Characterizing Optimal Allocations in Quantile-based Risk Sharing

Ruodu Wang* Yunran Wei†

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Abstract

Unlike classic risk sharing problems based on expected utilities or convex risk measures, quantile-based risk sharing problems exhibit two special features. First, quantile-based risk measures (such as the Value-at-Risk) are often not convex, and second, they ignore some part of the distribution of the risk. These features create technical challenges in establishing a full characterization of optimal allocations, a question left unanswered in the literature. In this paper, we address the issues on the existence and the characterization of (Pareto-)optimal allocations in risk sharing problems for the Range-Value-at-Risk family. It turns out that negative dependence, mutual exclusivity in particular, plays an important role in the optimal allocations, in contrast to positive dependence appearing in classic risk sharing problems. As a by-product of our main finding, we obtain some results on the optimization of the Value-at-Risk (VaR) and the Expected Shortfall, as well as a new result on the inf-convolution of VaR and a general distortion risk measure.

Keywords: Risk sharing, Value-at-Risk, Expected Shortfall, non-convexity, Pareto optimality

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1 Introduction

Quantile-based risk sharing problems, as studied by Embrechts et al. (2018), have recently drawn considerable interest in the literature of risk management, due to the popularity of quantile-based risk measures such as the Value-at-Risk (VaR) and the Expected Shortfall (ES) in current banking and insurance regulation (see, for instance, McNeil et al. (2015)). The key feature of these risk sharing problems is that each agent's preference is modelled by a quantile-based risk measure (more precisely, an RVaR), and these risk measures are often not convex; see Section 2 for

*Department of Statistics and Actuarial Science, University of Waterloo, Canada. Email: wang@uwaterloo.ca.

†Department of Statistics and Actuarial Science, Northern Illinois University, USA. Email: ywei1@niu.edu.

more details. This feature distinguishes quantile-based risk sharing problems from the classic ones based on utility functions or convex risk measures. The non-convexity of the preferences brings substantial challenges for studying risk sharing problems, as well as interesting mathematical and economic observations. For recent results and financial implications of quantile-based risk sharing, we refer to Embrechts et al. (2018) and the references therein. Weber (2018) contains results and discussions on quantile-based optimal risk sharing problem in the context of Solvency II.

The existing literature on this topic focuses on finding the minimum possible aggregate risk value and giving some optimal risk allocations, whereas existence and characterization issues are left partially or completely unaddressed. Embrechts et al. (2018) obtained some Pareto-optimal allocations and Weber (2018) generalized the underlying risk measures from the RVaR family to the so-called VaR-type distortion risk measures with concave active parts (see Section 5). The case of heterogeneous beliefs is analyzed by Embrechts et al. (2019). These papers give some solutions, but do not characterize the whole family of the optimal allocations.

In this paper, we provide a complete answer to the questions of the existence and the characterization of Pareto-optimal allocations in quantile-based risk sharing problems within the RVaR family. As noted by Embrechts et al. (2018), Pareto-optimal allocations (which we shall simply refer to as *optimal allocations*) are often equivalent to sum-optimal allocations. As we shall see from the main results, the characterization of all optimal allocations is highly non-trivial, since the quantile-based risk measures often ignore part of the distribution of the risk, creating a considerable amount of probabilistic freedom. A further complication arises when the total risk is not continuously distributed, leading to various issues with non-uniqueness of the quantile. Our results show that an optimal allocation exhibits a strong negative dependence, in sharp contrast to the classic risk sharing problems where an optimal allocation is always strongly positively dependent (see Section 6). Along our exploration, we obtain some technical lemmas on the optimizations of VaR and ES, which may be of interest in a different context.

This paper builds on the main results of Embrechts et al. (2018) on quantile-based risk sharing. As mentioned before, techniques in this framework are different from the classic risk sharing problems with convex risk measures or expected utilities; for the latter, we refer to Barrieu and El Karoui (2005), Acciaio (2007), Filipović and Svindland (2008), Anthropelos and Kardaras (2017) and the references therein. See also Tsanakas (2009) for the risk sharing problem in the context of capital allocation. The RVaR family of risk measures are introduced by Cont et al. (2010) featuring its robustness properties, and Li et al. (2018) and Embrechts et al. (2018) contain more discussions on its properties and financial applications. In this paper, the term "risk sharing problem" refers to the search for Pareto-optimal allocations. The study on competitive equilibria is beyond the scope of the current paper, and we refer to Xia and Zhou (2016), Boonen et al. (2018) and Embrechts et al. (2018, 2019) for recent related results. As a first attempt to characterize the forms of optimal allocations, the risk sharing problems that we consider are formulated in a static setting with homogeneous beliefs, as opposed to the more sophisticated settings of dynamic equilibrium (see e.g. Beissner and Riedel (2018)) or heterogeneous beliefs (see e.g. Embrechts et al. (2019)).

Our paper is a technical one in nature, and as such we concentrate our discussions on mathematical results. For a comprehensive economic discussion on the optimal allocations in quantile-based risk sharing problems as well as their practical implications, we refer the reader to Embrechts et al. (2018, 2019) in the context of banking capital sharing and Weber (2018) in the context of insurance regulation.

The rest of the paper is organized as follows. In Section 2 we present preliminaries on quantile-based risk sharing, as well as some existing results. In Section 3, we address the existence issue of optimal allocations by showing that optimal allocations exist in exactly four cases (Theorem 3.6). Section 4 contains characterization results of optimal allocations as well as some technical lemmas. In particular, Propositions 4.10 and 4.11 characterize the optimal allocations for RVaR agents based on explicit results in the cases of VaR agents (Theorem 4.2), ES agents (Theorem 4.4), and one VaR plus one ES agents (Theorem 4.8). In Section 5, we generalize our main results to the risk sharing problem for the class of VaR-type risk measures, which goes beyond the RVaR family, and obtain an explicit formula for the inf-convolution of a VaR and a general distortion risk measure (Theorem 5.3). Section 6 concludes the paper by presenting a representative class of optimal allocations.

2 Preliminaries

2.1 Risk measures

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space, and $\mathcal{X}$ be the set of real integrable random variables (i.e. random variables with finite means) defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We treat almost surely equal random variables as identical in this paper; equalities, inequalities and set inclusions should always be understood in the almost sure sense (e.g., $B \subset A$ almost surely if $\mathbb{P}(B \setminus A) = 0$). A *risk measure* is a functional $\rho : \mathcal{X} \to [-\infty, \infty]$.

The Value-at-Risk (VaR) of $X \in \mathcal{X}$ at level $\alpha \in \mathbb{R}_+ := [0, \infty)$ is defined as the $100(1-\alpha)\%$ left quantile of X,

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in [-\infty, \infty] : \mathbb{P}(X \leqslant x) \geqslant 1 - \alpha\}. \tag{2.1}$$

The corresponding right quantile is denoted by VaR_α^+, namely,

$$\mathrm{VaR}_\alpha^+(X) = \inf\{x \in [-\infty, \infty] : \mathbb{P}(X \leqslant x) > 1 - \alpha\}. \tag{2.2}$$

In (2.1)-(2.2), we use the convention $\inf(\varnothing) = \infty$. Note that in (2.1), for $\alpha \geqslant 1$, $\mathrm{VaR}_\alpha(X) = -\infty$ for all $X \in \mathcal{X}$. Certainly, only the case $\alpha \in [0, 1)$ is relevant in risk management; in particular, practical values of α are close to 0 in banking and insurance regulation.

For $X \in \mathcal{X}$, the Range-Value-at-Risk (RVaR) at level $(\alpha, \beta) \in \mathbb{R}_+^2$ is defined as

$$\mathrm{RVaR}_{\alpha,\beta}(X) = \begin{cases} \frac{1}{\beta}\int_\alpha^{\alpha+\beta} \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma & \text{if } \beta > 0, \\ \mathrm{VaR}_\alpha(X) & \text{if } \beta = 0. \end{cases} \tag{2.3}$$

More intuitively, $\mathrm{RVaR}_{\alpha,\beta}$ is defined as the average of quantiles between levels $1-\alpha-\beta$ and $1-\alpha$. For mathematical rigor, we set $\mathrm{RVaR}_{\alpha,\beta}(X) = -\infty$ for $X \in \mathcal{X}$ and $\alpha+\beta > 1$. Besides VaR, another special case of RVaR is the Expected Shortfall (ES), defined as

$$\mathrm{ES}_\beta(X) = \mathrm{RVaR}_{0,\beta}(X) = \frac{1}{\beta}\int_0^\beta \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma, \quad \beta \geqslant 0.$$

Different from RVaR and VaR, an ES is subadditive.

Remark 2.1 (Terminological remark)**.** There are several different conventions used in the literature of risk measures. Some papers use the convention $\mathrm{VaR}_1 = \lim_{\alpha\to 1}\mathrm{VaR}_\alpha$, which corresponds to our VaR_1^+. The convention $\mathrm{VaR}_1 = -\infty$ used in this paper and Embrechts et al. (2018) unifies the notation in several technical results when some parameters exceed 1. In different contexts, ES has various alternative names, such as AVaR (Föllmer and Schied (2016)), CVaR (Rockafellar and Uryasev (2000)) and TVaR (Denuit et al. (2005)).

A useful optimization property linking VaR and ES obtained by Rockafellar and Uryasev (2000, 2002) is, for $\beta \in (0, 1)$,

$$\mathrm{ES}_\beta(X) = \min\left\{\frac{1}{\beta}\mathbb{E}[(X-x)_+] + x : x \in \mathbb{R}\right\}, \tag{2.4}$$

and

$$[\mathrm{VaR}_\beta(X), \mathrm{VaR}_\beta^+(X)] = \arg\min\left\{\frac{1}{\beta}\mathbb{E}[(X-x)_+] + x : x \in \mathbb{R}\right\}. \tag{2.5}$$

The second parameter β in $\mathrm{RVaR}_{\alpha,\beta}$ is referred to as the *tolerance parameter*; see the discussions in Embrechts et al. (2018) after Theorem 2. RVaR was first introduced by Cont et al. (2010) featuring its robustness properties (see Embrechts et al. (2018) for more details on the family of RVaR). The RVaR family of risk measures provide a flexible and tractable framework for the study of risk sharing, including the two most practical risk measures as special cases. Following the setup of Embrechts et al. (2018), we shall focus on the RVaR family of risk measures in this paper. As far as we know, there are very few results on risk sharing problems with other non-convex distortion risk measures; see Weber (2018) for some available results.

2.2 Risk sharing and inf-convolution

Similarly to Embrechts et al. (2018), we refer to a participant in the risk sharing transactions as an *agent*, which may represent an affiliate, a firm, an insured, an insurer, or an investor in various specific contexts. Let n be a positive integer which represents the number of agents. Given random variable $X \in \mathcal{X}$, we define the set of *allocations* of X as

$$\mathbb{A}_n(X) = \left\{ (X_1, \dots, X_n) \in \mathcal{X}^n : \sum_{i=1}^n X_i = X \right\}. \tag{2.6}$$

For $i = 1, \dots, n$, agent i is equipped with a risk measure $\rho_i : \mathcal{X} \to \mathbb{R}$, which is the agent's objective to minimize. In this paper, we consider Pareto-optimal allocations defined below.

Definition 2.2 (Pareto-optimal allocations)**.** Fix any risk measures $\rho_1, \dots, \rho_n$ and the total risk $X \in \mathcal{X}$. An allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is *Pareto-optimal* with respect to $(\rho_1, \dots, \rho_n)$ if for any allocation $(Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$, $\rho_i(Y_i) \leqslant \rho_i(X_i)$ for all $i = 1, \dots, n$ implies $\rho_i(Y_i) = \rho_i(X_i)$ for all $i = 1, \dots, n$. Throughout, we shall simply call a Pareto-optimal allocation an *optimal allocation*.

To study risk sharing problems for risk measures, define the *inf-convolution* of risk measures (see e.g. Delbaen (2012) and Rüschendorf (2013)) as

$$\mathop{\square}_{i=1}^{n} \rho_i(X) = \inf \left\{ \sum_{i=1}^n \rho_i(X_i) : (X_1, \cdots, X_n) \in \mathbb{A}_n(X) \right\}, \quad X \in \mathcal{X}. \tag{2.7}$$

Our choices of $\rho_1, \dots, \rho_n$ in this paper do not take the value $-\infty$ on $\mathcal{X}$ and hence the infimum in (2.7) is well posed. It is well-known that for *monetary risk measures* (Föllmer and Schied (2016)) including the RVaR family, Pareto optimality is equivalent to optimality with respect to the sum (Proposition 1 of Embrechts et al. (2018)). More precisely, assuming that each of $\rho_i(X_i)$, $i = 1, \dots, n$ is finite, $(X_1, \dots, X_n)$ is a Pareto-optimal allocation of X if and only if

$$\sum_{i=1}^n \rho_i(X_i) = \mathop{\square}_{i=1}^{n} \rho_i(X). \tag{2.8}$$

In general, there is a one-to-one connection between Pareto optimality and (weighted) convolution for objectives other than monetary risk measures; see e.g., Mas-Colell et al. (1995, Chapter 10). Inf-convolutions are also closely related to the Riesz-Kantorovich transform in Banach lattice theory; see e.g., Aliprantis and Border (2006, Chapter 8).

In the sequel, an allocation $(X_1,\dots,X_n)$ satisfying (2.8) is called a *sum-optimal allocation*. We will omit "with respect to $(\rho_1,\dots,\rho_n)$" in most cases as long as the underlying risk measures are clear. As mentioned above, unless $\square_{i=1}^n \rho_i(X)$ is infinite, optimal allocations and sum-optimal ones are equivalent; Lemma 3.1 and Remark 3.2 below contain more discussions on the subtle cases $\square_{i=1}^n \rho_i(X) = \pm\infty$.

