makarov (1981) and Rüschendorf (1982). It has become well known since then that comonotonicity does not maximize or minimize the probability $\mathbb{P}(X+Y>x)$, and hence it is not the right notion to describe the corresponding dependence structures.

4 Some properties of weak comonotonicity

In this section we explore some properties of weak comonotonicity, and its relation to notions of dependence structures and measures of association.

4.1 Point-masses and comonotonicity

We have already noted that point masses reduce weak comonotonicity to strong comonotonicity, but the class

$$\mathcal{R}_{g,h} = \big\{\rho_1 \times \rho_2 : h \text{ and } g \text{ are weakly comonotonic with respect to } \rho_1 \times \rho_2\big\}$$

depends, naturally, on the functions g and h. In a sense, we can circumvent this dependence by introducing certain classes of point masses. Define

$$\mathcal{R}_{\mathrm{c}} = \{\delta_x \times \delta_{x'} : x, x' \in \mathbb{R}\}$$

and

$$\mathcal{R}_{\mathrm{a}} = \{\delta_x \times \delta_x : x \in \mathbb{R}\}.$$

Note that $\mathcal{R}_{g,h}$ is the largest set of product measures $\rho_1 \times \rho_2$ with respect to which g and h are weakly comonotonic. The set $\mathcal{R}_{g,h}$ is never empty because $\mathcal{R}_\mathrm{a} \subseteq \mathcal{R}_{g,h}$. Finally, we note that for any two functions g and h, the inclusions $\mathcal{R}_\mathrm{a} \subseteq \mathcal{R}_{g,h}$ and $\mathcal{R}_\mathrm{a} \subseteq \mathcal{R}_\mathrm{c}$ always hold.

Theorem 4.1. *We have the following two statements:*

(i) $\mathcal{R}_{g,h} \supseteq \mathcal{R}_\mathrm{c}$ *if and only if g and h are strongly comonotonic.*

(ii) $\mathcal{R}_{g,h} = \mathcal{R}_\mathrm{a}$ *if and only if g and h are strongly antimonotonic and injective on $\mathbb{R}$.*

Proof. Statement (i) is trivial. To prove statement (ii), we first note that if $\mathcal{R}_{g,h} = \mathcal{R}_\mathrm{a}$, then for any two $x, x' \in \mathbb{R}$ which are not identical, we have $\delta_x \times \delta_{x'} \notin \mathcal{R}_{g,h}$. Thus, $(g(x) - g(x'))(h(x) - h(x')) < 0$, and the desired injectivity and antimonotonicity follow. Next, assume injectivity and antimonotonicity. Then, $(g(x)-g(x'))(h(x)-h(x')) < 0$ for all $x, x' \in \mathbb{R}$ that are not identical. For any product measure $\rho_1 \times \rho_2$, if condition (2.2) holds, then $\rho_1 \times \rho_2$ must be supported in the points (x, x') where either $g(x) = g(x')$ or $h(x) = h(x')$, and hence $x = x'$. Since $\rho_1 \times \rho_2$ is a product measure, we know that it has to be of the form $\delta_x \times \delta_x$ for $x \in \mathbb{R}$. This concludes the proof of Theorem 4.1. □

We now turn our attention to random variables X and Y. Similarly to $\mathcal{R}_{g,h}$, let

$$\mathcal{P}_{X,Y} = \big\{\pi_1 \times \pi_2 : X \text{ and } Y \text{ are weakly comonotonic with respect to } \pi_1 \times \pi_2\big\}.$$

In other words, $\mathcal{P}_{X,Y}$ is the largest set of product measures with respect to which X and Y are weakly comonotonic. It is a symmetric set with respect to X and Y, that is, we have $\mathcal{P}_{X,Y} = \mathcal{P}_{Y,X}$. The validity of this symmetry easily follows from the equation

$$\iint_{\Omega^2} (X(\omega) - X(\omega'))(Y(\omega) - Y(\omega'))\pi_1(\mathrm{d}\omega)\pi_2(\mathrm{d}\omega') \\ = \mathbb{E}^{\pi_1}[XY] + \mathbb{E}^{\pi_2}[XY] - \mathbb{E}^{\pi_1}[X]\mathbb{E}^{\pi_2}[Y] - \mathbb{E}^{\pi_2}[X]\mathbb{E}^{\pi_1}[Y]. \tag{4.1}$$

It also follows from the latter equation that if $\pi_1 = \pi_2 =: \pi$, then condition (2.4) means that the correlation of X and Y under the measure π is non-negative. Finally, we note that $\mathcal{P}_{X,Y}$ is invariant under all increasing linear marginal transforms, that is, the equation $\mathcal{P}_{\lambda_1 X + a_1, \lambda_2 Y + a_2} = \mathcal{P}_{X,Y}$ holds for all $\lambda_1, \lambda_2 > 0$ and $a_1, a_2 \in \mathbb{R}$.

Theorem 4.2. *Let $\mathcal{P}_\mathrm{a} = \{\delta_\omega \times \delta_\omega : \omega \in \Omega\}$ and $\mathcal{P}_\mathrm{c} = \{\delta_\omega \times \delta_{\omega'} : \omega, \omega' \in \Omega\}$. We have the following two statements:*

(i) $\mathcal{P}_{X,Y} \supseteq \mathcal{P}_{\mathrm{c}}$ *if and only if* X *and* Y *are strongly comonotonic.*

(ii) $\mathcal{P}_{X,Y} = \mathcal{P}_{\mathrm{a}}$ *if and only if* X *and* Y *are strongly antimonotonic and injective on* Ω.

Note that $\mathcal{P}_{\mathrm{a}} \subseteq \mathcal{P}_{X,Y}$ and $\mathcal{P}_{\mathrm{a}} \subseteq \mathcal{P}_{\mathrm{c}}$. The proof of Theorem 4.2 is analogous to that of Theorem 4.1 and is therefore omitted.

4.2 Set-masses and independence

We now go back to the integral, for a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\iint_{\Omega^2} (X(\omega) - X(\omega'))(Y(\omega) - Y(\omega'))\mathbb{P}(\mathrm{d}\omega)\mathbb{P}(\mathrm{d}\omega')$$

and distort, or rather weight, its probabilities. This gives rise to the integral

$$\iint_{\Omega^2} (X(\omega) - X(\omega'))(Y(\omega) - Y(\omega'))\mathbb{P}_{W_1}(\mathrm{d}\omega)\mathbb{P}_{W_2}(\mathrm{d}\omega'), \tag{4.2}$$

where, for two random variables $W_1 \geq 0$ and $W_2 \geq 0$, the probability measure $\mathbb{P}_{W_1}$ is defined via the equation

$$\mathbb{P}_{W_1}(\mathrm{d}\omega) = \frac{W_1(\omega)}{\mathbb{E}^{\mathbb{P}}[W_1]}\mathbb{P}(\mathrm{d}\omega),$$

with $\mathbb{P}_{W_2}$ defined analogously. We next explore the case when the weights W_1 and W_2 are the indicators $\mathbb{I}_A$ and $\mathbb{I}_B$, respectively, where A and B are elements of the σ-field $\mathcal{F}$.

Let $\sigma(X)$ denote the σ-field generated by X, and let

$$\sigma^+(X) = \{A \in \sigma(X) : \mathbb{P}(A) > 0\}.$$

For any event $A \in \sigma^+(X)$, let $\mathbb{P}_A$ be the conditional probability of $\mathbb{P}$ on A. We call these conditional probabilities set masses, which are natural extensions of the earlier explored point masses.

