An impossibility theorem on capital allocation

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Abstract

Two natural and potentially useful properties for capital allocation rules are top-down consistency and shrinking independence. Top-down consistency means that the total capital is determined by the aggregate portfolio risk. Shrinking independence means that the risk capital allocated to a given business line should not be affected by a proportional reduction of exposure in another business line. These two properties are satisfied by, respectively, the Euler allocation rule and the stress allocation rule. We prove an impossibility theorem that states that these two properties jointly lead to the trivial capital allocation based on the mean. When a subadditive risk measure is used, the same result holds for weaker versions of shrinking independence, which prevents the increase in risk capital in one line, when exposure to another is reduced. The impossibility theorem remains valid even if one assumes strong positive dependence among the risk vectors.

KEYWORDS: Euler allocation, stress scenarios, top-down consistency, shrinking independence

1 Capital allocation rules

Capital allocation is an active topic for researchers in risk management and practitioners in the financial industry. Capital allocation problems are often studied in the context of risk measures, as in the axiomatic settings of Denault (2001) and Kalkbrener (2005). We refer to Dhaene et al. (2012) and Furman and Zitikis (2008) for overviews of capital allocation methods based on risk measures, and to Scaillet (2004), Targino et al. (2015), Boonen et al. (2019) and Asimit et al. (2019) for examples of statistical studies.

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We first explain the mathematical setting for capital allocation. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and some $q \in [1, \infty]$. Let $\mathcal{X}$ be the set L^q of random variables X with finite q-th moment, i.e., $\mathbb{E}[|X|^q] < \infty$ if $q \in [1, \infty)$ and ess-sup$(X) < \infty$ if $q = \infty$.

Each random vector $\mathbf{X} = (X_1, \dots, X_d) \in \mathcal{X}^d$ represents risks from multiple business lines; positive outcomes of each X_i are understood as losses. An *allocation rule* Λ is a mapping from $\mathcal{X}^d$ to $\mathbb{R}^d$. For $\mathbf{X} \in \mathcal{X}^d$ and an allocation rule Λ, we denote by $\Lambda(\mathbf{X}) = (\Lambda_1(\mathbf{X}), \dots, \Lambda_d(\mathbf{X}))$ where $\Lambda_i(\mathbf{X})$ represents the amount of capital allocated to line $i \in \{1, \dots, d\}$. Capital allocation is intimately linked to risk measures. A *risk measure* in this paper is a continuous and law-invariant mapping $\rho : \mathcal{X} \to \mathbb{R}$. We do not require anything beyond continuity and law invariance, which is satisfied by all risk measures in the literature and risk management practice.[1] Even law invariance can be easily relaxed; see Section 4.3.

Examples of capital allocation rules include the proportional allocation, the Euler allocation, the Aumann-Shapley capital allocation, and those based on stress scenarios. Formal definitions of some capital allocation rules are put in Appendix A; below we give two specific examples which are sufficient to illustrate our main message. These two examples share the general form

$$\Lambda(\mathbf{X}) = \mathbb{E}^{Q_{\mathbf{X}}}[\mathbf{X}], \qquad \text{i.e., } \Lambda_i(\mathbf{X}) = \mathbb{E}^{Q_{\mathbf{X}}}[X_i] \text{ for } i = 1, \dots, d, \tag{1}$$

where $Q_{\mathbf{X}}$ is a probability measure determined by the risk vector $\mathbf{X}$.

1. The Euler allocation based on the Expected Shortfall (ES) (also known as the CTE allocation) is one of the most popular rules in capital allocation; see e.g., Kalkbrener (2005). It is defined as a special case of (1) by

$$\Lambda(\mathbf{X}) = \mathbb{E}^{Q_{\mathbf{X}}}[\mathbf{X}] \quad \text{and} \quad \frac{\mathrm{d}Q_{\mathbf{X}}}{\mathrm{d}\mathbb{P}} = \frac{1}{1-p}\mathbb{1}_{\{\sum_{i=1}^n X_i \geqslant s_p\}}, \quad \text{for some } p \in (0,1), \tag{2}$$

where s_p is the p-quantile of $S := \sum_{i=1}^n X_i$. Here we assume that S is continuously distributed.[2] This leads to $\Lambda_i(\mathbf{X}) = \mathbb{E}[X_i | S \geqslant s_p]$, $i = 1, \dots, d$. The total capital is $\sum_{i=1}^d \Lambda_i(\mathbf{X}) = \mathbb{E}[S | S \geqslant s_p]$, which is the ES of the total risk S at level p.