2.3 Existing results on optimal allocations

We first specify agents' preferences in the risk sharing problems in this paper. As these preferences will be used throughout the paper, we emphasize it in the following assumption. Throughout, for any constants $\beta_1,\dots,\beta_n \in \mathbb{R}$, write $\bigvee_{i=1}^n \beta_i = \max\{\beta_1,\dots,\beta_n\}$ and $\bigwedge_{i=1}^n \beta_i = \min\{\beta_1,\dots,\beta_n\}$.

Global Assumption. Unless otherwise specified, all optimal allocations are with respect to the risk measures $(\mathrm{RVaR}_{\alpha_1,\beta_1},\dots,\mathrm{RVaR}_{\alpha_n,\beta_n})$, where $\alpha_i \in [0,1)$, $\beta_i \in [0,1]$ and $\alpha_i+\beta_i \leqslant 1$, $i=1,\dots,n$.

We will always denote by $\alpha = \sum_{i=1}^n \alpha_i$ and $\beta = \bigvee_{i=1}^n \beta_i$. The above specification of parameters guarantees $\mathrm{RVaR}_{\alpha_i,\beta_i}(X) > -\infty$ for all $X \in \mathcal{X}$. If the assumption is not satisfied (i.e. $\alpha_i + \beta_i > 1$ or $\alpha_i = 1$), then $\mathrm{RVaR}_{\alpha_i,\beta_i}(X) = -\infty$ for all $X \in \mathcal{X}$, leading to a trivial case.

Below we summarize the main results from Embrechts et al. (2018) on optimal allocations. Let U_X be a uniform random variable on $[0,1]$ such that $F^{-1}(U_X) = X$ almost surely where F is the distribution function of the random variable X and $F^{-1}(p) = \inf\{x \in \mathbb{R}: F(x) \geqslant p\}$, $p \in (0,1)$. If X is continuously distributed, then $U_X = F(X)$. For a general random variable X, the existence of U_X is guaranteed; see, for instance, Lemma A.32 of Föllmer and Schied (2016).

Theorem 2.3 (Theorem 2 of Embrechts et al. (2018))**.** *We have*

$$\mathop{\square}_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X) = \mathrm{RVaR}_{\alpha,\beta}(X), \quad X \in \mathcal{X}. \tag{2.9}$$

Moreover, if $p := \alpha + \beta < 1$, then, assuming (without loss of generality) $\beta_n = \beta$, a sum-optimal allocation $(X_1,\dots,X_n)$ of $X \in \mathcal{X}$ is given by

$$X_i = (X-m)\,\mathbb{1}_{\{1-\sum_{k=1}^i \alpha_k < U_X \leqslant 1-\sum_{k=1}^{i-1}\alpha_k\}}, \quad i=1,\dots,n-1, \tag{2.10}$$

$$X_n = (X-m)\,\mathbb{1}_{\{U_X \leqslant 1-\sum_{k=1}^{n-1}\alpha_k\}} + m, \tag{2.11}$$

where $m \in (-\infty, \mathrm{VaR}_p(X)]$ is a constant.

Theorem 2.3 implies the following useful inequality, which is given in Theorem 1 of Embrechts et al. (2018). For all $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n \geqslant 0$ and $X_1,\dots,X_n \in \mathcal{X}$, we have

$$\sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) \geqslant \mathrm{RVaR}_{\alpha,\beta}\left(\sum_{i=1}^n X_i\right). \tag{2.12}$$

Theorem 2.3 and (2.12) will be used repeatedly in this paper. It is clear that the above results do not fully address the issue of existence, and no results on the unique forms of optimal allocations are provided. In particular, the following questions are unanswered:

(i) Theorem 2.3 implies that a sum-optimal allocation exists if $\alpha + \beta < 1$, and it does not exist if $\alpha + \beta > 1$. Under what conditions does a Pareto-optimal allocation exist?

(ii) When an optimal allocation exists, is it possible to identify all possible optimal allocations (unique form up to certain freedom)?

This paper is dedicated to complete answers to both questions above.

3 Existence of the optimal allocations

In this section, we analyze the existence of optimal allocations in a quantile-based risk sharing problem. The main results are that Pareto-optimal and sum-optimality are equivalent for RVaR, unless $\alpha = \beta = 0$, and the existence of a Pareto-optimal allocation can be characterized in four cases (A1)-(A4) below depending on the parameters $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ and the total risk X.

In the sequel, for $X \in \mathcal{X}$, we say that X is bounded from below (resp. above) if $\mathrm{VaR}_1^+(X) > -\infty$ (resp. $\mathrm{VaR}_0(X) < \infty$). The following lemma clarifies the subtle difference between optimal allocations and sum-optimal ones.

Lemma 3.1. *For $X \in \mathcal{X}$, the following hold.*

(i) If $\mathrm{RVaR}_{\alpha,\beta}(X) = \infty$, there does not exist an optimal allocation, whereas all allocations are sum-optimal.

(ii) If $-\infty < \mathrm{RVaR}_{\alpha,\beta}(X) < \infty$, an allocation is optimal if and only if it is sum-optimal.

(iii) If $\mathrm{RVaR}_{\alpha,\beta}(X) = -\infty$, there does not exist an optimal allocation or a sum-optimal allocation.

Proof. (i) Suppose that $(X_1, \dots, X_n)$ is an optimal allocation. As $\mathrm{RVaR}_{\alpha,\beta}(X) \leqslant \sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i)$, at least one of $\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i)$, $i = 1, \dots, n$ is equal to ∞. Without loss of generality, assume $\mathrm{RVaR}_{\alpha_1,\beta_1}(X_1) = \infty$. If $\mathrm{RVaR}_{\alpha_2,\beta_2}(X_2) = \infty$, then we take an allocation $(X_1 + X_2, 0, X_3 \dots, X_n) \in \mathbb{A}_n(X)$. It is clear that $\mathrm{RVaR}_{\alpha_1,\beta_1}(X_1 + X_2) \leqslant \infty = \mathrm{RVaR}_{\alpha_1,\beta_1}(X_1)$ and $\mathrm{RVaR}_{\alpha_2,\beta_2}(0) < \infty = \mathrm{RVaR}_{\alpha_2,\beta_2}(X_2)$. Hence, $(X_1, \dots, X_n)$ is not Pareto-optimal. If $\mathrm{RVaR}_{\alpha_2,\beta_2}(X_2) < \infty$, then we take an allocation $(X_1 + c, X_2 - c, X_3, \dots, X_n) \in \mathbb{A}_n(X)$ for some $c > 0$. It is clear that $\mathrm{RVaR}_{\alpha_1,\beta_1}(X_1+c) = \infty = \mathrm{RVaR}_{\alpha_1,\beta_1}(X_1)$ and $\mathrm{RVaR}_{\alpha_2,\beta_2}(X_2-c) < \mathrm{RVaR}_{\alpha_2,\beta_2}(X_2)$. Hence, $(X_1, \dots, X_n)$ is not Pareto-optimal.

On the other hand, as $\mathrm{RVaR}_{\alpha,\beta}(X) = \square_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X) = \infty$, any choice of $(X_1,\dots,X_n) \in \mathbb{A}_n(X)$ satisfies $\sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) = \infty$, and hence it is a sum-optimal allocation.

(ii) This is due to Proposition 1 of Embrechts et al. (2018).

(iii) Note that for our choices of parameters, $\mathrm{RVaR}_{\alpha_i,\beta_i}$ does not take the value $-\infty$. Suppose that $(Y_1,\cdots,Y_n)$ is an optimal allocation. Since $\mathrm{RVaR}_{\alpha,\beta}(X) = -\infty$, there exists $(X_1,\cdots,X_n) \in \mathbb{A}_n(X)$ such that $\sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) < \sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(Y_i)$. Then at least one $\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) < \mathrm{RVaR}_{\alpha_i,\beta_i}(Y_i)$. Without loss of generality, assume $\mathrm{RVaR}_{\alpha_1,\beta_1}(X_1) < \mathrm{RVaR}_{\alpha_1,\beta_1}(Y_1)$. Let $c_i = \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) - \mathrm{RVaR}_{\alpha_i,\beta_i}(Y_i)$ for $i = 2,\dots,n$. Clearly $\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i - c_i) = \mathrm{RVaR}_{\alpha_i,\beta_i}(Y_i)$, $i = 2,\dots,n$. Moreover,

$$\mathrm{RVaR}_{\alpha_1,\beta_1}\left(X_1 + \sum_{i=2}^n c_i\right) = \sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) - \sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(Y_i) + \mathrm{RVaR}_{\alpha_1,\beta_1}(Y_1)$$
$$< \mathrm{RVaR}_{\alpha_1,\beta_1}(Y_1).$$

This means that $(X_1 + \sum_{i=2}^n c_i, X_2 - c_2, \cdots, X_n - c_n) \in \mathbb{A}_n(X)$ strictly dominates $(Y_1,\cdots,Y_n)$, and the latter is not Pareto-optimal.

On the other hand, since for $i = 1,\dots,n$, $\mathrm{RVaR}_{\alpha_i,\beta_i}$ does not take the value $-\infty$, one always have, for any $(X_1,\dots,X_n) \in \mathbb{A}_n(X)$, $\sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) > -\infty = \square_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X)$. Therefore, no sum-optimal allocations exist. $\square$

Remark 3.2. The only possible difference between an optimal allocation and a sum-optimal one is case (i) in Lemma 3.1. More precisely, it corresponds to $\alpha = \beta = 0$ (implying $\alpha_1 = \cdots = \alpha_n = \beta_1 = \cdots = \beta_n = 0$) and $\mathrm{VaR}_0(X) = \infty$. As any allocation $(X_1,\dots,X_n) \in \mathbb{A}_n(X)$ is sum-optimal in this case and no Pareto-optimal allocation exists, it is not interesting for further study.

Lemma 3.1 implies that, unless $\alpha = \beta = 0$ and X is unbounded from above, there is no difference between optimal allocations and sum-optimal ones. By (2.8) and Theorem 2.3, an allocation $(X_1,\dots,X_n)$ is optimal if and only if

$$\sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) = \mathrm{RVaR}_{\alpha,\beta}(X) \quad \text{and} \quad \mathrm{RVaR}_{\alpha,\beta}(X) < \infty.$$

Below we illustrate four cases where an optimal allocation can be explicitly formulated:

(A1) $\alpha = \beta = 0$ and X is bounded from above;

(A2) $0 < \alpha + \beta < 1$;

(A3) $\alpha+\beta=1$, $\beta>0$ and X is bounded from below;

(A4) $\alpha+\beta=1$, $\beta>0$ and there exists $i\in\{1,\dots,n\}$ such that $\alpha_i=\alpha$ and $\beta_i=\beta$.

To describe the corresponding optimal allocations under (A1)-(A4), assume, without loss of generality, $\beta_n=\beta$, i.e. β_n is the largest among $\beta_1,\dots,\beta_n$.

Case (A1): A sum-optimal allocation is provided by (2.10) and (2.11) in Theorem 2.3, which is optimal by Lemma 3.1 (ii) due to $-\infty<\mathrm{RVaR}_{\alpha,\beta}(X)<\infty$.

Case (A2): Same as Case (A1).

Case (A3): Let $(X_1,\dots,X_n)$ be given by (2.10) and (2.11), where $m=\mathrm{VaR}_1^+(X)$. One can check that $\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i)=0$ for $i=1,\dots,n-1$ and $\mathrm{RVaR}_{\alpha_n,\beta_n}(X_n)=\mathrm{RVaR}_{\alpha,\beta}(X)$, and hence $\sum_{i=1}^n\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i)=\mathrm{RVaR}_{\alpha,\beta}(X)$, i.e. $(X_1,\dots,X_n)$ is a sum-optimal allocation.

Case (A4): Let $X_i=X$ and $X_j=0$ for $j\neq i$. Recall that our specification of (α_i,β_i) guarantees $\alpha_i+\beta_i\leqslant 1$ and $\alpha_i<1$; thus $\mathrm{RVaR}_{\alpha_i,\beta_i}(X)>-\infty$. We can easily see $\sum_{i=1}^n\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i)=\mathrm{RVaR}_{\alpha,\beta}(X)$, and hence $(X_1,\dots,X_n)$ is a sum-optimal allocation.

In all four cases, since $\mathrm{RVaR}_{\alpha,\beta}(X)<\infty$, sum-optimal allocations are optimal.

We briefly explain the economic meaning of the four cases (A1)-(A4) below. In (A1), all agents are using the essential supremum as their risk measure, and thus they are all extremely risk-averse and not willing to take any risk. (A2) covers the most practical situations where agents have non-zero but small parameters so that the aggregated parameter $\alpha+\beta$ is less than 1; this would include the important situation where each of the (several or dozens of) financial institutions is regulated by $\mathrm{VaR}_{0.01}$, $\mathrm{ES}_{0.02}$ or $\mathrm{ES}_{0.025}$, as specified by Basel II, III, Solvency II or the Swiss Solvency Test. (A3) and (A4) are special cases where the parameter $\alpha+\beta$ happens to hit the boundary value 1, and the inf-convolution $\mathrm{RVaR}_{\alpha,1-\alpha}$ becomes a *superadditive* risk measure (see Remark 3.5). (A3) further requires the risk X to be bounded from below. (A4) requires $\alpha_j=0$ for $j\neq i$, and in fact all other agents except for agent i are essentially not participating in the risk sharing transactions, because they have very conservative risk attitude (each of them uses an ES with a smaller tolerance parameter β_j compared to β_i). Obviously, both cases (A3) and (A4) are quite special and not very practically relevant; nevertheless, as the boundary cases, they do offer delicate mathematical features.