We shall next connect weak comonotonicity with (in)dependence of random variables X and Y. It is instructive to start with the bivariate Gaussian case, and the following proposition is akin to the classical result which says that the equivalence of uncorrelatedness and independence characterizes Gaussian random variables.

Proposition 4.1. *Let* (X, Y) *be jointly Gaussian with standard margins and correlation* $c \in [-1, 1]$. *Then the following three statements are equivalent:*

(i) $c \geq 0$;

(ii) $\{\mathbb{P}_A \times \mathbb{P}_B : A, B \in \sigma^+(X)\} \subseteq \mathcal{P}_{X,Y}$;

(iii) $\{\mathbb{P}_A \times \mathbb{P}_A : A \in \sigma^+(X)\} \subseteq \mathcal{P}_{X,Y}$.

Proof. We first write $Y = cX + \sqrt{1-c^2}Z$ for some standard Gaussian Z independent of X. For any $A \in \sigma^+(X)$, we have $\mathbb{E}[XY|A] = \mathbb{E}[cX^2|A]$ and $\mathbb{E}[Y|A] = \mathbb{E}[cX|A]$. Therefore, the following holds if and only if $c \geq 0$:

$$\mathbb{E}[XY|A] = c\mathbb{E}[X^2|A] \geq c(\mathbb{E}[X|A])^2 = \mathbb{E}[X|A]\mathbb{E}[Y|A].$$

Furthermore, we check that, for $c \geq 0$,

$$\begin{aligned}
&\mathbb{E}[XY|A] + \mathbb{E}[XY|B] - \mathbb{E}[X|A]\mathbb{E}[Y|B] - \mathbb{E}[Y|A]\mathbb{E}[X|B] \\
&\qquad = c\mathbb{E}[X^2|A] + c\mathbb{E}[X^2|B] - 2c\mathbb{E}[X|A]\mathbb{E}[X|B] \\
&\qquad \geq c\left(\mathbb{E}[X^2|A] + \mathbb{E}[X^2|B] - (\mathbb{E}[X|A])^2 - (\mathbb{E}[X|B])^2\right) \geq 0.
\end{aligned}$$

This establishes the proposition. □

Generally, $\{\mathbb{P}_A \times \mathbb{P}_B : A, B \in \sigma^+(X)\} \subseteq \mathcal{P}_{X,Y}$ and $\{\mathbb{P}_A \times \mathbb{P}_A : A \in \sigma^+(X)\} \subseteq \mathcal{P}_{X,Y}$ are not equivalent conditions, although they are in the Gaussian case, as we have just seen in Proposition 4.1.

Proposition 4.2. *We have the following statements:*

(i) *If X and Y are independent, then $\{\mathbb{P}_A \times \mathbb{P}_B : A, B \in \sigma^+(X)\} \subseteq \mathcal{P}_{X,Y}$ and, by symmetry, $\{\mathbb{P}_A \times \mathbb{P}_B : A, B \in \sigma^+(Y)\} \subseteq \mathcal{P}_{X,Y}$.*

(ii) *If $\{\mathbb{P}_A \times \mathbb{P}_B : A, B \in \sigma^+(X)\} \subseteq \mathcal{P}_{X,Y}$, then, for $A, B \in \sigma^+(X)$, we have the property*

$$\mathbb{E}[XY|A] + \mathbb{E}[XY|B] - \mathbb{E}[X|A]\mathbb{E}[Y|B] - \mathbb{E}[Y|A]\mathbb{E}[X|B] \geq 0,$$

which in the "diagonal" case $A = B$ reduces to non-negativity of the conditional correlation $\mathrm{Corr}[X, Y|A]$ *for every event $A \in \sigma^+(X)$.*

Proof. To prove part (i), we use equation (4.1) and have

$$\begin{aligned}
&\iint_{\Omega^2} (X(\omega) - X(\omega'))(Y(\omega) - Y(\omega'))\mathbb{P}_A(\mathrm{d}\omega)\mathbb{P}_B(\mathrm{d}\omega') \\
&= \mathbb{E}[XY|A] + \mathbb{E}[XY|B] - \mathbb{E}[X|A]\mathbb{E}[Y|B] - \mathbb{E}[Y|A]\mathbb{E}[X|B] \\
&= \mathbb{E}[X|A]\mathbb{E}[Y] + \mathbb{E}[X|B]\mathbb{E}[Y] - \mathbb{E}[X|A]\mathbb{E}[Y] - \mathbb{E}[Y]\mathbb{E}[X|B] = 0.
\end{aligned}$$

Hence $\{\mathbb{P}_A \times \mathbb{P}_B : A, B \in \sigma^+(X)\} \subseteq \mathcal{P}_{X,Y}$. The other half of (i) is by symmetry. The proof of statement (ii) is a straightforward verification. □

4.3 Weak comonotonicity and measures of association

The notion of weak comonotonicity has enabled us to establish a whole spectrum of comonotonicity notions, ranging from the classical (strong) comonotonicity under the pairs of all point masses to weaker comonotonicity notions under the pairs of more elaborate measures. As we shall see next, this flexibility enables us to capture a whole array of measures of association.

(S1) The Pearson correlation $\mathrm{Corr}(X, Y)$ is non-negative if and only if X and Y are weakly comonotonic with respect to $\mathbb{P} \times \mathbb{P}$.

(S2) Two random variables X and Y are positively associated (also called positively function dependent; see Joe (1997) for details) if and only if for all non-decreasing functions h and g, the random variables $h(X)$ and $g(Y)$ are weakly comonotonic with respect to $\mathbb{P} \times \mathbb{P}$.

(S3) Assuming that X and Y have continuous cdf's F_X and F_Y, respectively, the Spearman correlation is non-negative if and only if $F_X(X)$ and $F_Y(Y)$ are weakly comonotonic with respect to the product $\mathbb{P} \times \mathbb{P}$.

(S4) Two random variables X and Y are independent if and only if, for all $A, B \in \mathcal{B}$, the indicators $\mathbb{I}_{\{X \in A\}}$ and $\mathbb{I}_{\{Y \in B\}}$ are weakly comonotonic with respect to $\mathbb{P} \times \mathbb{P}$. The same statement holds if we replace weak comonotonicity by weak antimonotonicity.

All the above statements are straightforward and follow from the equivalence of weak comonotonicity (with respect to $\mathbb{P} \times \mathbb{P}$) and covariance non-negativity. The fourth property, however, warrants a simple comment-like proof.

Proof of (S4). It is obvious that independence implies weak comonotonicity, as well as weak antimonotonicity, of $\mathbb{I}_{\{X \in A\}}$ and $\mathbb{I}_{\{Y \in B\}}$. For the other direction, let (X', Y') be an independent copy of (X, Y). For all $A, B \in \mathcal{B}$, we have

$$\mathbb{E}[(\mathbb{I}_{\{X \in A\}} - \mathbb{I}_{\{X' \in A\}})(\mathbb{I}_{\{Y \in B\}} - \mathbb{I}_{\{Y' \in B\}})] = 2\mathbb{P}(X \in A, Y \in B) - 2\mathbb{P}(X \in A)\mathbb{P}(Y \in B),$$

which is non-negative. Likewise, we have

$$\mathbb{E}[(\mathbb{I}_{\{X \in A\}} - \mathbb{I}_{\{X' \in A\}})(\mathbb{I}_{\{Y \in B^c\}} - \mathbb{I}_{\{Y' \in B^c\}})] = 2\mathbb{P}(X \in A, Y \in B^c) - 2\mathbb{P}(X \in A)\mathbb{P}(Y \in B^c),$$

which is also non-negative. Adding the left-hand sides of the two equations gives zero, which, due to the just established non-negativity statements, implies that the right-hand sides are also zeros, which implies independence. □

It is convenient to have probability-based quantities expressed in terms of distribution functions, and we next do so expressly for the purpose of checking whether or not the random variables $h(X)$ and $g(Y)$ are weakly comonotonic with respect to $\mathbb{P}\times\mathbb{P}$. To this end, we write the equations

$$
\begin{aligned}
\iint_{\Omega^2} \big(g(X(\omega)) - g(X(\omega'))\big)\big(h(Y(\omega)) - h(Y(\omega'))\big)\mathbb{P}(\mathrm{d}\omega)\mathbb{P}(\mathrm{d}\omega') \\
&= \mathbb{E}\big[(g(X) - g(X'))(h(Y) - h(Y'))\big] \\
&= \mathbb{E}\big[(g(X) - g(X'))(h^*(X) - h^*(X'))\big] \\
&= \iint_{\mathbb{R}^2} \big(g(x) - g(x')\big)\big(h^*(x) - h^*(x')\big) F_X(\mathrm{d}x)F_X(\mathrm{d}x'), \qquad (4.3)
\end{aligned}
$$

where

$$h^*(x) := \mathbb{E}\big[h(Y)|X = x\big].$$

Consequently, $h(X)$ and $g(Y)$ are weakly comonotonic with respect to $\mathbb{P}\times\mathbb{P}$ if and only if the functions g and h^* are weakly comonotonic with respect to $F_X \times F_X$, that is,

$$\iint_{\mathbb{R}^2} \big(g(x) - g(x')\big)\big(h^*(x) - h^*(x')\big) F_X(\mathrm{d}x)F_X(\mathrm{d}x') \geq 0. \qquad (4.4)$$

From this we arrive at the following interpretation of positive association in terms of weak comonotonicity.

Proposition 4.3. *The following two statements are equivalent:*

(1) *The random variables X and Y are positively associated.*

(2) *For all non-decreasing Borel functions g and h, the functions g and $h^*(x) := \mathbb{E}[h(Y)|X = x]$ are weakly comonotonic with respect to $F_X \times F_X$.*

From Proposition 4.3 we see that if we require the functions g and h^* to be weakly comonotonic with respect to *all* product measures $\varrho_1 \times \varrho_2$, and thus in particular with respect to the products $\delta_x \times \delta_{x'}$ for all $x, x' \in \mathbb{R}$, then this is tantamount to the functions g and h^* being strongly comonotonic. The next theorem connects the notion of weak comonotonicity of g and h^* with the notion of positive regression dependence (Lehmann, 1966).

Proposition 4.4. *The following two statements are equivalent:*

(i) For all non-decreasing Borel functions g and h, the functions g and h^* are weakly comonotonic with respect to all product measures $\varrho_1 \times \varrho_2$.

(ii) The random variable Y is positively regression dependent on X, that is, for every $y \in \mathbb{R}$, the function $x \mapsto F_{Y|X}(y|x)$ is non-increasing.

Proof. Statement (i) means that g and h^* are strongly comonotonic for all non-decreasing Borel functions g and h. With this in mind, the equivalence of statements (i) and (ii) follows by noting that $h^*(x)$ and $1 - F_{Y|X}(y|x)$ are equal to $\mathbb{E}[h(Z_x)]$ and $\mathbb{E}[h_y(Z_x)]$, respectively, where $Z_x := [Y|X = x]$ and $h_y = \mathbb{I}_{(y,\infty)}$. It now remains to recall that the class of all non-decreasing functions h and the class $\{h_y,\ y \in \mathbb{R}\}$ give rise to two equivalent ways for defining stochastic ordering (e.g., Pflug and Römisch, 2007; Rüschendorf, 2013; Föllmer and Schied, 2016). $\square$

5 Maximality of product measures

Definition 2.2 is based on the product measure $\pi_1 \times \pi_2$, which is a natural choice in view of the examples that have given rise to the notion of weak comonotonicity. There are, however, situations when the need for more generality arises, and for this we introduce an extension of integral (2.4):

$$\iint_{\Omega^2} (X(\omega) - X(\omega'))(Y(\omega) - Y(\omega'))\pi_W(\mathrm{d}\omega, \mathrm{d}\omega'), \tag{5.1}$$

where, π is a measure on $(\Omega, \mathcal{F})$, and for any random variable W on $(\Omega^2, \mathcal{F}^2)$,

$$\pi_W(\mathrm{d}\omega, \mathrm{d}\omega') = \frac{W(\omega, \omega')}{\mathbb{E}^{\pi\times\pi}[W]}\pi(\mathrm{d}\omega)\pi(\mathrm{d}\omega').$$

Definition 5.1. We say that random variables X and Y are weakly comonotonic with respect to a set $\mathcal{P}$ of (not necessarily product) measures π on $(\Omega^2, \mathcal{F}^2)$ whenever

$$\iint_{\Omega^2} (X(\omega) - X(\omega'))(Y(\omega) - Y(\omega'))\pi(\mathrm{d}\omega, \mathrm{d}\omega') \geq 0$$

for all $\pi \in \mathcal{P}$.

This generalization provides a context within which we can better understand the role of the product measure $\pi_1 \times \pi_2$, which happens to enjoy the following maximality property:

$$\begin{aligned}\iint_{\Omega^2} &(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega'))\pi(\mathrm{d}\omega, \mathrm{d}\omega') \\ &\leq \iint_{\Omega^2} (X(\omega) - X(\omega'))(Y(\omega) - Y(\omega'))\pi_1(\mathrm{d}\omega)\pi_2(\mathrm{d}\omega'),\end{aligned} \tag{5.2}$$

provided that

$$\begin{aligned}\mathbb{C}^\pi(X, Y) := &\frac{1}{2}\left\{\iint_{\Omega^2} X(\omega)Y(\omega')\pi(\mathrm{d}\omega, \mathrm{d}\omega') - \int_\Omega X(\omega)\pi_1(\mathrm{d}\omega)\int_\Omega Y(\omega')\pi_2(\mathrm{d}\omega')\right\} \\ &+ \frac{1}{2}\left\{\iint_{\Omega^2} Y(\omega)X(\omega')\pi(\mathrm{d}\omega, \mathrm{d}\omega') - \int_\Omega Y(\omega)\pi_1(\mathrm{d}\omega)\int_\Omega X(\omega')\pi_2(\mathrm{d}\omega')\right\} \geq 0,\end{aligned} \tag{5.3}$$

where $\pi_1(A) := \int_\Omega \pi(A, \mathrm{d}\omega')$ and $\pi_2(A') := \int_\Omega \pi(\mathrm{d}\omega, A')$. If the measure π is symmetric, that is, $\pi(A, A') = \pi(A', A)$ for all $A, A' \in \mathcal{F}$, then $\pi_1 = \pi_2$. Note also that the covariance-looking quantities inside the first braces and inside the second braces are *not*, in general, symmetric with respect to X and Y, but their sum $\mathbb{C}^\pi(X, Y)$ is always symmetric, irrespective of the measure π. Finally, we note that in the "diagonal" case $X = Y$, we have

$$\mathbb{C}^\pi(X, X) = \iint_{\Omega^2} X(\omega)X(\omega')\pi(\mathrm{d}\omega, \mathrm{d}\omega') - \int_\Omega X(\omega)\pi_1(\mathrm{d}\omega) \int_\Omega X(\omega')\pi_2(\mathrm{d}\omega').$$