Next, we shall show that (A1)-(A4) are precisely the only possible cases where an optimal allocation may exist. We first present a lemma on the sum of a VaR and an ES, which may be of independent interest.

Lemma 3.3. *For $\alpha \in (0,1)$ and $X, Y \in \mathcal{X}$ such that $X+Y$ is unbounded from below, we have*

$$\mathrm{VaR}_\alpha(X) + \mathrm{ES}_{1-\alpha}(Y) > \mathrm{RVaR}_{\alpha,1-\alpha}(X+Y). \tag{3.1}$$

Proof. Since one can freely replace X by $X+c$ for any constant $c \in \mathbb{R}$ in (3.1), without loss of generality we assume $\mathrm{VaR}_\alpha(X) = 0$. It suffices to show $\mathrm{ES}_{1-\alpha}(Z-X) > \mathrm{RVaR}_{\alpha,1-\alpha}(Z)$ for all $X, Z \in \mathcal{X}$ such that Z is unbounded from below and $\mathrm{VaR}_\alpha(X) = 0$. Note that $\mathrm{ES}_{1-\alpha}(Z-X) \geqslant \mathrm{ES}_{1-\alpha}(Z-X_+)$, and $\mathrm{VaR}_\alpha(X) = 0$ can be loosened to $\mathrm{VaR}_\alpha(X) \geqslant 0$, which is equivalent to $\mathbb{P}(X>0) \leqslant \alpha$. Therefore, to prove the lemma, it suffices to show

$$\mathrm{ES}_{1-\alpha}(Z-X) > \mathrm{RVaR}_{\alpha,1-\alpha}(Z) \tag{3.2}$$

for all $X, Z \in \mathcal{X}$ such that $X \geqslant 0$, $\mathbb{P}(X>0) \leqslant \alpha$ and Z is unbounded from below.

Fix arbitrary $X, Z \in \mathcal{X}$ satisfying the above conditions. Write $Y = Z - X$ and note that $\mathbb{P}(Z = Y) \geqslant 1-\alpha$. As a consequence, for all $x \in \mathbb{R}$, $\mathbb{P}(Y \leqslant x) - \mathbb{P}(Z \leqslant x) \leqslant \alpha$. Using the above relation and the definition of VaR, for $\gamma \leqslant 1-\alpha$, we have $\mathrm{VaR}_\gamma(Y) \geqslant \mathrm{VaR}_{\gamma+\alpha}(Z)$. Also note that $\mathrm{VaR}_\gamma(Y) \geqslant \mathrm{VaR}_{1-\alpha}(Y)$. Therefore, we have

$$\begin{aligned}
\mathrm{ES}_{1-\alpha}(Y) &= \frac{1}{1-\alpha}\int_0^{1-\alpha} \mathrm{VaR}_\gamma(Y)\mathrm{d}\gamma \\
&\geqslant \frac{1}{1-\alpha}\int_0^{1-\alpha} (\mathrm{VaR}_{\gamma+\alpha}(Z) \vee \mathrm{VaR}_{1-\alpha}(Y))\mathrm{d}\gamma \\
&= \frac{1}{1-\alpha}\int_\alpha^{1} (\mathrm{VaR}_{\gamma}(Z) \vee \mathrm{VaR}_{1-\alpha}(Y))\mathrm{d}\gamma \\
&> \frac{1}{1-\alpha}\int_\alpha^{1} \mathrm{VaR}_{\gamma}(Z)\mathrm{d}\gamma = \mathrm{RVaR}_{\alpha,1-\alpha}(Z),
\end{aligned} \tag{3.3}$$

where the last inequality is due to the fact that Z is unbounded from below and $\mathrm{VaR}_{1-\alpha}(Y)$ is a constant. Therefore, (3.2) holds and the proof is complete. □

Remark 3.4. The condition that $X+Y$ is unbounded from below is essential to the statement of Lemma 3.3. In fact, from the proof of Lemma 3.3 we can see that, if Z is bounded from below, then one can choose X such that $\mathrm{VaR}_{1-\alpha}(Y)$ is small enough so that the last inequality in (3.3) is an equality, leading to $\mathrm{VaR}_\alpha(X) + \mathrm{ES}_{1-\alpha}(Y) \geqslant \mathrm{RVaR}_{\alpha,1-\alpha}(X+Y)$, a special case of the inequality (2.12).

Remark 3.5. Obtained from the definition of RVaR, for any random variable Z, one has

$$\mathrm{RVaR}_{\alpha,1-\alpha}(Z) = -\mathrm{ES}_{1-\alpha}(-Z).$$

Therefore, Lemma 3.3 is equivalent to the following statement: For any $\alpha \in (0,1)$ and $(X,Y,Z) \in \mathbb{A}_3(0)$ with Z unbounded from above,

$$\mathrm{VaR}_\alpha(X) + \mathrm{ES}_{1-\alpha}(Y) + \mathrm{ES}_{1-\alpha}(-Z) > 0.$$

With the help of Lemma 3.3, we are ready to give a full characterization of the existence of an optimal allocation. Recall that all optimal allocations, if not specified otherwise, are with respect to the RVaR risk measures specified in the Global Assumption in Section 2.3.

Theorem 3.6. *For $X \in \mathcal{X}$, an optimal allocation exists if and only if one of (A1)-(A4) holds.*

Proof. As explicitly constructed above, under each of the conditions (A1)-(A4), an optimal allocation exists. We only need to show that no optimal allocation exists when none of (A1)-(A4) holds. First, note that if $\alpha+\beta > 1$, $\alpha = 1$, or $\mathrm{RVaR}_{\alpha,\beta}(X) = \infty$, no optimal allocation may exist according to Lemma 3.1. Hence, we only need to consider the case where $\alpha + \beta = 1$ and $\beta \in (0, 1)$. As (A3) does not hold, X is unbounded from below. Furthermore, $\alpha + \beta = 1$ and (A4) does not hold, we have $\alpha_i + \beta_i < 1$ for each $i = 1, \dots, n$.

Take an arbitrary $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$. For $i = 1, \dots, n$, we assert that there exists $(Y_i, Z_i) \in \mathbb{A}_2(X_i)$ such that

$$\mathrm{VaR}_{\alpha_i}(Y_i) + \mathrm{ES}_{\beta_i}(Z_i) = \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i).$$

This assertion is shown by noticing the fact that, for the risk sharing problem of two agents with VaR_{α_i} and ES_{β_i} as their preferences, an optimal allocation always exists, because either $\beta_i = 0$ in which $(Y_i, Z_i) = (X_i, 0)$ gives the equality, or condition (A2) is satisfied for this problem (since $0 < \alpha_i + \beta_i < 1$) and Theorem 2.3 gives the explicit construction. Write $Y = \sum_{i=1}^n Y_i$ and $Z = \sum_{i=1}^n Z_i$. We have

$$\sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) = \sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(Y_i) + \sum_{i=1}^n \mathrm{ES}_{\beta_i}(Z_i) \geqslant \mathrm{VaR}_{\alpha}(Y) + \mathrm{ES}_{\beta}(Z) > \mathrm{RVaR}_{\alpha,\beta}(X),$$

where the first inequality is an application of (2.12) and the second inequality is due to Lemma 3.3 by noting that X is unbounded from below, $Y + Z = X$, and $\alpha + \beta = 1$. Therefore, no optimal allocation exists if none of (A1)-(A4) holds. □

Note that the cases (A1)-(A3) are mutually exclusive, but (A4) may overlap with (A3). To obtain mutually exclusive cases, one can replace (A4) by

(A4') $\alpha + \beta = 1$, $\beta > 0$, X is unbounded from below, and there exists $i \in \{1, \dots, n\}$ such that $\alpha_i = \alpha$ and $\beta_i = \beta$.

With this modification, Theorem 3.6 reads as, for $X \in \mathcal{X}$, an optimal allocation exists if and only if precisely one of (A1)-(A3) and (A4') holds.

4 Characterizing optimal allocations

4.1 An overview

In this section, we characterize all optimal allocations in a quantile-based risk sharing problem. We first make an intuitive statement. Due to the fact that each risk measure in the RVaR family ignores part of the distribution, one might naturally expect that the class of optimal allocations has a lot of freedom. As we shall see in this section, this is indeed the case.

We outline the key ideas behind our main results. In order to characterize optimal allocations for RVaR agents, we note the following relationship from Theorem 2 of Embrechts et al. (2018),

$$\mathop{\square}_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i} = \mathrm{RVaR}_{\alpha,\beta} = \mathrm{VaR}_\alpha \square \mathrm{ES}_\beta = \left(\mathop{\square}_{i=1}^{n} \mathrm{VaR}_{\alpha_i}\right) \square \left(\mathop{\square}_{i=1}^{n} \mathrm{ES}_{\beta_i}\right). \tag{4.1}$$

Intuitively, a risk sharing problem for RVaR agents may be decomposed into two steps: first, allocate X to $(Y,Z) \in \mathbb{A}_2(X)$ such that $\mathrm{RVaR}_{\alpha,\beta}(X) = \mathrm{VaR}_\alpha(Y) + \mathrm{ES}_\beta(Z)$, and second, allocate Y and Z to $(Y_1,\dots,Y_n) \in \mathbb{A}_n(Y)$ and $(Z_1,\dots,Z_n) \in \mathbb{A}_n(Z)$ such that $\sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(Y_i) = \mathrm{VaR}_\alpha(Y)$ and $\sum_{i=1}^n \mathrm{ES}_{\alpha_i}(Z_i) = \mathrm{ES}_\beta(Z)$. If all of the above allocations exist, then by letting $X_i = Y_i + Z_i$, $i = 1,\dots,n$, we obtain an optimal allocation for the RVaR agents. Note that the above allocations are optimal with respect to the corresponding risk sharing problems, namely, the case of one VaR and one ES agent, the case of n VaR agents, and the case of n ES agents.

Following the above plan, we analyze the special case $\alpha = \beta = 0$ in Section 4.2, the case of n VaR agents ($\beta = 0$) in Section 4.3 and the case of n ES agents ($\alpha = 0$) in Section 4.4. In Section 4.5, we study the case of one VaR agent and one ES agent. Finally, in Propositions 4.10 and 4.11 in Section 4.6, we characterize optimal allocations for RVaR agents based on the above two-step decomposition and the results obtained in Sections 4.2-4.5.

The following notation will be useful in this section. For a set $A \in \mathcal{F}$, let $\pi_n(A)$ be the set of n-partitions of A in $\mathcal{F}^n$, namely,

$$\pi_n(A) = \left\{(A_1,\dots,A_n) \in \mathcal{F}^n : \bigcup_{i=1}^n A_i = A, \text{ and } A_1,\dots,A_n \text{ are mutually disjoint}\right\}.$$

Let $\mathbb{A}_n^+(X)$ (resp. $\mathbb{A}_n^-(X)$) be the set of non-negative (resp. non-positive) allocations of a random variable X, namely,

$$\mathbb{A}_n^+(X) = \{(X_1,\dots,X_n) \in \mathbb{A}_n(X) : X_i \geqslant 0,\ i = 1,\dots,n\}, \quad X \geqslant 0,$$

and

$$\mathbb{A}_n^-(X) = \{(X_1,\dots,X_n) \in \mathbb{A}_n(X) : X_i \leqslant 0,\ i = 1,\dots,n\}, \quad X \leqslant 0.$$

For a constant x, let $\mathbb{A}_n^c(x)$ be the set of constant allocations, namely,

$$\mathbb{A}_n^c(x) = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 + \dots + x_n = x\}, \quad x \in \mathbb{R}.$$

To simplify the notation, for a specified X, we always write $y_\alpha = \mathrm{VaR}_\alpha(X)$ and $y_\alpha^+ = \mathrm{VaR}_\alpha^+(X)$ for $\alpha \in [0,1]$. For given $\alpha_1, \dots, \alpha_n \geqslant 0$, write $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$. If $y_\alpha \in \mathbb{R}$, let

$$\mathcal{Z}_n^{\boldsymbol{\alpha}} = \left\{(Z_1, \dots, Z_n) \in \mathcal{X}^n : Z_i \geqslant 0,\ \mathbb{P}(Z_i > 0) \leqslant \alpha_i,\ i = 1, \dots, n \text{ and } \sum_{i=1}^n Z_i \geqslant (X - y_\alpha)_+\right\}.$$

We can verify that $\mathcal{Z}_n^{\boldsymbol{\alpha}}$ is non-empty since $\mathbb{P}((X - y_\alpha)_+ > 0) \leqslant \alpha$.

The set $\mathcal{Z}_n^{\boldsymbol{\alpha}}$ can be explained as follows. For the sake of illustration, we assume $\mathbb{P}(X > y_\alpha) = \alpha$, which is satisfied by all X with a continuous distribution function. In this case, for any $(Z_1, \dots, Z_n) \in \mathcal{Z}_n^{\boldsymbol{\alpha}}$, we can write

$$Z_i = Z\mathbb{1}_{A_i} \quad \text{a.s.,} \quad i = 1, \dots, n$$

for some $Z \geqslant (X - y_\alpha)_+$, $(A_1, \dots, A_n) \in \pi_n(\{X > y_\alpha\})$ with $\mathbb{P}(A_i) = \alpha_i$, $i = 1, \dots, n$. Note that the random vector $(Z_1, \dots, Z_n)$ is *mutually exclusive* (or *pair-wise countermonotonic*; see Section 3.2 of Puccetti and Wang (2015)), showing a strongest form of negative dependence. Therefore, an intuitive explanation of the set $\mathcal{Z}_n^{\boldsymbol{\alpha}}$ is that each vector in $\mathcal{Z}_n^{\boldsymbol{\alpha}}$ is strongly negatively dependent with non-negative components, and their sum covers the excess loss $(X - y_\alpha)_+$. This interpretation explains the negative dependence structure in an optimal allocation for VaR agents; see Remark 4.3.