To get a deeper insight into the above notion, and to also connect it to weak comonotonicity and positive association, we shift our focus to 1) the measurable space $(\mathbb{R}^2, \mathcal{B}^2)$, 2) Borel functions g and h, and 3) the joint cdf $F_{V,W}$ generated by two random variables V and W, whose marginal cdf's we denote by F_V and F_W, respectively. Under this scenario, bound (5.2) takes on the following form

$$\iint_{\mathbb{R}^2} \big(g(v) - g(w)\big)\big(h(v) - h(w)\big) F_{V,W}(\mathrm{d}v, \mathrm{d}w) \\ \le \iint_{\mathbb{R}^2} \big(g(v) - g(w)\big)\big(h(v) - h(w)\big) F_V(\mathrm{d}v) F_W(\mathrm{d}w), \quad (5.4)$$

which holds (cf. condition (5.3)) if and only if

$$\mathbb{C}^\pi(g, h) := \frac{1}{2}\mathrm{Cov}[g(V), h(W)] + \frac{1}{2}\mathrm{Cov}[h(V), g(W)] \ge 0, \quad (5.5)$$

where $\pi = F_{V,W}$. Obviously, $\mathbb{C}^\pi(g, h) = \mathbb{C}^\pi(h, g)$ irrespective of the measure π, and we also have the equation $\mathbb{C}^\pi(g, g) = \mathrm{Cov}[g(V), g(W)]$.

From the above notes we conclude that within the class of measures $\pi = F_{V,W}$ generated by positively-associated random variables V and W, the product measure $\pi_0 = F_V \times F_V$ is maximal in the sense of bound (5.4) within the class of all pairs of non-decreasing Borel functions g and h. But the assumptions that 1) V and W are positively associated and 2) g and h are non-decreasing are rather strong: they ensure non-negativity of the two covariances on the right-hand side of equation (5.5) and thus, in turn, imply the required non-negativity of $\mathbb{C}^\pi(g, h)$.

Due to the notion of weak comonotonicity, we can specify necessary and sufficient conditions for non-negativity of the two covariances on the right-hand side of equation (5.5). For this, we write

$$\mathbb{C}^\pi(g, h) = \frac{1}{2}\mathrm{Cov}[g(V), h^*(V)] + \frac{1}{2}\mathrm{Cov}[g^*(V), h(V)], \quad (5.6)$$

where $h^*(v) = \mathbb{E}[h(W)|V = v]$ and $g^*(v) = \mathbb{E}[g(W)|V = v]$. The two covariances on the right-hand side of equation (5.6) are non-negative if and only if the two pairs (g, h^*) and (g^*, h) are weakly comonotonic with respect to the measure $\pi_0 = F_V \times F_V$.

Note, however, that the covariance $\mathbb{C}^\pi(g,h)$ can be non-negative without making the two covariances on the right-hand side of equation (5.6) non-negative. To show this, we next construct an example when one of the two covariances is negative but $\mathbb{C}^\pi(g,h)$ is positive.

Example 5.1. Let $g(x) = \sin(x)$ and $h(x) = \cos(x)$. Furthermore, let V and W be random variables whose marginal distributions are

$$V = \begin{cases} 0 & \text{with } \ 3/10 \\ \pi/2 & \text{with } \ 7/10 \end{cases}$$

and

$$W = \begin{cases} 2\pi/3 & \text{with } \ 3/10 \\ \pi & \text{with } \ 7/10 \end{cases}$$

and let the dependence structure be given by the matrix

$$\begin{array}{c} \\ 0 \\ \pi/2 \end{array} \begin{array}{c} \begin{array}{cc} 2\pi/3 & \pi \end{array} \\ \begin{pmatrix} 1/10 & 2/10 \\ 2/10 & 5/10 \end{pmatrix} \end{array}$$

with Archimedes' constant $\pi \approx 3.14159$ not be confused with the earlier used notation for measures. We have

$$\begin{aligned} \mathrm{Cov}[g(V), h(W)] &= -\frac{1}{200} = -0.005, \\ \mathrm{Cov}[h(V), g(W)] &= \frac{\sqrt{3}}{200} \approx 0.00866, \end{aligned}$$

and thus

$$\mathbb{C}^\pi(g,h) = \frac{1}{2}\Big(-\frac{1}{200} + \frac{\sqrt{3}}{200}\Big) \approx 0.00183.$$

This concludes Example 5.1.

6 An application to quantile-based risk sharing

In this section, we illustrate the above developed theory by studying an optimization problem arising in the context of risk sharing, where weak comonotonicity provides a natural constraint on the dependence structure of admissible risk allocations. We follow the framework of Embrechts et al. (2018, 2019), who studied risk sharing problems with quantile-based risk measures.

Let $\mathcal{X}$ be the set of all random variables in an atomless probability space. The random variable $X \in \mathcal{X}$ represents a total random loss, and $\rho_1, \dots, \rho_n$ are risk measures (e.g., VaR or ES) used by n economic agents (e.g., firms or investors). Denote

$$\mathbb{A}_n(X) = \left\{ (X_1, \dots, X_n) \in \mathcal{X}^n : \sum_{i=1}^{n} X_i \geq X \right\}, \tag{6.1}$$

which is the set of all possible allocations of losses to the agents, summing up to at least the total loss X. By Embrechts et al. (2018, Proposition 1), Pareto-optimal allocations for the risk sharing problem are solutions to the following optimization problem

$$\min \left\{ \sum_{i=1}^{n} \rho_i(X_i) : (X_1, \dots, X_n) \in \mathbb{A}_n(X) \right\}. \tag{6.2}$$

In problem (6.2), the dependence structure among the allocation $(X_1, \dots, X_n)$ is arbitrary. Embrechts et al. (2018) also consider the constrained problem

$$\min \left\{ \sum_{i=1}^{n} \rho_i(X_i) : (X_1, \dots, X_n) \in \mathbb{A}_n(X),\ X_i \uparrow X,\ i = 1, \dots, n \right\}. \tag{6.3}$$

where $X_i \uparrow X$ means that X_i and X are strongly comonotonic.

For a practical situation, the assumption of arbitrary dependence in the admissible allocations as in problem (6.2) may be too weak, and the assumption of strongly comonotonic allocations in problem (6.3) may be too strong. Therefore, we can consider an intermediate assumption on the dependence structure of the admissible allocations in the risk sharing problem, which is modelled by weak comonotonicity.

To this end, we construct a spectrum of weak comonotonicity indexed by $\beta \in [0, 1]$, such that $\beta = 0$ corresponds to no dependence constraint and $\beta = 1$ corresponds to strong comonotonicity. For this purpose, recall that in Section 3 above, for a random variable X and for any $p \in [0, 1)$, we defined

$$A_p^X = \{\omega \in \Omega : X(\omega) > \mathrm{VaR}_p(X)\}$$

and

$$\mathcal{P}_p^X = \{\delta_\omega \times \delta_{\omega'} : \omega \in A_p^X,\ \omega' \in (A_p^X)^c\}.$$

In what follows, for two random variables Y and Z, we shall use the notation $Y \uparrow_\beta Z$ when Y and Z are weakly comonotonic with respect to $\bigcup_{p \in [1-\beta, 1)} \mathcal{P}_p^Z$.

The interpretation of $Y \uparrow_\beta Z$ is that Y and Z are comonotonic and both take large values on the event $A_{1-\beta}^Z$, and there is no dependence assumption on $(A_{1-\beta}^Z)^c$. Note also that the requirement

$Y \uparrow_\beta Z$ gets stronger when β increases. In particular, assuming that Z is continuously distributed, for $\beta = 0$, $Y \uparrow_\beta Z$ imposes no dependence assumption, and for $\beta = 1$, it means that Y and Z are strongly comonotonic. Using this connection, we will impose $X_i \uparrow_\beta X$, $i = 1, \dots, n$ as a constraint on the admissible allocations in our risk sharing problem, so that $\beta = 0$ corresponds to (6.2) and $\beta = 1$ corresponds to (6.3).

For the purpose of illustration, we focus on an important special case studied by Embrechts et al. (2018), when the risk measures $\rho_1, \dots, \rho_n$ are quantiles at different levels. Following the setup of Embrechts et al. (2018), for $\alpha \in (0, 1)$ and $Y \in \mathcal{X}$, we define

$$\mathrm{Q}_\alpha(Y) = \inf\{x \in \mathbb{R} : \mathbb{P}(Y \le x) \ge 1 - \alpha\}.$$

Remark 6.1. Note that Q_α is the left $(1-\alpha)$-quantile, which is different from the VaR (right quantile) defined in Section 3. The choice of the left quantile here and in Embrechts et al. (2018, 2019) is intentional. For minimization problems, we need to work with left quantiles to guarantee the existence of optimal allocations. Recall that in Section 3 we study maximization problems, and hence right quantiles are natural choices there. On the other hand, using $(1-\alpha)$-quantile instead of α-quantile leads to concise statements of the results; this will be clear from statements (6.4)–(6.5) below.

Let $\rho_i = \mathrm{Q}_{\alpha_i}$, $i = 1, \dots, n$, where $\alpha_1, \dots, \alpha_n$ are positive constants such that $\sum_{i=1}^n \alpha < 1$. For this choice of risk measures, both problems (6.2) and (6.3) admit analytical solutions, given in Theorem 2 and Proposition 5 of Embrechts et al. (2018), respectively. These results imply

$$\min\left\{\sum_{i=1}^n \mathrm{Q}_{\alpha_i}(X_i) : (X_1, \dots, X_n) \in \mathbb{A}_n(X)\right\} = \mathrm{Q}_{\sum_{i=1}^n \alpha_i}(X) \tag{6.4}$$

and

$$\min\left\{\sum_{i=1}^n \mathrm{Q}_{\alpha_i}(X_i) : (X_1, \dots, X_n) \in \mathbb{A}_n(X),\ X_i \uparrow X,\ i = 1, \dots, n\right\} = \mathrm{Q}_{\bigvee_{i=1}^n \alpha_i}(X), \tag{6.5}$$

and the corresponding optimal allocations can be explicitly constructed as well. Note that result (6.4) implies

$$\mathrm{Q}_{\sum_{i=1}^n \alpha_i}\left(\sum_{i=1}^n X_i\right) \le \sum_{i=1}^n \mathrm{Q}_{\alpha_i}(X_i) \tag{6.6}$$

for all $X_1, \dots, X_n \in \mathcal{X}$ (Embrechts et al., 2018, Corollary 1), which will be useful in our analysis below.

Remark 6.2. Embrechts et al. (2018) formulate the admissible allocations in (6.1) using $\sum_{i=1}^n X_i = X$ instead of $\sum_{i=1}^n X_i \geq X$. It is easy to see that in problems (6.2) and (6.3), these two setups are equivalent for monotone risk measures such as the quantiles. In this paper, we use inequality in definition (6.1) because our dependence constraint would make the two formulations generally no longer equivalent, and analytical solutions are found for the current formulation.

For a continuously distributed X and a parameter $\beta \in [0,1]$, we consider the optimization problem

$$V_\beta(X) = \inf \left\{ \sum_{i=1}^n \mathrm{Q}_{\alpha_i}(X_i) : (X_1, \dots, X_n) \in \mathbb{A}_n(X),\ X_i \uparrow_\beta X,\ i = 1, \dots, n \right\}. \tag{6.7}$$

It is clear that $\beta = 0$ corresponds to problem (6.2) and $\beta = 1$ corresponds to problem (6.3). Therefore, the use of weak comonotonicity yields a bridge between the two risk sharing problems (6.2) and (6.3) considered by Embrechts et al. (2018), and it offers more flexibility as one can impose a partial dependence constraint on the admissible allocations.

Similarly to many other optimization problems involving quantiles (or VaR), problem (6.7) is not convex as Q_α is generally not convex, and thus a specialized analysis of the problem is needed. Nevertheless, via some auxiliary technical results, we will show below that problem (6.7) admits an analytical solution, and an optimal allocation will be obtained in explicit form.

Theorem 6.1. *Suppose that X is a continuously distributed random variable, $\alpha_1, \dots, \alpha_n > 0$, $\sum_{i=1}^n \alpha_i < 1$, and $\beta \in [0,1]$. We have*

$$V_\beta(X) = \mathrm{Q}_\gamma(X),$$

where $\gamma = \beta \wedge (\bigvee_{i=1}^n \alpha_i) + \sum_{i=1}^n (\alpha_i - \beta)_+$.

Proof. We first note that $\gamma = \sum_{i=1}^n \alpha_i$ if $\beta = 0$, and $\gamma = \bigvee_{i=1}^n \alpha_i$ if $\beta = 1$, corresponding to statements (6.4) and (6.5), respectively. Thus, it suffices to consider $\beta \in (0,1)$. To proceed, we need the following lemma, whose proof will be given in the appendix.

Lemma 6.1. *Let $\beta \in (0,1)$ and $Y \uparrow_\beta X$. Denote $B = A^X_{1-\beta}$. We have the statements:*

(i) *$B \subset \{Y \geq \mathrm{Q}_\beta(Y)\}$ and $B^c \subset \{Y \leq \mathrm{Q}_\beta(Y)\}$ a.s.*

(ii) *If $\alpha > \beta$, then $\mathrm{Q}_\alpha(Y) = \mathrm{Q}_{\alpha-\beta}(z\mathbb{1}_B + Y\mathbb{1}_{B^c})$ for all $z \leq \mathrm{Q}_\alpha(Y)$.*

(iii) *If $\alpha > \beta$, then $\mathrm{Q}_\alpha(Y) = \mathrm{Q}_\alpha(z\mathbb{1}_B + Y\mathbb{1}_{B^c})$ for all $z \geq \mathrm{Q}_\alpha(Y)$.*

(iv) *If* $\alpha \le \beta$*, then* $\mathrm{Q}_\alpha(Y) = \mathrm{Q}_\alpha(Y\mathbb{1}_B + z\mathbb{1}_{B^c})$ *for all* $z \le \mathrm{Q}_\alpha(Y)$*.*

(v) *If* $\alpha + \beta < 1$*, then* $\mathrm{Q}_\alpha(Z) \ge \mathrm{Q}_{\alpha+\beta}(z\mathbb{1}_B + Z\mathbb{1}_{B^c})$ *for all* $Z \in \mathcal{X}$ *and* $z \in \mathbb{R}$*.*

We can now continue the proof of Theorem 6.1. Let $\beta \in (0,1)$ and take an arbitrary admissible allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ such that $X_i \uparrow_\beta X$, $i = 1, \dots, n$. We need additional notation: $B = A^X_{1-\beta}$, $J = \{i \in \{1, \dots, n\} : \alpha_i > \beta\}$, and $K = \{1, \dots, n\} \setminus J$. Moreover, let $x_i = \mathrm{Q}_{\alpha_i}(X_i)$, $y_i = \mathrm{Q}_\beta(X_i)$, $i = 1, \dots, n$, $y_J = \sum_{i\in J} y_i$, $y_K = \sum_{i\in K} y_i$, $X_J = \sum_{i\in J} X_i$, and $X_K = \sum_{i\in K} X_i$.