4.2 The special case of essential supremum agents

We first consider the special case where $\alpha_1 = \dots = \alpha_n = \beta_1 = \dots = \beta_n = 0$, corresponding to Case (A1) in Section 3. In this case an optimal allocation exists if and only if $y_0 < \infty$, according to Theorem 3.6. This special case is obviously the simplest, and it is treated separately since its solution form is different from any of the later, more complicated, cases.

Proposition 4.1. *Suppose that $\alpha = \beta = 0$, $X \in \mathcal{X}$, and $y_0 < \infty$. $(X_1, \dots, X_n) \in \mathcal{X}^n$ is an optimal allocation of X if and only if*

$$X_i = Y_i + c_i,\ i = 1, \dots, n \tag{4.2}$$

for some $(Y_1, \dots, Y_n) \in \mathbb{A}_n^-(X - y_0)$ and $(c_1, \dots, c_n) \in \mathbb{A}_n^c(y_0)$.

Proof. Recall that by Theorem 2.3, $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is optimal if and only if $\sum_{i=1}^n \mathrm{VaR}_0(X_i) = \mathrm{VaR}_0(X) = y_0$. It is easy to see that (4.2) defines an optimal allocation since $\sum_{i=1}^n \mathrm{VaR}_0(X_i) \leqslant \sum_{i=1}^n \mathrm{VaR}_0(c_i) = y_0$. It remains to show that any optimal allocation $(X_1, \dots, X_n)$ admits the form

(4.2). Note that $\sum_{i=1}^n \mathrm{VaR}_0(X_i) = y_0 < \infty$ implies $\mathrm{VaR}_0(X_i) < \infty$ for each $i = 1, \dots, n$. Take $c_i = \mathrm{VaR}_0(X_i)$, $i = 1, \dots, n$. It is clear that $\sum_{i=1}^n (X_i - c_i) = X - y_0$ and hence (4.2) holds by taking $Y_i = X_i - c_i$, $i = 1, \dots, n$. $\square$

The allocation in (4.2) is quite intuitive. First, as in any other cases, the constants $c_1, \dots, c_n$ are not relevant since our risk measures are all cash-invariant, meaning that shifting constants among components of an allocation does not affect its optimality. We will omit discussing $c_1, \dots, c_n$ here and in all cases below. Omitting the constants $c_1, \dots, c_n$, the requirement that $(Y_1, \dots, Y_n) \in \mathbb{A}_n^-(X - y_0)$ means that each agent takes a portion of the non-positive excess loss $X - y_0$ and this portion should not be positive, which is naturally a reasonable way to allocate a non-positive loss. We remark that a non-positive loss here means a surplus mathematically. Since our problem is invariant under constant shifts, for simplicity we call each component in a risk allocation a loss, regardless of it being positive or negative.

4.3 VaR agents

We consider the case where $\beta_1 = \cdots = \beta_n = 0$ and $\alpha \in (0, 1)$, that is, the objective of each agent is a VaR. In this case, by Theorem 3.6, an optimal allocation exists if and only if $\alpha = \sum_{i=1}^n \alpha_i$ is less than 1, i.e. (A2) holds. We introduce the following class of allocations. Let $(X_1, \dots, X_n)$ be given by

$$
\begin{aligned}
&X_i = Z_i + Y_i + c_i, \quad i = 1, \dots, n, \\
&\text{where } (Z_1, \dots, Z_n) \in \mathcal{Z}_n^{\boldsymbol{\alpha}},\ (Y_1, \dots, Y_n) \in \mathbb{A}_n^-(X - y_\alpha - \textstyle\sum_{i=1}^n Z_i) \text{ and } (c_1, \dots, c_n) \in \mathbb{A}_n^c(y_\alpha).
\end{aligned} \tag{4.3}
$$

We assert that (4.3) gives a properly defined allocation of X by verifying a few facts:

1. As we have seen above, $\mathcal{Z}_n^{\boldsymbol{\alpha}}$ is non-empty.

2. Since $\sum_{i=1}^n Z_i \geqslant (X - y_\alpha)_+$, we have $X - y_\alpha - \sum_{i=1}^n Z_i \leqslant 0$, and hence $\mathbb{A}_n^-(X - y_\alpha - \sum_{i=1}^n Z_i)$ is non-empty.

3. It is easy to see $\sum_{i=1}^n X_i = X$ for all choices of $(Z_1, \dots, Z_n)$, $(Y_1, \dots, Y_n)$ and $(c_1, \dots, c_n)$ in (4.3).

To interpret (4.3), omitting the constants $c_1, \dots, c_n$, each agent i is allocated two pieces of losses: Z_i, which covers part of the excess loss $(X - y_\alpha)_+$, and Y_i, which covers part of the non-positive loss $X - y_\alpha - \sum_{i=1}^n Z_i$. Since each agent is a VaR agent, the loss Z_i with a small probability does not contribute to the calculation of the corresponding VaR; that is, $\mathrm{VaR}_{\alpha_i}(Z_i) = 0$.

Below we show the optimality of (4.3) and that any optimal allocation of X has the form (4.3).

Theorem 4.2. *Assume $\beta = 0$ and $\alpha \in (0,1)$. For $X \in \mathcal{X}$, $(X_1, \ldots, X_n) \in \mathcal{X}^n$ is an optimal allocation of X if and only if it has the form (4.3).*

Proof. We first show the "if" part. For $i = 1, \ldots, n$, we have

$$\mathrm{VaR}_{\alpha_i}(X_i) = \mathrm{VaR}_{\alpha_i}(Z_i + Y_i + c_i) \leqslant \mathrm{VaR}_{\alpha_i}(Z_i + c_i) \leqslant c_i.$$

Therefore,

$$\sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(X_i) \leqslant \sum_{i=1}^n c_i = y_\alpha = \mathrm{VaR}_\alpha(X).$$

Using Theorem 2.3, we have

$$\sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(X_i) \leqslant \mathrm{VaR}_\alpha(X) = \mathop{\square}_{i=1}^n \mathrm{VaR}_{\alpha_i}(X) \leqslant \sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(X_i).$$

Noting that sum-optimality is equivalent to Pareto-optimality, we conclude that $(X_1, \ldots, X_n)$ is optimal.

Next we show the "only-if" part in two steps.

(i) Let $Y \in \mathcal{X}$ be such that $\mathrm{VaR}_\alpha(Y) = 0$ and $(X_1, \ldots, X_n)$ be an optimal allocation of Y such that $\mathrm{VaR}_{\alpha_i}(X_i) = 0$ for $i = 1, \ldots, n$. Write

$$X_i = \mathbb{1}_{\{X_i > 0\}} X_i + \mathbb{1}_{\{X_i \leqslant 0\}} X_i, \quad i = 1, \ldots, n.$$

Write $Z_i = X_i \mathbb{1}_{\{X_i > 0\}}$, $i = 1, \ldots, n$. Note that $\mathbb{P}(Z_i > 0) = \mathbb{P}(X_i > 0) \leqslant \alpha_i$ since $\mathrm{VaR}_{\alpha_i}(X_i) = 0$, $i = 1, \ldots, n$. We have

$$Y_+ = \left(\sum_{i=1}^n X_i\right)_+ \leqslant \left(\sum_{i=1}^n Z_i\right)_+ = \sum_{i=1}^n Z_i.$$

and $Y_i = \mathbb{1}_{\{X_i \leqslant 0\}} X_i$, $i = 1, \ldots, n$. Since $X_1 + \cdots + X_n = Y$, we have $(Y_1, \ldots, Y_n) \in \mathbb{A}_n^-(Y - \sum_{i=1}^n Z_i)$. Therefore, we have

$$X_i = Z_i + Y_i, \quad i = 1, \ldots, n$$

for some $Z_1, \ldots, Z_n$ and $Y_1, \ldots, Y_n$ satisfying $\mathbb{P}(Z_i > 0) \leqslant \alpha_i$, $i = 1, \ldots, n$, $\sum_{i=1}^n Z_i \geqslant Y_+$, and $(Y_1, \ldots, Y_n) \in \mathbb{A}_n^-(Y - \sum_{i=1}^n Z_i)$.

(ii) Let $(X_1, \ldots, X_n)$ be an optimal allocation of X. Recall the notation $y_\alpha = \mathrm{VaR}_\alpha(X)$ and we further write $x_i = \mathrm{VaR}_{\alpha_i}(X)$, $i = 1, \dots, n$. Note that by Theorem 2.3,

$$
\begin{aligned}
&(X_1, \dots, X_n) \text{ is an optimal allocation of } X \\
&\Rightarrow \sum_{i=1}^{n} \mathrm{VaR}_{\alpha_i}(X_i) = y_\alpha \\
&\Rightarrow \sum_{i=1}^{n} \mathrm{VaR}_{\alpha_i}(X_i - x_i) = y_\alpha - \sum_{i=1}^{n} x_i = \mathrm{VaR}_\alpha(X - y_\alpha) \\
&\Rightarrow (X_1 - x_1, \dots, X_n - x_n) \text{ is an optimal allocation of } X - y_\alpha.
\end{aligned}
$$

Therefore, $(X_1 - x_1, \dots, X_n - x_n)$ is an optimal allocation of $X - y_\alpha$. Observing $\mathrm{VaR}_\alpha(X - y_\alpha) = 0$ and $\mathrm{VaR}_{\alpha_i}(X_i - x_i) = 0$, $i = 1, \dots, n$, by letting $Y = X - y_\alpha$ in part (i), we obtain

$$X_i - x_i = Z_i + Y_i, \quad i = 1, \dots, n$$

where $(Z_1, \dots, Z_n) \in \mathcal{Z}_n^{\boldsymbol{\alpha}}$ and $(Y_1, \dots, Y_n) \in \mathbb{A}_n^-(X - y_\alpha - \sum_{i=1}^n Z_i)$. Therefore, $(X_1, \dots, X_n)$ has the form in (4.3). □

Assuming $\beta_1 = \dots = \beta_n = 0$, the optimal allocation (2.10)-(2.11) in Embrechts et al. (2018) is a special case of (4.3), by taking $Z_i = (X - m)\mathbb{1}_{\{1 - \sum_{k=1}^{i} \alpha_k < U_X \leqslant 1 - \sum_{k=1}^{i-1} \alpha_k\}}$, $i = 1, \dots, n$, $Y_1 = \dots = Y_{n-1} = 0$, $Y_n = (X - m)\mathbb{1}_{\{U_X \leqslant 1 - \sum_{k=1}^{n} \alpha_k\}} + m - y_\alpha$, $c_1 = \dots = c_{n-1} = 0$, and $c_n = y_\alpha$.

Remark 4.3. As explained above, each random vector in $\mathcal{Z}_n^{\boldsymbol{\alpha}}$ is mutually exclusive in the general case $\mathbb{P}(X > y_\alpha) = \alpha$. Theorem 4.2 is the first result showing that *an optimal allocation for VaR agents has to have a mutually exclusive part*, whereas in the literature (Embrechts et al. (2018, 2019)) we only know that *some* optimal allocations for VaR agents have a mutually exclusive part.

4.4 ES agents

Next, we consider the case where $\alpha_1 = \dots = \alpha_n = 0$ and $\beta > 0$, that is, the objective of each agent is an ES. In this case, by Theorem 3.6, an optimal allocation exists for $\beta \in (0, 1]$ since (A2) holds in case $\beta \in (0, 1)$ and (A4) holds in case $\beta = 1$. We introduce the following class of allocations. Let $J = \{i \in \{1, \dots, n\} : \beta_i = \beta\}$, that is, J is the set of agents with the largest tolerance parameter. If $0 < \beta < 1$, let $(X_1, \dots, X_n)$ be given by

$$
\begin{aligned}
&X_i = Z_i \mathbb{1}_{\{i \in J\}} + Y_i + c_i, \quad i = 1, \dots, n, \\
&\text{where } x \in [y_\beta, y_\beta^+],\ (Z_i)_{i \in J} \in \mathbb{A}_{\#J}^+((X - x)_+),\ (Y_1, \dots, Y_n) \in \mathbb{A}_n^-(-(x - X)_+), \\
&\qquad \text{and } (c_1, \dots, c_n) \in \mathbb{A}_n^c(x).
\end{aligned}
\tag{4.4}
$$

If $\beta = 1$, let $(X_1, \dots, X_n)$ be given by

$$
\begin{aligned}
&X_i = Z_i \mathbb{1}_{\{i \in J\}} + c_i, \quad i = 1, \dots, n, \\
&\text{where } (Z_i)_{i \in J} \in \mathbb{A}_{\#J}(X), \text{ and } (c_1, \dots, c_n) \in \mathbb{A}_n^c(0).
\end{aligned}
\tag{4.5}
$$

Intuitively, the allocations in (4.4)-(4.5) mean that the most risk-tolerant agent (or agents) picks up all the tail risk, and all other agents share the remaining non-positive part of the loss (again, up to constant shifts). Below we show the optimality of (4.4)-(4.5) and that any optimal allocation of X has the forms (4.4)-(4.5).

Theorem 4.4. *Assume $\alpha = 0$ and $\beta \in (0,1]$. For $X \in \mathcal{X}$, $(X_1,\dots,X_n) \in \mathcal{X}^n$ is an optimal allocation of X if and only if it has the form* (4.4)-(4.5).

Proof. We first show the "if" part. Let $(X_1,\dots,X_n)$ be an optimal allocation of X. If $\beta < 1$, using the VaR-ES relation (2.4), we have

$$\begin{aligned}
\sum_{i=1}^n \mathrm{ES}_{\beta_i}(X_i) &= \sum_{i\in J} \mathrm{ES}_{\beta_i}(Z_i+Y_i) + \sum_{i\in\{1,\dots,n\}\setminus J} \mathrm{ES}_{\beta_i}(Y_i) + x \\
&\leqslant \sum_{i\in J} \mathrm{ES}_{\beta_i}(Z_i) + x \\
&= \sum_{i\in J} \min_{z_i\in\mathbb{R}} \left\{ \frac{1}{\beta}\mathbb{E}[(Z_i-z_i)_+] + z_i \right\} + x \\
&= \min_{z_i\in\mathbb{R},\ i\in J} \left\{ \frac{1}{\beta}\sum_{i\in J}\mathbb{E}[(Z_i-z_i)_+] + \sum_{i\in J} z_i \right\} + x \\
&\leqslant \frac{1}{\beta}\sum_{i\in J}\mathbb{E}[(Z_i)_+] + x = \frac{1}{\beta}\mathbb{E}[(X-x)_+] + x = \mathrm{ES}_\beta(X),
\end{aligned}$$

where the last equality is because $x \in [y_\beta, y_\beta^+]$. If $\beta = 1$, then

$$\begin{aligned}
\sum_{i=1}^n \mathrm{ES}_{\beta_i}(X_i) &= \sum_{i\in J} \mathbb{E}[Z_i+Y_i+c_i] + \sum_{i\in\{1,\dots,n\}\setminus J} \mathrm{ES}_{\beta_i}(Y_i+c_i) \\
&\leqslant \sum_{i\in J}\mathbb{E}[Z_i] + \sum_{i=1}^n c_i = \mathbb{E}[(X-x)_+] + x = \mathbb{E}[X] = \mathrm{ES}_\beta(X).
\end{aligned}$$

In both cases, $(X_1, ..., X_n)$ is optimal.

Next we show the "only-if" part. Let $(X_1,\dots,X_n)$ be an optimal allocation of X. By Theorem 2.3, this means

$$\sum_{i=1}^n \mathrm{ES}_{\beta_i}(X_i) = \mathrm{ES}_\beta(X). \tag{4.6}$$

(i) First we assume $\beta \in (0,1)$. Using the VaR-ES relation (2.4), there exist $x_1,\dots,x_n \in \mathbb{R}$, such that

$$\sum_{i=1}^n \mathrm{ES}_{\beta_i}(X_i) = \sum_{i=1}^n \left(\frac{1}{\beta_i}\mathbb{E}[(X_i-x_i)_+] + x_i \right).$$

It follows that

$$
\begin{aligned}
\mathrm{ES}_\beta(X) &= \sum_{i=1}^n \left(\frac{1}{\beta_i}\mathbb{E}[(X_i - x_i)_+] + x_i\right) \\
&\geqslant \sum_{i=1}^n \left(\frac{1}{\beta}\mathbb{E}[(X_i - x_i)_+] + x_i\right) \\
&= \frac{1}{\beta}\mathbb{E}\left[\sum_{i=1}^n (X_i - x_i)_+\right] + \sum_{i=1}^n x_i \\
&\geqslant \frac{1}{\beta}\mathbb{E}\left[\left(\sum_{i=1}^n (X_i - x_i)\right)_+\right] + \sum_{i=1}^n x_i \\
&= \frac{1}{\beta}\mathbb{E}\left[\left(X - \sum_{i=1}^n x_i\right)_+\right] + \sum_{i=1}^n x_i \geqslant \min_{x\in\mathbb{R}}\left\{\frac{1}{\beta}\mathbb{E}\left[(X-x)_+\right] + x\right\} = \mathrm{ES}_\beta(X).
\end{aligned}
$$

Therefore, the three inequalities above are all equalities, namely

$$
\sum_{i=1}^n \left(\frac{1}{\beta_i}\mathbb{E}[(X_i - x_i)_+] + x_i\right) = \sum_{i=1}^n \left(\frac{1}{\beta}\mathbb{E}[(X_i - x_i)_+] + x_i\right), \tag{4.7}
$$

$$
\frac{1}{\beta}\mathbb{E}\left[\sum_{i=1}^n (X_i - x_i)_+\right] + \sum_{i=1}^n x_i = \frac{1}{\beta}\mathbb{E}\left[\left(X - \sum_{i=1}^n x_i\right)_+\right] + \sum_{i=1}^n x_i, \tag{4.8}
$$

and

$$
\frac{1}{\beta}\mathbb{E}\left[\left(X - \sum_{i=1}^n x_i\right)_+\right] + \sum_{i=1}^n x_i = \min_{x\in\mathbb{R}}\left\{\frac{1}{\beta}\mathbb{E}\left[(X-x)_+\right] + x\right\}. \tag{4.9}
$$

Note that the equalities of expectations in (4.7) and (4.8) are indeed almost surely point-wise equality.

Next, write $x = \sum_{i=1}^n x_i$, $Z_i = (X_i - x_i)_+$ and $Y_i = -(x_i - X_i)_+$ for $i = 1, \dots, n$. Recall that we treat almost equal random variables as identical. By (4.7), we have $Z_i = (X_i - x_i)_+ = 0$ for each $i \notin J$. By (4.8), we have,

$$
\sum_{i\in J} Z_i = \sum_{i\in J}(X_i - x_i)_+ = \sum_{i=1}^n (X_i - x_i)_+ = \left(X - \sum_{i=1}^n x_i\right)_+ = (X - x)_+.
$$

Consequently, $(Z_i)_{i\in J} \in \mathbb{A}^+_{\#J}((X-x)_+)$. Since $\sum_{i=1}^n X_i = X$, we have

$$
\sum_{i=1}^n Y_i = X - \sum_{i=1}^n Z_i - x = X - x - (X-x)_+ = -(x-X)_+,
$$

which gives $(Y_1, \dots, Y_n) \in \mathbb{A}^-_n(-(x-X)_+)$. By (4.9) and using the VaR-ES relation (2.5), we have $x \in [y_\beta, y_\beta^+]$. Note that $X_i = (X_i - x_i)_+ - (x_i - X_i)_+ + x_i = Z_i + Y_i + x_i$ for $i = 1, \dots, n$ and $Z_i = 0$ for $i \notin J$. Therefore, $(X_1, \dots, X_n)$ has the form (4.4).

(ii) Assume $\beta = 1$. Equation (4.6) reads as

$$\sum_{i=1}^{n} \mathrm{ES}_{\beta_i}(X) = \mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i].$$

Note that for $Y \in \mathcal{X}$ and $\gamma \in [0,1)$, $\mathrm{ES}_\gamma(Y) \geqslant \mathbb{E}[Y]$ holds, and $\mathrm{ES}_\gamma(Y) = \mathbb{E}[Y]$ if and only if Y is a constant. Therefore, X_i is a constant for all $i \notin J$. This leads to the conclusion that $(X_1, \dots, X_n)$ has the form (4.5). □

Remark 4.5. It is trivial to observe that, if there is only one agent whose tolerance parameter is the largest, that is, $\#J = 1$, then $(Z_1, \dots, Z_n)$ is mutually exclusive. Combined with the observation in Remark 4.3, in both the case of ES agents and that of VaR agents, $(X_1, \dots, X_n)$ is mutually exclusive on an event that X is large. We will continue discussing this phenomenon in Section 6.

4.5 One VaR agent and one ES agent

We move on to consider the combined case of one VaR agent and one ES agent. For this purpose, assume $n = 2$, $\alpha_1 > 0$, $\beta_1 = \alpha_2 = 0$, and $\beta_2 > 0$. Recall that $\alpha = \alpha_1$ and $\beta = \beta_2$. According to Theorem 3.6, for a fixed $X \in \mathcal{X}$, there are two cases where an optimal allocation exists: either (A2) $\alpha + \beta < 1$ or (A3) $\alpha + \beta = 1$ and X is bounded from below. In both cases, $y^+_{\alpha+\beta} > -\infty$. To characterize all optimal allocations, we define the following set

$$\mathcal{A}_{\alpha,\beta} = \{A \in \mathcal{F} : \{X > y_\alpha\} \subset A,\ \mathbb{P}(A) = \alpha;\ \text{moreover, } A \subset \{X \geqslant y_\alpha\} \text{ if } y^+_{\alpha+\beta} \neq y_\alpha\}.$$

In the above notation we omit the reliance on X, which should be clear throughout this section. It is easy to see that $\mathcal{A}_{\alpha,\beta}$ is non-empty as $\mathbb{P}(X > y_\alpha) \leqslant \alpha \leqslant \mathbb{P}(X \geqslant y_\alpha)$.

A set A in $\mathcal{A}_{\alpha,\beta}$ represents an event of probability α on which X takes the largest values. It is clear that, $A = \{X > y_\alpha\}$ if $\mathbb{P}(X > y_\alpha) = \alpha$, and $\{X > y_\alpha\} \subset A \subset \{X \geqslant y_\alpha\}$ if $y^+_{\alpha+\beta} < y_\alpha$. A small complication arises when $\mathbb{P}(X > y_\alpha) \neq \alpha$ and $y^+_{\alpha+\beta} = y_\alpha$, in which case $A \setminus \{X > y_\alpha\}$ can be arbitrary as long as $\mathbb{P}(A) = \alpha$. The reason for this complication can be seen from the proof of Lemma 4.6, where an optimization for ES relies on the set $\mathcal{A}_{\alpha,\beta}$. For risk management practice such a special case is irrelevant; it is included in our main results for the completeness of this study.

We first present two lemmas useful in characterizing the optimal allocations, and they may be of independent interest in optimizing ES.

Lemma 4.6. *For any $X \in \mathcal{X}$, $\alpha > 0$ and $\beta > 0$ with $\alpha + \beta \leqslant 1$ and $y^+_{\alpha+\beta} > -\infty$, $(Y, B) \in \mathcal{X} \times \mathcal{F}$ is a solution to the problem*

$$\text{to minimize } \ \mathrm{ES}_\beta(X - Y\mathbb{1}_B) \ \text{ subject to } B \in \mathcal{F},\ \mathbb{P}(B) = \alpha \text{ and } Y \in \mathcal{X}, \tag{4.10}$$

if and only if $B \in \mathcal{A}_{\alpha,\beta}$ *and* $Y\mathbb{1}_B \geqslant (X - y^+_{\alpha+\beta})\mathbb{1}_B$. *Moreover, the minimum of* (4.10) *is* $\mathrm{RVaR}_{\alpha,\beta}(X)$.

Proof. We first show the "if" part. Note that by Theorem 2.3, for any $B \in \mathcal{F}$, $\mathbb{P}(B) = \alpha$ and $Y \in \mathcal{X}$,

$$\mathrm{RVaR}_{\alpha,\beta}(X) \leqslant \mathrm{VaR}_\alpha(Y\mathbb{1}_B) + \mathrm{ES}_\beta(X - Y\mathbb{1}_B) \leqslant \mathrm{ES}_\beta(X - Y\mathbb{1}_B).$$

Suppose $B \in \mathcal{A}_{\alpha,\beta}$ and $Y\mathbb{1}_B \geqslant (X - y^+_{\alpha+\beta})\mathbb{1}_B$. We have

$$\begin{aligned}\mathrm{ES}_\beta(X - Y\mathbb{1}_B) \leqslant \mathrm{ES}_\beta(X - (X - y^+_{\alpha+\beta})\mathbb{1}_B) &= \frac{1}{\beta}\int_0^\beta \mathrm{VaR}_\alpha(X - (X - y^+_{\alpha+\beta})\mathbb{1}_B)\mathrm{d}\gamma \\ &= \frac{1}{\beta}\int_\alpha^{\alpha+\beta} \mathrm{VaR}_{\gamma-\alpha}(X - (X - y^+_{\alpha+\beta})\mathbb{1}_B)\mathrm{d}\gamma.\end{aligned}$$

In both the case $y^+_{\alpha+\beta} < y_\alpha$ and the case $y^+_{\alpha+\beta} = y_\alpha$, we have $\mathrm{VaR}_{\gamma-\alpha}(X - (X - y^+_{\alpha+\beta})\mathbb{1}_B) \leqslant \mathrm{VaR}_\gamma(X)$ holds for $\gamma \in [\alpha, \alpha+\beta)$. Hence, $\mathrm{ES}_\beta(X - Y\mathbb{1}_B) \leqslant \mathrm{RVaR}_{\alpha,\beta}(X)$. This shows that (Y, B) satisfying $B \in \mathcal{A}_{\alpha,\beta}$ and $Y\mathbb{1}_B \geqslant (X - y^+_{\alpha+\beta})\mathbb{1}_B$ minimizes (4.10). Moreover, the corresponding minimum is $\mathrm{ES}_\beta(X - Y\mathbb{1}_B) = \mathrm{RVaR}_{\alpha,\beta}(X)$.

We next show the "only-if" direction. Suppose that (Y, B) is such that $\mathrm{ES}_\beta(X - Y\mathbb{1}_B) = \mathrm{RVaR}_{\alpha,\beta}(X)$, namely,

$$\int_\alpha^\beta \mathrm{VaR}_{\gamma-\alpha}(X - Y\mathbb{1}_B)\mathrm{d}\gamma = \int_\alpha^{\alpha+\beta} \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma. \tag{4.11}$$

Observe that for $\gamma \in (\alpha, \alpha+\beta)$,

$$\mathrm{VaR}_{\gamma-\alpha}(X - Y\mathbb{1}_B) \geqslant \mathrm{VaR}_{\gamma-\alpha}(X - Y\mathbb{1}_B) + \mathrm{VaR}_\alpha(Y\mathbb{1}_B) \geqslant \mathrm{VaR}_{\gamma-\alpha}\square\mathrm{VaR}_\alpha(X) = \mathrm{VaR}_\gamma(X).$$

To make (4.11) hold, we need $\mathrm{VaR}_{\gamma-\alpha}(X - Y\mathbb{1}_B) = \mathrm{VaR}_\gamma(X)$ for $\gamma \in (\alpha, \alpha+\beta)$ a.e. By the right-continuity of the left-quantile (VaR), this requires

$$\mathrm{VaR}_{\gamma-\alpha}(X - Y\mathbb{1}_B) = \mathrm{VaR}_\gamma(X) \tag{4.12}$$

holds for $\gamma \in [\alpha, \alpha+\beta)$.