By Lemma 6.1(i), we have (all statements are in the sense of a.s.)

$$B \subset \{X_i \ge y_i\} \quad \text{and} \quad B^c \subset \{X_i \le y_i\} \quad \text{for each } i = 1, \dots, n. \tag{6.8}$$

Using statements (6.8), we see that the random vector $(X_i\mathbb{1}_B + y_i\mathbb{1}_{B^c})_{i\in K}$ is strongly comonotonic, because $(X_1, \dots, X_n)$ is strongly comonotonic on the event B by assumption. Hence, using $y_i \le x_i$ for $i \in K$, Lemma 6.1(iv) and statement (6.5), we get

$$\begin{aligned}
\sum_{i\in K} \mathrm{Q}_{\alpha_i}(X_i) &= \sum_{i\in K} \mathrm{Q}_{\alpha_i}(X_i\mathbb{1}_B + y_i\mathbb{1}_{B^c}) \\
&\ge \mathrm{Q}_{\bigvee_{i\in K}\alpha_i}\left(\sum_{i\in K} X_i\mathbb{1}_B + \sum_{i\in K} y_i\mathbb{1}_{B^c}\right) \\
&= \mathrm{Q}_{\bigvee_{i\in K}\alpha_i}\left(X_K\mathbb{1}_B + y_K\mathbb{1}_{B^c}\right).
\end{aligned} \tag{6.9}$$

Further, statements (6.8) also imply

$$B \subset \{X_K \ge y_K\}, \quad B \subset \{X_J \ge y_J\}, \quad B^c \subset \{X_K \le y_K\} \quad \text{and} \quad B^c \subset \{X_J \le y_J\}. \tag{6.10}$$

We split the following considerations into two cases.

Case 1. Assume $\beta \ge \bigvee_{i=1}^n \alpha_i$, which means $K = \{1, \dots, n\}$ and $\gamma = \bigvee_{i=1}^n \alpha_i$. Note that statements (6.10) imply $X_K\mathbb{1}_B + y_K\mathbb{1}_{B^c} \ge X_K \ge X$. Using bound (6.9) and the fact that $\mathrm{Q}_\gamma(Y)$ is increasing in Y, we have

$$\sum_{i=1}^n \mathrm{Q}_{\alpha_i}(X_i) \ge \mathrm{Q}_{\bigvee_{i=1}^n \alpha_i}\left(X_K\mathbb{1}_B + y_K\mathbb{1}_{B^c}\right) \ge \mathrm{Q}_\gamma\left(X_K\right) \ge \mathrm{Q}_\gamma(X).$$

Therefore, $V_\beta(X) \ge \mathrm{Q}_\gamma(X)$. On the other hand, by statement (6.5), we have

$$V_\beta(X) \le \mathrm{Q}_{\bigvee_{i=1}^n \alpha_i}(X) = \mathrm{Q}_\gamma(X).$$

Putting the above observations together, we get $V_\beta(X) = \mathrm{Q}_\gamma(X)$.

Case 2. Assume $\beta < \bigvee_{i=1}^n \alpha_i$, which means $\gamma = \beta + \sum_{i=1}^n (\alpha_i - \beta)_+ > \beta$, and $J \neq \varnothing$. Using Lemma 6.1(ii) and (v), and bound (6.6), we get

$$\begin{aligned}\sum_{i\in J} \mathrm{Q}_{\alpha_i}(X_i) &= \sum_{i\in J} \mathrm{Q}_{\alpha_i-\beta}(x_i \mathbb{1}_B + X_i \mathbb{1}_{B^c}) \\ &\geq \mathrm{Q}_{\sum_{i\in J}(\alpha_i-\beta)}\left(\sum_{i\in J} x_i \mathbb{1}_B + \sum_{i\in J} X_i \mathbb{1}_{B^c}\right) \\ &\geq \mathrm{Q}_{\beta+\sum_{i\in J}(\alpha_i-\beta)}\left(y_J \mathbb{1}_B + X_J \mathbb{1}_{B^c}\right) \\ &= \mathrm{Q}_\gamma\left(y_J \mathbb{1}_B + X_J \mathbb{1}_{B^c}\right). \end{aligned} \tag{6.11}$$

Therefore, $y_J \mathbb{1}_B + X_J \mathbb{1}_{B^c}$ and $X_K \mathbb{1}_B + y_K \mathbb{1}_{B^c}$ are strongly comonotonic. Putting inequalities (6.9) and (6.11) together, and using statements (6.5) and (6.10), we obtain

$$\begin{aligned}\sum_{i=1}^n \mathrm{Q}_{\alpha_i}(X_i) &= \sum_{i\in J} \mathrm{Q}_{\alpha_i}(X_i) + \sum_{i\in K} \mathrm{Q}_{\alpha_i}(X_i) \\ &\geq \mathrm{Q}_\gamma\left(y_J \mathbb{1}_B + X_J \mathbb{1}_{B^c}\right) + \mathrm{Q}_\gamma\left(X_K \mathbb{1}_B + y_K \mathbb{1}_{B^c}\right) \\ &\geq \mathrm{Q}_\gamma\left((X_K + y_J) \mathbb{1}_B + (X_J + y_K) \mathbb{1}_{B^c}\right) \\ &\geq \mathrm{Q}_\gamma\left((y_K + y_J) \mathbb{1}_B + (X_J + X_K) \mathbb{1}_{B^c}\right) \\ &\geq \mathrm{Q}_\gamma\left((y_K + y_J) \mathbb{1}_B + X \mathbb{1}_{B^c}\right). \end{aligned} \tag{6.12}$$

Note that X is continuously distributed, implying $\mathrm{Q}_\gamma(X) < \mathrm{Q}_\beta(X)$. Moreover, $y_K + y_J \geq X_J + X_K \geq X$ on B^c, and by Lemma 6.1(i), we have

$$\{X \leq \mathrm{Q}_\gamma(X)\} \subset \{X < \mathrm{Q}_\beta(X)\} \subset B^c \subset \{X \leq y_K + y_J\}.$$

This shows $y_K + y_J \geq \mathrm{Q}_\gamma(X)$. Using Lemma 6.1(iii) and bounds (6.12), we obtain

$$\sum_{i=1}^n \mathrm{Q}_{\alpha_i}(X_i) \geq \mathrm{Q}_\gamma\left((y_K + y_J) \mathbb{1}_B + X \mathbb{1}_{B^c}\right) = \mathrm{Q}_\gamma(X).$$

This proves $V_\beta(X) \geq \mathrm{Q}_\gamma(X)$.

Next, we show $V_\beta(X) \leq \mathrm{Q}_\gamma(X)$ by an explicit construction of an optimal allocation. Let $y = \mathrm{Q}_\beta(X)$ and $z = \mathrm{Q}_\gamma(X)$. Without loss of generality, assume $1 \in J$. Recall that

$$\mathbb{P}(A^X_{1-\gamma} \setminus A^X_{1-\beta}) = \gamma - \beta = \sum_{i\in J} (\alpha_i - \beta),$$

and hence we can find a partition $(A_i)_{i\in J}$ of $A^X_{1-\gamma} \setminus A^X_{1-\beta}$ such that $\mathbb{P}(A_i) = \alpha_i - \beta_i$ for each $i \in J$.