Suppose for the purpose of contradiction that $\mathbb{P}(Y < X - y^+_{\alpha+\beta}|B) > 0$. Then,

$$\mathbb{P}(X - Y\mathbb{1}_B > y^+_{\alpha+\beta}) > \mathbb{P}(X - (X - y^+_{\alpha+\beta})\mathbb{1}_B > y^+_{\alpha+\beta}).$$

As a consequence, there exists some $\gamma \in (\alpha, \alpha+\beta)$ such that $\mathrm{VaR}_{\gamma-\alpha}(X - Y\mathbb{1}_B) > \mathrm{VaR}_{\gamma-\alpha}(X - (X - y^+_{\alpha+\beta})\mathbb{1}_B)$. It follows that

$$\begin{aligned}\mathrm{VaR}_{\gamma-\alpha}(X - Y\mathbb{1}_B) &> \mathrm{VaR}_{\gamma-\alpha}\left(X - (X - y^+_{\alpha+\beta})\mathbb{1}_B\right) \\ &\geqslant \mathrm{VaR}_{\gamma-\alpha}\left(X - (X - y^+_{\alpha+\beta})\mathbb{1}_B\right) + \mathrm{VaR}_\alpha\left((X - y^+_{\alpha+\beta})\mathbb{1}_B\right) \geqslant \mathrm{VaR}_\gamma(X),\end{aligned}$$

contradicting (4.12). Therefore, we have $\mathbb{P}(Y < X - y^+_{\alpha+\beta}|B) = 0$, namely, $Y\mathbb{1}_B \geqslant (X - y^+_{\alpha+\beta})\mathbb{1}_B$.

Next we show $B \in \mathcal{A}_{\alpha,\beta}$. Note that we treat two sets as equal if the difference of the two sets is of measure zero. Equation (4.12) implies $\mathbb{P}(X - Y\mathbb{1}_B \leqslant \mathrm{VaR}_\gamma(X)) \geqslant 1 - \gamma + \alpha$. Taking $\gamma = \alpha$, we have $\mathbb{P}(X - Y\mathbb{1}_B \leqslant \mathrm{VaR}_\alpha(X)) = 1$. This implies $\{X > \mathrm{VaR}_\alpha\} \subset B$.

It remains to show $B \subset \{X \geqslant \mathrm{VaR}_\alpha(X)\}$ if $y^+_{\alpha+\beta} \neq y_\alpha$. Take $(Y^*, B^*) \in \mathcal{X} \times \mathcal{F}$ such that $B^* \in \mathcal{A}_{\alpha,\beta}$ and $Y^*\mathbb{1}_{B^*} \geqslant (X - y^+_{\alpha+\beta})\mathbb{1}_{B^*}$. From the first part of the proof, we know that (Y^*, B^*) minimizes (4.10). Since (Y, B) also minimizes (4.10), by (4.12), we know

$$\mathrm{VaR}_{\gamma-\alpha}(X - Y\mathbb{1}_B) = \mathrm{VaR}_{\gamma-\alpha}(X - Y^*\mathbb{1}_{B^*}) = \mathrm{VaR}_\gamma(X),$$

for $\gamma \in [\alpha, \alpha + \beta)$. Since $y^+_{\alpha+\beta} < y_\alpha$, the above equation implies

$$\mathbb{P}(X - Y\mathbb{1}_B \geqslant y_\alpha) = \mathbb{P}(X - Y^*\mathbb{1}_{B^*} \geqslant y_\alpha). \tag{4.13}$$

Since $Y\mathbb{1}_B \geqslant (X - y^+_{\alpha+\beta})\mathbb{1}_B$ and $Y^*\mathbb{1}_{B^*} \geqslant (X - y^+_{\alpha+\beta})\mathbb{1}_{B^*}$, $\mathbb{P}(X - Y \geqslant y_\alpha, B) = \mathbb{P}(X - Y^* \geqslant y_\alpha, B^*) = 0$. Using this relation and noting that $B^* \subset \{X \geqslant y_\alpha\}$, (4.13) implies

$$\mathbb{P}(X \geqslant y_\alpha, B^c) = \mathbb{P}(X \geqslant y_\alpha, (B^*)^c) = \mathbb{P}(X \geqslant y_\alpha) - \mathbb{P}(B^*) = \mathbb{P}(X \geqslant y_\alpha) - \alpha = \mathbb{P}(X \geqslant y_\alpha) - \mathbb{P}(B).$$

Therefore, $B \subset \{X \geqslant y_\alpha\}$. This shows $B \in \mathcal{A}_{\alpha,\beta}$. □

Lemma 4.7. *For any $X, Y \in \mathcal{X}$ with $Y \geqslant 0$ and $\beta \in (0, 1)$, $\mathrm{ES}_\beta(X + Y) = \mathrm{ES}_\beta(X)$ if and only if $Y \leqslant (\mathrm{VaR}^+_\beta(X) - X)_+$.*

Proof. We first show the "if" direction. Note that

$$X + Y \leqslant X + (\mathrm{VaR}^+_\beta(X) - X)_+ = X \vee \mathrm{VaR}^+_\beta(X).$$

It is easy to see, for $\gamma \in (0, \beta)$, that

$$\mathrm{VaR}_\gamma(X \vee \mathrm{VaR}^+_\beta(X)) = \mathrm{VaR}_\gamma(X).$$

Therefore, $\mathrm{ES}_\beta(X) = \mathrm{ES}_\beta(X \vee \mathrm{VaR}^+_\beta(X)) \geqslant \mathrm{ES}_p(X + Y)$, which implies $\mathrm{ES}_\beta(X) = \mathrm{ES}_\beta(X + Y)$.

Next we show the "only-if" direction. By the fact that

$$\mathrm{ES}_\beta(X) = \frac{1}{\beta}\int_0^\beta \mathrm{VaR}^+_\gamma(X)\mathrm{d}\gamma = \frac{1}{\beta}\int_0^\beta \mathrm{VaR}^+_\gamma(X + Y)\mathrm{d}\gamma = \mathrm{ES}_\beta(X + Y),$$

and $\mathrm{VaR}^+_\gamma(X) \leqslant \mathrm{VaR}^+_\gamma(X + Y)$ for all $\gamma \in (0, \beta]$, we have $\mathrm{VaR}^+_\gamma(X) = \mathrm{VaR}^+_\gamma(X + Y)$ a.e. on $(0, \beta)$. Since $\mathrm{VaR}^+_\gamma(Z)$ is left-continuous in γ for any fixed $Z \in \mathcal{X}$, $\mathrm{VaR}^+_\gamma(X) = \mathrm{VaR}^+_\gamma(X + Y)$ holds for all $\gamma \in (0, \beta]$. Using the VaR-ES relation (2.4), it follows that

$$\mathbb{E}\left[\left(X - \mathrm{VaR}^+_\beta(X)\right)_+\right] = \mathbb{E}\left[\left(X + Y - \mathrm{VaR}^+_\beta(X)\right)_+\right]. \tag{4.14}$$

Since $Y \geqslant 0$, (4.14) means

$$\left(X - \mathrm{VaR}^+_\beta(X)\right)_+ = \left(X + Y - \mathrm{VaR}^+_\beta(X)\right)_+,$$

and therefore $Y \leqslant (\mathrm{VaR}^+_\beta(X) - X)_+$. $\square$

Now we are ready to characterize the optimal allocations in the setting of this section. Let (X_1, X_2) be given by

$$\begin{aligned} &X_1 = Y\mathbb{1}_B - Z + c, \quad X_2 = X - X_1 \\ &\text{where } B \in \mathcal{A}_{\alpha,\beta},\ Y \geqslant X - y^+_{\alpha+\beta},\ 0 \leqslant Z \leqslant (y^+_{\alpha+\beta} - X + Y\mathbb{1}_B)_+, \text{ and } c \in \mathbb{R}. \end{aligned} \tag{4.15}$$

Intuitively, the allocation (4.15) means that the VaR agent picks up all the tail risk, and this tail risk is measured 0 by VaR_α. This leads to the regulatory arbitrage of VaR as discussed by Wang (2016) and Embrechts et al. (2018).

Theorem 4.8. *Assume $\alpha_1 > 0$, $\beta_1 = \alpha_2 = 0$, $\beta_2 > 0$, and either (A2) or (A3) holds. For $X \in \mathcal{X}$, $(X_1, X_2) \in \mathcal{X}^2$ is an optimal allocation of X if and only if it has the form (4.15).*

Proof. Obviously, the constant c does not matter in terms of the optimality of (X_1, X_2), and we set $c = 0$ for simplicity.

We first show that (4.15) gives an optimal allocation. It is easy to verify that

$$\mathrm{VaR}_\alpha(X_1) = \mathrm{VaR}_\alpha(Y\mathbb{1}_B - Z) \leqslant \mathrm{VaR}_\alpha(Y\mathbb{1}_B) = 0.$$

Using Lemmas 4.6 and 4.7, and noting that $\mathrm{VaR}^+_\beta(X - Y\mathbb{1}_B) = y^+_{\alpha+\beta}$ as implied by (4.12), we have

$$\mathrm{ES}_\beta(X_2) = \mathrm{ES}_\beta(X - Y\mathbb{1}_B + Z) = \mathrm{ES}_\beta(X - Y\mathbb{1}_B) = \mathrm{RVaR}_{\alpha,\beta}(X).$$

Therefore, (X_1, X_2) is an optimal allocation.

Next, suppose that (X_1, X_2) is an optimal allocation. Without loss of generality, assume $\mathrm{VaR}_\alpha(X_1) = 0$, which implies $\mathbb{P}(X_1 \geqslant 0) \geqslant \alpha \geqslant \mathbb{P}(X_1 > 0)$. Therefore, there exists $B \in \mathcal{F}$ such that $\{X_1 \geqslant 0\} \subset B \subset \{X_1 > 0\}$ with $\mathbb{P}(B) = \alpha$. Write $X_1 = X_1\mathbb{1}_B - Z$ where $Z = -X_1\mathbb{1}_{B^c}$. Note that $Z = -X_1\mathbb{1}_{B^c} \geqslant 0$. Since (X_1, X_2) is an optimal allocation, we know

$$\mathrm{RVaR}_{\alpha,\beta}(X) = \mathrm{VaR}_\alpha(X_1) + \mathrm{ES}_\beta(X_2) = \mathrm{ES}_\beta(X - X_1\mathbb{1}_B + Z) \geqslant \mathrm{ES}_\beta(X - X_1\mathbb{1}_B).$$

From Lemma 4.6, we know that (X_1, B) minimizes (4.10). The results of Lemma 4.6 imply $B \in \mathcal{A}_{\alpha,\beta}$, $X_1\mathbb{1}_B \geqslant (X - y^+_{\alpha+\beta})\mathbb{1}_B$, and

$$\mathrm{RVaR}_{\alpha,\beta}(X) = \mathrm{ES}_\beta(X - X_1\mathbb{1}_B) = \mathrm{ES}_\beta(X - X_1\mathbb{1}_B + Z).$$

Using Lemma 4.7, we have $Z \leqslant (y^+_{\alpha+\beta} - X + X_1 \mathbb{1}_B)_+$. Let $Y = X_1 \mathbb{1}_B + (X - y^+_{\alpha+\beta})\mathbb{1}_{B^c}$. It is clear that $Y\mathbb{1}_B = X_1\mathbb{1}_B$, $Y \geqslant X - y^+_{\alpha+\beta}$. Therefore, $X_1 = X_1\mathbb{1}_B + X_1\mathbb{1}_{B^c} = Y\mathbb{1}_B - Z$, which has the form (4.15). $\square$

Remark 4.9. Asimit et al. (2013, Theorem 2.2) studied the risk sharing problem of ES and VaR under the condition of comonotonicity, which is different from Theorem 4.8 where the optimal allocation is generally not comonotonic such as the one given in Theorem 2.3. This also distinguishes our framework to many other papers in the literature of optimal (re)insurance, where comonotonicity is important as either an assumption or a result; see e.g., Cai et al. (2016). Another technical difference is that allocations in optimal (re)insurance problems (representing insurance contracts) are commonly assumed non-negative.

4.6 RVaR agents

Finally, based on the results in Sections 4.2-4.5, we are able to present some general result for the case of RVaR agents. The main idea here is that, for each $i = 1, \dots, n$, we write $\mathrm{RVaR}_{\alpha_i,\beta_i} = \mathrm{VaR}_{\alpha_i} \square \mathrm{ES}_{\beta_i}$, and reduce the risk sharing problem to the above studied cases. We summarize this methodology in the following proposition. Since we need to translate between different cases, below we emphasize the risk measures with respect to which we speak of optimality. The case (A4') requires a special treatment which will be discussed later.

Proposition 4.10. *Assume (A4') does not hold.* $(X_1, \dots, X_n)$ *is an optimal allocation of* $X \in \mathcal{X}$ *with respect to (w.r.t.)* $(\mathrm{RVaR}_{\alpha_1,\beta_1}, \dots, \mathrm{RVaR}_{\alpha_n,\beta_n})$ *if and only if there exist an optimal allocation* (Y, Z) *of* X *w.r.t.* $(\mathrm{VaR}_\alpha, \mathrm{ES}_\beta)$*, an optimal allocation* $(Y_1, \dots, Y_n)$ *of* Y *w.r.t.* $(\mathrm{VaR}_{\alpha_1}, \dots, \mathrm{VaR}_{\alpha_n})$*, and an optimal allocation* $(Z_1, \dots, Z_n)$ *of* Z *w.r.t.* $(\mathrm{ES}_{\beta_1}, \dots, \mathrm{ES}_{\beta_n})$*, such that*

$$X_i = Y_i + Z_i, \quad i = 1, \dots, n.$$

Proof. We first show the "if" part. From the construction of X, it is easy to calculate

$$\begin{aligned}
\sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) &= \sum_{i=1}^n \mathrm{VaR}_{\alpha_i} \square \mathrm{ES}_{\beta_i}(X_i) \\
&\leqslant \sum_{i=1}^n (\mathrm{VaR}_{\alpha_i}(Y_i) + \mathrm{ES}_{\beta_i}(Z_i)) \\
&= \sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(Y_i) + \sum_{i=1}^n \mathrm{ES}_{\beta_i}(Z_i) \\
&= \mathrm{VaR}_\alpha(Y) + \mathrm{ES}_\beta(Z) = \mathrm{RVaR}_{\alpha,\beta}(X).
\end{aligned}$$

Therefore, $(X_1, \dots, X_n)$ is an optimal allocation of X with respect to $(\mathrm{RVaR}_{\alpha_1,\beta_1}, \dots, \mathrm{RVaR}_{\alpha_n,\beta_n})$. Next we show the "only-if" part. Suppose that $(X_1, \dots, X_n)$ is an optimal allocation of $\mathcal{X}$ with respect to $(\mathrm{RVaR}_{\alpha_1,\beta_1}, \dots, \mathrm{RVaR}_{\alpha_n,\beta_n})$.

Since (A4') does not hold, it is easy to see from the existence of the optimal allocation and Theorem 3.6 that for each $i = 1, \dots, n$, $\alpha_i + \beta_i < 1$ and $\mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) \neq \infty$. As a consequence, for each $i = 1, \dots, n$, we can use Theorem 3.6 on X_i to conclude that there exists $(Y_i, Z_i) \in \mathbb{A}_2(X_i)$ such that $\mathrm{VaR}_{\alpha_i}(Y_i) + \mathrm{ES}_{\beta_i}(Z_i) = \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i)$. Write $Y = \sum_{i=1}^n Y_i$ and $Z = \sum_{i=1}^n Z_i$. Clearly $Y + Z = X$. It follows that

$$
\begin{aligned}
\mathrm{RVaR}_{\alpha,\beta}(X) &= \sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) \\
&= \sum_{i=1}^n \mathrm{VaR}_{\alpha_i} \square \mathrm{ES}_{\beta_i}(X_i) \\
&= \sum_{i=1}^n (\mathrm{VaR}_{\alpha_i}(Y_i) + \mathrm{ES}_{\beta_i}(Z_i)) \\
&= \sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(Y_i) + \sum_{i=1}^n \mathrm{ES}_{\beta_i}(Z_i) \geqslant \mathrm{VaR}_{\alpha}(Y) + \mathrm{ES}_{\beta}(Z) \geqslant \mathrm{RVaR}_{\alpha,\beta}(X),
\end{aligned}
$$

where the two inequalities are due to Theorem 2.3. Noting that

$$
\mathrm{RVaR}_{\alpha,\beta}(X) \geqslant \mathrm{VaR}_{\alpha}(Y) + \mathrm{ES}_{\beta}(Z) \geqslant \mathrm{RVaR}_{\alpha,\beta}(X),
$$

the inequalities herein are equalities. Therefore $\sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(Y_i) = \mathrm{VaR}_{\alpha}(Y)$, $\sum_{i=1}^n \mathrm{ES}_{\beta_i}(Z_i) = \mathrm{ES}_{\beta}(Z)$, and $\mathrm{VaR}_{\alpha}(Y) + \mathrm{ES}_{\beta}(Z) = \mathrm{RVaR}_{\alpha,\beta}(X)$. In other words, (Y, Z) is an optimal allocation of X with respect to $(\mathrm{VaR}_{\alpha}, \mathrm{ES}_{\beta})$, $(Y_1, \dots, Y_n)$ is an optimal allocation of Y with respect to $(\mathrm{VaR}_{\alpha_1}, \dots, \mathrm{VaR}_{\alpha_n})$, and $(Z_1, \dots, Z_n)$ is an optimal allocation of Z with respect to $(\mathrm{ES}_{\beta_1}, \dots, \mathrm{ES}_{\beta_n})$. $\square$

The reason why case (A4') requires a special treatment can also be seen from the proof. A key step in the proof is to write $X_i = Y_i + Z_i$ where $(Y_i, Z_i) \in \mathbb{A}_2(X_i)$ satisfies $\mathrm{VaR}_{\alpha_i}(Y_i) + \mathrm{ES}_{\beta_i}(Z_i) = \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i)$. If (A4') holds, then such (Y_i, Z_i) may not exist, as shown in Lemma 3.3. Below we analyze the case of (A4'), which is different from all other cases. Recall that, (A4') implies that there exists $j \in \{1, \dots, n\}$ such that $\alpha_j = \alpha$ and $\beta_j = \beta = 1 - \alpha$.

Proposition 4.11. *Assume that (A4') holds, and without loss of generality, $\alpha_n = \alpha$ and $\beta_n = \beta$. Then, $(X_1, \dots, X_n)$ is an optimal allocation of $X \in \mathcal{X}$ with respect to $(\mathrm{RVaR}_{\alpha_1,\beta_1}, \dots, \mathrm{RVaR}_{\alpha_n,\beta_n})$ if and only if $(X_1, \dots, X_{n-1}, -X)$ is an optimal allocation of $-X_n$ with respect to $(\mathrm{ES}_{\beta_1}, \dots, \mathrm{ES}_{\beta_n})$.*

Proof. We shall use the fact that, for all $Y \in \mathcal{X}$,

$$\mathrm{RVaR}_{\alpha,\beta}(Y) = \mathrm{RVaR}_{\alpha,1-\alpha}(Y) = -\mathrm{ES}_{1-\alpha}(-Y) = -\mathrm{ES}_{\beta_n}(-Y), \tag{4.16}$$

which is immediate from the definition of RVaR. Note that for any $(X_1, \ldots, X_n) \in \mathcal{X}^n$,

$$\sum_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) = \sum_{i=1}^{n-1} \mathrm{ES}_{\beta_i}(X_i) + \mathrm{RVaR}_{\alpha,1-\alpha}(X_n) = \sum_{i=1}^{n-1} \mathrm{ES}_{\beta_i}(X_i) - \mathrm{ES}_{\beta_n}(-X_n). \tag{4.17}$$

By (4.16) and (4.17), the equality

$$\sum_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) = \mathrm{RVaR}_{\alpha,\beta}(X) \tag{4.18}$$

is equivalent to

$$\sum_{i=1}^{n-1} \mathrm{ES}_{\beta_i}(X_i) - \mathrm{ES}_{\beta_n}(-X_n) = -\mathrm{ES}_{\beta_n}(-X).$$

Rearranging terms, it is

$$\sum_{i=1}^{n-1} \mathrm{ES}_{\beta_i}(X_i) + \mathrm{ES}_{\beta_n}(-X) = \mathrm{ES}_{\beta_n}(-X_n). \tag{4.19}$$

As (4.18) is equivalent to (4.19), the proposition holds. $\square$

Before ending this section, we remark that, although we are able to translate the general case of RVaR agents to the completely characterized cases in Sections 4.2-4.5, we were not able to write down an elegant unifying form of the optimal allocations, due to the complications raised in the two-step characterization in Proposition 4.10.

5 Risk sharing for VaR-type risk measures

In this section, we discuss how the techniques developed in Section 4 can be applied to more general risk measures outside the RVaR family. In particular, we consider the VaR-type risk measures as studied by Weber (2018). To avoid cases of $\infty - \infty$, we choose the underlying space $\mathcal{Y}$ as the set of bounded random variables as in Weber (2018). Our main result in this section is an explicit formula of the inf-convolution of VaR and another distortion risk measure, which will be useful in constructing optimal allocations for VaR-type risk measures. Nevertheless, a full characterization of optimal allocations is not yet available and it requires future research.

We first give the necessary definitions.

Definition 5.1. (i) A *distortion function* g is a left-continuous and non-decreasing function on $[0, 1]$ with $g(0) = 0$ and $g(1) = 1$. We denote by $\mathcal{G}$ the set of distortion functions.

(ii) A *distortion risk measure* ρ_g on $\mathcal{Y}$ with distortion function g is defined as the Choquet integral

$$\rho_g(X) = \int X \mathrm{d}(g \circ \mathbb{P}) = \int_{-\infty}^{0} (g \circ \mathbb{P}(X > x) - 1)\mathrm{d}x + \int_{0}^{\infty} g \circ \mathbb{P}(X > x)\mathrm{d}x, \quad X \in \mathcal{Y}. \tag{5.1}$$

(iii) For a distortion function g, the number $\alpha = \sup\{t \in [0,1] : g(t) = 0\} \in [0,1)$ is called the *parameter* of g.

(iv) A distortion risk measure is said to be *VaR-type* if the parameter α of its distortion function is positive.

(v) For a distortion function g with parameter α, the function $\hat{g}$, defined by $\hat{g}(t) = g((t+\alpha) \wedge 1)$, $t \in [0,1]$, is called the *active part* of g.

Remark 5.2. In the literature, the distortion risk measure ρ_g defined by (5.1) does not require g to be left-continuous. Here we consider the case of left-continuous function g as in Weber (2018). It is well known that if g is left-continuous, ρ_g can be written in a Lebesgue-Stieltjes integral form

$$\rho_g(X) = \int_0^1 \mathrm{VaR}_\gamma(X)\mathrm{d}g(\gamma). \tag{5.2}$$

Clearly, (5.2) includes the RVaR family by definition.

Weber (2018) studied the (sum-)optimal risk sharing problem with respect to the distortion risk measures $\rho_{g_1}, \dots, \rho_{g_n}$, where the distortion functions have concave active parts. For $\alpha \in (0,1)$ and $\beta \in [0, 1-\alpha)$, the risk measure $\mathrm{RVaR}_{\alpha,\beta}$ is a VaR-type distortion risk measure with parameter α, and its distortion function has a concave active part. Below, we illustrate how the technique developed in Section 4 can be applied to investigate optimal allocations for VaR-type risk measures. In what follows, the set of allocations and the inf-convolution are defined as in (2.6) and (2.7) with $\mathcal{X}$ replaced by $\mathcal{Y}$.

Below, we establish a connection between a VaR-type distortion risk measure and a corresponding VaR, generalizing the formula $\mathrm{RVaR}_{\alpha,\beta} = \mathrm{VaR}_\alpha \square \mathrm{ES}_\beta$ which we used repeatedly in this paper. To make the presentation concise, for $h \in \mathcal{G}$ and $\alpha \in [0,1)$, we define $h_\alpha(t) = h((t-\alpha)_+)$, $t \in [0,1]$. Clearly, $h_0 = h$, and $h_\alpha \in \mathcal{G}$ if $h(1-\alpha) = 1$. Moreover, for $g \in \mathcal{G}$ with parameter α, we can easily get $\hat{g}_\alpha = g$.

Theorem 5.3. *For any $h \in \mathcal{G}$ and $\alpha \in [0,1)$, we have*

$$\mathrm{VaR}_\alpha \square \rho_h = \begin{cases} \rho_{h_\alpha} & \text{if } h(1-\alpha) = 1, \\ -\infty & \text{if } h(1-\alpha) < 1. \end{cases}$$

In particular, for any $g \in \mathcal{G}$ with parameter α, we have

$$\rho_g = \mathrm{VaR}_\alpha \square \rho_{\hat{g}}.$$

Proof. Take any $X \in \mathcal{Y}$. We first consider the case $h(1-\alpha) < 1$. Take $A \in \mathcal{F}$ with $\mathbb{P}(A) = \alpha$ and $m > 0$. Note that $\int_0^\infty h \circ \mathbb{P}(X > x)\mathrm{d}x < \infty$ because X is bounded. Using (5.1), we have

$$\begin{aligned}
\rho_h\left(X\mathbb{1}_{A^c} - m\mathbb{1}_A\right) &= \int_{-\infty}^0 (h \circ \mathbb{P}(X\mathbb{1}_{A^c} - m\mathbb{1}_A > x) - 1)\mathrm{d}x + \int_0^\infty h \circ \mathbb{P}(X\mathbb{1}_{A^c} - m\mathbb{1}_A > x)\mathrm{d}x \\
&\leqslant \int_{-m}^0 (h \circ \mathbb{P}(X\mathbb{1}_{A^c} - m\mathbb{1}_A > x) - 1)\mathrm{d}x + \int_0^\infty h \circ \mathbb{P}(X > x)\mathrm{d}x \\
&\leqslant \int_{-m}^0 (h \circ \mathbb{P}(A^c) - 1)\mathrm{d}x + \int_0^\infty h \circ \mathbb{P}(X > x)\mathrm{d}x \\
&= \int_{-m}^0 (h(1-\alpha) - 1)\mathrm{d}x + \int_0^\infty h \circ \mathbb{P}(X > x)\mathrm{d}x \\
&= m(h(1-\alpha) - 1) + \int_0^\infty h \circ \mathbb{P}(X > x)\mathrm{d}x \to -\infty \quad \text{as } m \to \infty.
\end{aligned}$$

On the other hand, $\mathrm{VaR}_\alpha((X+m)\mathbb{1}_A) \leqslant 0$ because $\mathbb{P}((X+m)\mathbb{1}_A > 0) \leqslant \mathbb{P}(A) = \alpha$. Combining the above observations, we have $\rho_h\left(X\mathbb{1}_{A^c} - m\mathbb{1}_A\right) + \mathrm{VaR}_\alpha((X+m)\mathbb{1}_A) \to -\infty$ as $m \to \infty$, and hence $\mathrm{VaR}_\alpha \Box \rho_h(X) = -\infty$ for all $X \in \mathcal{Y}$.