Define

$$X_i = \begin{cases} (X-z)\left(\mathbb{1}_B + \mathbb{1}_{A_1} + \mathbb{1}_{(A^X_{1-\gamma})^c}\right) + z & \text{if } i = 1, \\ y_+\mathbb{1}_B + (X-z)\mathbb{1}_{A_i} & \text{if } i \in J \setminus \{1\}, \\ 0 & \text{if } i \in K. \end{cases} \tag{6.13}$$

We easily verify that $\sum_{i=1}^n X_i = (\#J-1)y_+\mathbb{1}_B + X \geq X$ and $X_i \uparrow_\beta X$, $i = 1, \dots, n$. Hence, $(X_1, \dots, X_n)$ is an admissible allocation for problem (6.7). Furthermore, we check that $\mathrm{Q}_{\alpha_1}(X) = z$ and $\mathrm{Q}_{\alpha_i}(X_i) = 0$ for $i \neq 1$. Therefore,

$$\sum_{i=1}^n \mathrm{Q}_{\alpha_i}(X_i) = z = \mathrm{Q}_\gamma(X),$$

showing that $V_\beta(X) \leq \mathrm{Q}_\gamma(X)$.

With this, we finish the proof of Theorem 6.1. □

An explicit construction of an optimal allocation to problem (6.7) has been obtained in the proof of Theorem 6.1. Specifically, and without loss of generality, let $\alpha_1 = \bigvee_{i=1}^n \alpha_i$. If $\beta < \bigvee_{i=1}^n \alpha_i$, then an optimal allocation is given by equation (6.13). On the other hand, if $\beta \geq \bigvee_{i=1}^n \alpha_i$, then an optimal allocation is trivially given by $X_1 = X$ and $X_i = 0$ for $i \neq 1$. The optimal allocations are generally not unique, similarly to the case of problems (6.2) and (6.3) in Embrechts et al. (2018).

Finally, we discuss the implication of the values of the parameter β in problem (6.7). Recall that $V_\beta(X)$ represents the smallest total risk measure after risk redistribution. In Theorem 6.1, $\gamma = \gamma(\beta)$ is a piece-wise linear decreasing function of β, with $\gamma(0) = \sum_{i=1}^n \alpha_i$ and $\gamma(\beta) = \bigvee_{i=1}^n \alpha_i$ if $\beta \geq \bigvee_{i=1}^n \alpha_i$. Thus, if there is no dependence constraint, we arrive at (6.4), the minimum possible total risk measure obtained by Embrechts et al. (2018, Theorem 2). If the dependence constraint is strong enough (i.e., $\beta \geq \bigvee_{i=1}^n \alpha_i$), then we arrive at the same value of the minimum total risk measure to (6.5), obtained by Embrechts et al. (2018, Proposition 5). If the dependence constraint is intermediate, then the total risk measure $V_\beta(X)$ varies between the two values, decreasing in β. This suggests that the use of weak comonotonicity as a dependence constraint yields a spectrum of flexible formulations of the risk sharing problem.

7 Summary and concluding notes

In this paper, we introduced the notion of weak comonotonicity. Via the analysis of several properties and applications, we show the encompassing nature of weak comonotonicity, which contains

– as a special case – the classical notion of comonotonicity. The new notion serves a bridge that connects the classical notion of comonotonicity of random variables with a number of well-known notions of (in)dependence and association (e.g., Joe, 2014; Durante and Sempi, 2015, and references therein). More importantly, we illustrate that introduced weak comonotonicity provides necessary and sufficient conditions for a number of problems in economics, banking, and insurance, and in particular to those dealing with risk aggregation and risk sharing. Specifically, it is shown that the notion of weak comonotonicity yields a sufficient condition for the maximum VaR aggregation, and a necessary and sufficient condition for the maximum ES aggregation. As far as we are aware of, such conditions have been elusive. In addition, we provided analytical solutions to a risk sharing problem whose constraint on the dependence structure of admissible allocations has been most naturally described by weak comonotonicity, bridging the gap between strong comonotonicity and no dependence assumption studied in the literature. We finally remark that, as weak comonotonicity depends on the set $\mathcal{P}$ of product measures, its spectrum is very wide, including many types of dependence.

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A Appendix: Proof of Lemma 6.1

Proof of statement (i). By definition of $Y \uparrow_\beta X$, for a.s. all $\omega \in B$ and $\omega' \in B^c$, we have $Y(\omega) \geq Y(\omega')$. Therefore, there exists a constant $z \in \mathbb{R}$ such that $B \subset \{Y \geq z\}$ and $B^c \subset \{Y \leq z\}$ a.s. It is easy to see that this constant can be chosen as $z = \mathrm{Q}_\beta(Y)$ because $\mathbb{P}(Y \geq \mathrm{Q}_\beta(Y)) \geq 1 - \beta$ and $\mathbb{P}(Y \leq \mathrm{Q}_\beta(Y)) \geq \beta$.

Proof of statement (ii). By bound (6.6), we have

$$\mathrm{Q}_{\alpha-\beta}(z\mathbb{1}_B + Y\mathbb{1}_{B^c}) + \mathrm{Q}_\beta((Y - z)\mathbb{1}_B) \geq \mathrm{Q}_\alpha(Y).$$

Note that $\mathrm{Q}_\beta((Y-z)\mathbb{1}_B) = 0$ since $Y \geq z$ on B by statement (i) and $\mathbb{P}(B) = \beta$. Hence,

$$\mathrm{Q}_{\alpha-\beta}(z\mathbb{1}_B + Y\mathbb{1}_{B^c}) \geq \mathrm{Q}_\alpha(Y). \tag{A.1}$$

To show the other direction, we consider two cases. If $\mathrm{Q}_\alpha(Y) < \mathrm{Q}_\beta(Y)$, then $\{Y \leq \mathrm{Q}_\alpha(Y)\} \subset \{Y < \mathrm{Q}_\beta(Y)\} \subset B^c$ by statement (i). In this case,

$$\begin{aligned}\mathbb{P}(z\mathbb{1}_B + Y\mathbb{1}_{B^c} \leq \mathrm{Q}_\alpha(Y)) &= \mathbb{P}(z \leq \mathrm{Q}_\alpha(Y),\ B) + \mathbb{P}(Y \leq \mathrm{Q}_\alpha(Y),\ B^c)\\ &= \mathbb{P}(B) + \mathbb{P}(Y \leq \mathrm{Q}_\alpha(Y))\\ &\geq \beta + 1 - \alpha.\end{aligned}$$

If $\mathrm{Q}_\alpha(Y) = \mathrm{Q}_\beta(Y)$, then $B^c \subset \{Y \leq \mathrm{Q}_\beta(Y)\} = \{Y \leq \mathrm{Q}_\alpha(Y)\}$ by statement (i). In this case,

$$\begin{aligned}\mathbb{P}(z\mathbb{1}_B + Y\mathbb{1}_{B^c} \leq \mathrm{Q}_\alpha(Y)) &= \mathbb{P}(z \leq \mathrm{Q}_\alpha(Y),\ B) + \mathbb{P}(Y \leq \mathrm{Q}_\alpha(Y),\ B^c)\\ &= \mathbb{P}(B) + \mathbb{P}(B^c) = 1.\end{aligned}$$

In both cases,

$$\mathbb{P}(z\mathbb{1}_B + Y\mathbb{1}_{B^c} \leq \mathrm{Q}_\alpha(Y)) \geq 1 - (\alpha - \beta),$$

which implies $\mathrm{Q}_{\alpha-\beta}(z\mathbb{1}_B + Y\mathbb{1}_{B^c}) \leq \mathrm{Q}_\alpha(Y)$. By bound (A.1), we get $\mathrm{Q}_{\alpha-\beta}(z\mathbb{1}_B + Y\mathbb{1}_{B^c}) = \mathrm{Q}_\alpha(Y)$.

Proof of statement (iii). Note that

$$\begin{aligned}\mathbb{P}(z\mathbb{1}_B + Y\mathbb{1}_{B^c} \geq \mathrm{Q}_\alpha(Y)) &= \mathbb{P}(z \geq \mathrm{Q}_\alpha(Y),\ B) + \mathbb{P}(Y \geq \mathrm{Q}_\alpha(Y),\ B^c)\\ &= \mathbb{P}(B) + \mathbb{P}(Y \geq \mathrm{Q}_\alpha(Y),\ B^c)\\ &\geq \mathbb{P}(Y \geq \mathrm{Q}_\alpha(Y),\ B) + \mathbb{P}(Y \geq \mathrm{Q}_\alpha(Y),\ B^c)\\ &= \mathbb{P}(Y \geq \mathrm{Q}_\alpha(Y)) \geq \alpha.\end{aligned}$$

This shows

$$\mathrm{Q}_\alpha(z\mathbb{1}_B + Y\mathbb{1}_{B^c}) \geq \mathrm{Q}_\alpha(Y). \tag{A.2}$$

For the other direction, we consider two cases, similarly to statement (ii). If $\mathrm{Q}_\alpha(Y) < \mathrm{Q}_\beta(Y)$, then $\{Y \leq \mathrm{Q}_\alpha(Y)\} \subset \{Y < \mathrm{Q}_\beta(Y)\} \subset B^c$ by statement (i). In this case,

$$\begin{aligned}\mathbb{P}(z\mathbb{1}_B + Y\mathbb{1}_{B^c} \leq \mathrm{Q}_\alpha(Y)) &= \mathbb{P}(z \leq \mathrm{Q}_\alpha(Y),\ B) + \mathbb{P}(Y \leq \mathrm{Q}_\alpha(Y),\ B^c)\\ &\geq \mathbb{P}(Y \leq \mathrm{Q}_\alpha(Y)) \geq 1 - \alpha.\end{aligned}$$

If $\mathrm{Q}_\alpha(Y) = \mathrm{Q}_\beta(Y)$, then $B^c \subset \{Y \le \mathrm{Q}_\beta(Y)\} = \{Y \le \mathrm{Q}_\alpha(Y)\}$ by statement (i). In this case,

$$\begin{aligned}\mathbb{P}(z\mathbb{1}_B + Y\mathbb{1}_{B^c} \le \mathrm{Q}_\alpha(Y)) &= \mathbb{P}(z \le \mathrm{Q}_\alpha(Y),\ B) + \mathbb{P}(Y \le \mathrm{Q}_\alpha(Y),\ B^c)\\ &\ge \mathbb{P}(B^c) = 1-\beta \ge 1-\alpha.\end{aligned}$$

In both cases,

$$\mathbb{P}(z\mathbb{1}_B + Y\mathbb{1}_{B^c} \le \mathrm{Q}_\alpha(Y)) \ge 1-\alpha$$

which implies $\mathrm{Q}_\alpha(z\mathbb{1}_B + Y\mathbb{1}_{B^c}) \le \mathrm{Q}_\alpha(Y)$. By bound (A.2), we get $\mathrm{Q}_\alpha(z\mathbb{1}_B + Y\mathbb{1}_{B^c}) = \mathrm{Q}_\alpha(Y)$.

Proof of statement (iv). If $\alpha \le \beta$, then $\mathrm{Q}_\alpha(Y) = \mathrm{Q}_\alpha(Y\mathbb{1}_B + z\mathbb{1}_{B^c})$ for all $z \le y$. Note that

$$\begin{aligned}\mathbb{P}(Y\mathbb{1}_B + z\mathbb{1}_{B^c} \le \mathrm{Q}_\alpha(Y)) &= \mathbb{P}(Y \le \mathrm{Q}_\alpha(Y),\ B) + \mathbb{P}(z \le \mathrm{Q}_\alpha(Y),\ B^c)\\ &= \mathbb{P}(Y \le \mathrm{Q}_\alpha(Y),\ B) + \mathbb{P}(B^c)\\ &\ge \mathbb{P}(Y \le \mathrm{Q}_\alpha(Y)) \ge 1-\alpha.\end{aligned}$$

This shows

$$\mathrm{Q}_\alpha(Y\mathbb{1}_B + z\mathbb{1}_{B^c}) \le \mathrm{Q}_\alpha(Y). \tag{A.3}$$

For the other direction, we again consider two cases. If $\mathrm{Q}_\alpha(Y) > \mathrm{Q}_\beta(Y)$, then $\{Y \ge \mathrm{Q}_\alpha(Y)\} \subset \{Y > \mathrm{Q}_\beta(Y)\} \subset B$ by statement (i). In this case,

$$\begin{aligned}\mathbb{P}(Y\mathbb{1}_B + z\mathbb{1}_{B^c} \ge \mathrm{Q}_\alpha(Y)) &= \mathbb{P}(Y \ge \mathrm{Q}_\alpha(Y),\ B) + \mathbb{P}(z \ge \mathrm{Q}_\alpha(Y),\ B^c)\\ &\ge \mathbb{P}(Y \ge \mathrm{Q}_\alpha(Y)) \ge \alpha.\end{aligned}$$

If $\mathrm{Q}_\alpha(Y) = \mathrm{Q}_\beta(Y)$, then $B \subset \{Y \ge \mathrm{Q}_\beta(Y)\} = \{Y \ge \mathrm{Q}_\alpha(Y)\}$ by statement (i). In this case,

$$\begin{aligned}\mathbb{P}(Y\mathbb{1}_B + z\mathbb{1}_{B^c} \ge \mathrm{Q}_\alpha(Y)) &= \mathbb{P}(Y \ge \mathrm{Q}_\alpha(Y),\ B) + \mathbb{P}(z \ge \mathrm{Q}_\alpha(Y),\ B^c)\\ &\ge \mathbb{P}(B) = \beta \ge \alpha.\end{aligned}$$

In both cases,

$$\mathbb{P}(Y\mathbb{1}_B + z\mathbb{1}_{B^c} \ge \mathrm{Q}_\alpha(Y)) \ge \alpha$$

which implies $\mathrm{Q}_\alpha(Y\mathbb{1}_B + z\mathbb{1}_{B^c}) \ge \mathrm{Q}_\alpha(Y)$. By bound (A.3), we get $\mathrm{Q}_\alpha(Y\mathbb{1}_B + z\mathbb{1}_{B^c}) = \mathrm{Q}_\alpha(Y)$.

Proof of statement (v). Using (6.6), we have $\mathrm{Q}_\alpha(Z) + \mathrm{Q}_\beta((Z-z)\mathbb{1}_B) \ge \mathrm{Q}_{\alpha+\beta}(z\mathbb{1}_B + Z\mathbb{1}_{B^c})$. Note that $\mathrm{Q}_\beta((Z-z)\mathbb{1}_B) \le 0$ since $\mathbb{P}((Z-z)\mathbb{1}_B \ge 0) \le \mathbb{P}(B) = 1-\beta$. Hence $\mathrm{Q}_\alpha(Z) \ge \mathrm{Q}_{\alpha+\beta}(z\mathbb{1}_B + Z\mathbb{1}_{B^c})$. This establishes statement (v) and concludes the entire proof of Lemma 6.1 □

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