Next, we consider the case $h(1-\alpha) = 1$. Assume $X \geqslant 0$; this is without loss of generality since both ρ_{h_α} and $\mathrm{VaR}_\alpha \Box \rho_h$ satisfy the property (called cash-additivity) $\rho(X + c) = \rho(X) + c$ for any constant $c \in \mathbb{R}$. The case $\alpha = 0$ follows from the simple fact that, for all $Y \in \mathcal{Y}$,

$$\begin{aligned}
\mathrm{VaR}_0 \Box \rho_h(X) \leqslant \mathrm{VaR}_0(0) + \rho_h(X) = \rho_h(X) &= \mathrm{VaR}_0(Y) + \rho_h(X - \mathrm{VaR}_0(Y)) \\
&\leqslant \mathrm{VaR}_0(Y) + \rho_h(X - Y),
\end{aligned}$$

and thus $\mathrm{VaR}_0 \Box \rho_h(X) = \rho_h(X)$. In the following we assume $\alpha > 0$.

(i) We first show $\rho_{h_\alpha}(X) \geqslant \mathrm{VaR}_\alpha \Box \rho_h(X)$. Note that $\mathrm{VaR}_\alpha(X\mathbb{1}_{\{U_X > 1-\alpha\}}) = 0$. We have

$$\begin{aligned}
\rho_h\left(X\mathbb{1}_{\{U_X \leqslant 1-\alpha\}}\right) + \mathrm{VaR}_\alpha\left(X\mathbb{1}_{\{U_X > 1-\alpha\}}\right) &= \rho_h\left(X\mathbb{1}_{\{U_X \leqslant 1-\alpha\}}\right) \\
&= \int_0^\infty h \circ \mathbb{P}(X\mathbb{1}_{\{U_X \leqslant 1-\alpha\}} > x)\mathrm{d}x \\
&= \int_0^\infty h \circ \mathbb{P}(\{X > x\} \cup \{U_X \leqslant 1-\alpha\})\mathrm{d}x \\
&= \int_0^\infty h((\mathbb{P}(X > x) - \alpha)_+)\mathrm{d}x \\
&= \int_0^\infty h_\alpha(\mathbb{P}(X > x))\mathrm{d}x = \rho_{h_\alpha}(X).
\end{aligned}$$

By the definition of inf-convolution, we have $\rho_{h_\alpha}(X) \geqslant \mathrm{VaR}_\alpha \Box \rho_h(X)$.

(ii) Next we show $\rho_{h_\alpha}(X) \leqslant \mathrm{VaR}_\alpha \Box \rho_h(X)$. For this, it suffices to show $\rho_{h_\alpha}(X) \leqslant \rho_h(X - Y)$ for all $Y \in \mathcal{Y}$ with $\mathrm{VaR}_\alpha(Y) = 0$, again due to cash-additivity. Since $\mathrm{VaR}_\alpha(Y) = 0$ implies

$\mathbb{P}(Y > 0) \leqslant \alpha$, we have

$$\mathbb{P}(X - Y > x) \geqslant (\mathbb{P}(X > x) - \mathbb{P}(Y > 0))_+ \geqslant (\mathbb{P}(X > x) - \alpha)_+, \quad x \in \mathbb{R}.$$

As a consequence,

$$\begin{aligned}
\rho_h(X - Y) &= \int_{-\infty}^{0} (h \circ \mathbb{P}(X - Y > x) - 1)\mathrm{d}x + \int_{0}^{\infty} h \circ \mathbb{P}(X - Y > x)\mathrm{d}x \\
&\geqslant \int_{-\infty}^{0} (h((\mathbb{P}(X > x) - \alpha)_+) - 1)\mathrm{d}x + \int_{0}^{\infty} h((\mathbb{P}(X > x) - \alpha)_+)\mathrm{d}x \\
&= \int_{-\infty}^{0} (h_\alpha \circ \mathbb{P}(X > x) - 1)\mathrm{d}x + \int_{0}^{\infty} h_\alpha \circ \mathbb{P}(X > x)\mathrm{d}x = \rho_{h_\alpha}(X).
\end{aligned}$$

Therefore, we know $\rho_{h_\alpha}(X) \leqslant \mathrm{VaR}_\alpha \square \rho_h(X)$. $\square$

As shown in the proof of Theorem 5.3, for $h \in \mathcal{G}$ with $h(1 - \alpha) = 1$, a sum-optimal allocation (Y, Z) of $X \geqslant 0$ with respect to $(\mathrm{VaR}_\alpha, \rho_h)$ is given by $Y = X\mathbb{1}_{\{U_X > 1-\alpha\}}$ and $Z = X\mathbb{1}_{\{U_X \leqslant 1-\alpha\}}$.

Theorem 5.3 suggests that a VaR-type distortion risk measure is simply the inf-convolution of a VaR and another distortion risk measure. Using Theorem 5.3, we can apply the results in Section 4 to study forms of optimal allocations for VaR-type distortion risk measures. For $g_1, \dots, g_n \in \mathcal{G}$ with parameters $\alpha_1, \dots, \alpha_n$, respectively, assume that $\square_{i=1}^n \rho_{g_i}$ is finite on $\mathcal{Y}$. Write $\alpha = \sum_{i=1}^n \alpha_i$ and $\rho^* = \square_{i=1}^n \rho_{\hat{g}_i}$. Similarly to (4.1), noting that the inf-convolution is associative,

$$\mathop{\square}_{i=1}^{n} \rho_{g_i} = \mathop{\square}_{i=1}^{n} (\mathrm{VaR}_{\alpha_i} \square \rho_{\hat{g}_i}) = \left(\mathop{\square}_{i=1}^{n} \mathrm{VaR}_{\alpha_i} \right) \square \left(\mathop{\square}_{i=1}^{n} \rho_{\hat{g}_i} \right) = \mathrm{VaR}_\alpha \square \rho^*. \tag{5.3}$$

According to (5.3) and following the idea of Proposition 4.10, the problem of finding optimal allocations for the risk measures $\rho_{g_1}, \dots, \rho_{g_n}$ can be decomposed into two steps: first, allocate X to $(Y, Z) \in \mathbb{A}_2(X)$ such that $\mathrm{VaR}_\alpha \square \rho^*(X) = \mathrm{VaR}_\alpha(Y) + \rho^*(Z)$, and second, allocate Y and Z to $(Y_1, \dots, Y_n) \in \mathbb{A}_n(Y)$ and $(Z_1, \dots, Z_n) \in \mathbb{A}_n(Z)$ such that $\sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(Y_i) = \mathrm{VaR}_\alpha(Y)$ and $\sum_{i=1}^n \rho_{g_i}(Z_i) = \rho^*(Z)$. If all of the above allocations exist, then by letting $X_i = Y_i + Z_i$, $i = 1, \dots, n$, we obtain an optimal allocation for the agents with risk measures $\rho_{g_1}, \dots, \rho_{g_n}$. In the above procedure, there are three optimal allocation problems:

(i) Theorem 4.2 gives all forms of optimal allocations $(Y_1, \dots, Y_n) \in \mathbb{A}_n(Y)$ with respect to $(\mathrm{VaR}_{\alpha_1}, \dots, \mathrm{VaR}_{\alpha_n})$.

(ii) Solution of the optimal allocations $(Z_1, \dots, Z_n) \in \mathbb{A}_n(Z)$ with respect to general choices of $\rho_{\hat{g}_1}, \dots, \rho_{\hat{g}_n}$ is not available in the literature. In the special case that each of $g_1, \dots, g_n$ has a concave active part, as studied by Weber (2018), $\rho_{\hat{g}_1}, \dots, \rho_{\hat{g}_n}$ are coherent risk measures (Artzner et al. (1999)). In this case, there always exist comonotonic optimal allocations, and

$$\rho^* = \mathop{\square}_{i=1}^{n} \rho_{\hat{g}_i} = \rho_{g*}, \qquad \text{where } g^* = \textstyle\bigwedge_{i=1}^n g_i; \tag{5.4}$$

see e.g. Proposition 5 of Embrechts et al. (2018). For coherent distortion risk measures, the forms of optimal allocations $(Z_1, \dots, Z_n) \in \mathbb{A}_n(Z)$ are extensively studied in the literature; see Jouini et al. (2008) and Chapter 11 of Rüschendorf (2013). For similar results on quasi-convex risk measures, see Mastrogiacomo and Rosazza Gianin (2015).

(iii) The determination of optimal allocations $(Y, Z) \in \mathbb{A}_2(X)$ with respect to $(\mathrm{VaR}_\alpha, \rho^*)$ requires a result that is similar to Theorem 4.8, which depends highly on the form of ρ^*. The proof of Theorem 5.3 gives an optimal allocation when ρ^* is also a distortion risk measure (which is true if $g_1, \dots, g_n$ have concave active parts).

Although the above arguments do not give explicit forms of all optimal allocations for the VaR-type distortion risk measures, they offer technical tools as well as new interpretation of VaR-type risk measures and their optimal allocations. A full characterization of optimal allocations with respect to VaR-type risk measures requires future research.

Remark 5.4. If ρ_h is chosen as ES_β, Theorem 5.3 recovers the formula $\mathrm{RVaR}_{\alpha,\beta} = \mathrm{VaR}_\alpha \square \mathrm{ES}_\beta$ by checking the distortion functions of $\mathrm{RVaR}_{\alpha,\beta}$, VaR_α and ES_β in (5.2). Hence, Theorem 5.3 can be seen as a generalization of Theorem 2 of Embrechts et al. (2018). Theorem 5.3 also implies the result on $\square_{i=1}^n \rho_{g_i}$ in Theorem 11 of Weber (2018) for $g_1, \dots, g_n$ with concave active parts via (5.3) and (5.4).

6 A representative class of optimal allocations

As is seen from Section 4, optimal allocations for the RVaR family may take various forms, and this is due to the fact that risk measures in this family only uses partial information of the underlying distribution. Among many choices of optimal allocations, the allocation (2.10)-(2.11) obtained by Embrechts et al. (2018) is a rather simple and intuitive choice. One small disadvantage of (2.10)-(2.11) is that it is not symmetric with respect to the order of agents. Below we present a slightly more general class, which is also simple, and generalizes (2.10)-(2.11) to a symmetric form. We consider the most relevant case (A2), that is, $0 < \alpha + \beta < 1$. Define

$$\mathcal{P}_n = \left\{ (A_1, \dots, A_n) \in \mathcal{F}^n : \{X > y_\alpha\} \subset \bigcup_{i=1}^n A_i \subset \{X \geqslant y_\alpha\},\ \mathbb{P}(A_i) = \alpha_i,\ A_1, \dots, A_n \text{ are disjoint} \right\}.$$

Recall that $J = \{i \in \{1, \dots, n\} : \beta_i = \beta\}$. Let $(X_1, \dots, X_n)$ be given by

$$\begin{gathered} X_i = (X - y_{\alpha+\beta}) \mathbb{1}_{A_i} + \frac{1}{\# J} (X - y_{\alpha+\beta}) \mathbb{1}_{\{i \in J\}} \mathbb{1}_{(\bigcup_{i=1}^n A_i)^c} + c_i, \quad i = 1, \dots, n, \\ \text{where } (A_1, \dots, A_n) \in \mathcal{P}_n, \text{ and } (c_1, \dots, c_n) \in \mathbb{A}_n^c(y_{\alpha+\beta}). \end{gathered} \tag{6.1}$$

One can easily verify that (6.1) defines a class of optimal allocations, and it includes (2.10)-(2.11) if $\#J = 1$. Comparing with the results in Section 4, (6.1) gives one of the the simplest forms of optimal allocations.

The economic intuition behind (6.1) is also simple: the agents first share the most dangerous outcomes, modelled by the event $\bigcup_{i=1}^n A_i$ with probability α. They divide the event into pieces so that each agent feels safe, because the probability of loss is equal to α_i for agent i, thus insensitive to the agent. Then, they share the rest of the risk among agents in J, the most tolerant agents (with the biggest β_i value). Finally, they make some side-payments $c_1, \dots, c_n$.

The dependence structure of (6.1) may be worth noting. Assume $\mathbb{P}(X > y_\alpha) = \alpha$, as satisfied by all continuously distributed X. In this case, $\bigcup_{i=1}^n A_i = \{X > y_\alpha\}$. The optimal allocation $(X_1, \dots, X_n)$ is *mutually exclusive* on the event $\{X > y_\alpha\}$, a consistent observation with the ones made in Remarks 4.3 and 4.5. Mutual exclusivity represents the strongest form of negative dependence (see e.g. Puccetti and Wang (2015)). This is in sharp contrast to the classic risk sharing problems with law-invariant and convex objective functionals, where an optimal allocations is always *comonotonic* (based on a result of Landsberger and Meilijson (1994); see Rüschendorf (2013)), representing the strongest form of positive dependence. For related discussions on this phenomenon in the context of heterogeneous beliefs, see Embrechts et al. (2019).

We remark that in order to uniquely pin down a specific form of optimal allocations, one may need to involve a second-step optimization. Here, we give the representative allocation (6.1) only for the simplicity in its form, economic intuition and dependence structure.

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