2. The mixture-stress allocation proposed by Millossovich et al. (2021) is based on stress scenarios

[1] Continuity is with respect to the norm on $\mathcal{X} = L^q$. A mapping ρ is law invariant if $\rho(X) = \rho(Y)$ for identically distributed $X, Y \in \mathcal{X}$. All law-invariant convex risk measures are continuous on L^q for $q \geqslant 1$; see Rüschendorf (2013). Moreover, all cash-subadditive risk measures, including the Value-at-Risk, are continuous on L^∞; see Cerreia-Vioglio et al. (2011).

[2] More precisely, we require $\{S \geqslant s_p\}$ to have probability $1 - p$. If this does not hold, then we need to replace the event $\{S \geqslant s_p\}$ with a p-tail event of S introduced by Wang and Zitikis (2021).

generated directly by the risk vector $\mathbf{X}$. It is defined as a special case of (1) by

$$\Lambda(\mathbf{X}) = \mathbb{E}^{Q_{\mathbf{X}}}[\mathbf{X}] \quad \text{and} \quad \frac{\mathrm{d}Q_{\mathbf{X}}}{\mathrm{d}\mathbb{P}} = \frac{\theta+1}{d}\sum_{i=1}^{d}(F_i(X_i))^{\theta}, \quad \text{for some } \theta \geqslant 0, \tag{3}$$

where each X_i is assumed to have a continuous distribution function F_i.[3] The total capital for the stress allocation rule is given by $\sum_{i=1}^{d}\Lambda_i(\mathbf{X}) = \mathbb{E}^{Q_{\mathbf{X}}}[S]$. The mixture-stress allocation rule belongs to the class of stress allocation rules of Millossovich et al. (2021); see Appendix A.

Our main result does not need to assume any specific form of allocation rules such as (1), (2) or (3); the above examples are introduced only to motivate the two important properties in the next section.

2 Three properties for an allocation rule

We introduce three properties for an allocation rule Λ. All statements are meant to hold for all $\mathbf{X} = (X_1, \dots, X_d) \in \mathcal{X}^d$.

(i) *Vanishing continuity*: $\Lambda(\varepsilon\mathbf{X}) \to \mathbf{0}$ as $\varepsilon \downarrow 0$.

(ii) *Top-down consistency*: $\sum_{i=1}^{n}\Lambda_i(\mathbf{X}) = \rho(\sum_{i=1}^{n}X_i)$ for some risk measure ρ with $\rho(1) = 1$.

(iii) *Shrinking independence*: $\Lambda_i(X_1, \dots, X_{j-1}, aX_j, X_{j+1}, \dots, X_d) = \Lambda_i(\mathbf{X})$ for all $j \neq i$ and $a \in (0, 1)$.

Vanishing continuity (i), meaning that the allocated capital shrinks to 0 for a vanishing risk, is satisfied by any sensible capital allocation rule. For instance, it is weaker than positive homogeneity: $\Lambda(\varepsilon\mathbf{X}) = \varepsilon\Lambda(\mathbf{X})$ for $\varepsilon > 0$, and positive homogeneity is satisfied by almost all capital allocation rules, including the ones mentioned in Section 1 and Appendix A.

Top-down consistency (ii)[4] means that the total capital requirement can be calculated from a risk measure that depends solely on the model of the aggregate position. All top-down methods generated from a pre-specified risk measure satisfy this property. Indeed, it is the starting point of many studies on capital allocation; see e.g., Denault (2001), Kalkbrener (2005) and Tsanakas (2009), where this property is part of the definition of an allocation based on a risk measure. In

[3]The case of discontinuity can be addressed using a suitable uniform transform; see Millossovich et al. (2021).

[4]This property is also referred to as the *full allocation* requirement or, particularly in a game theoretic context, as *efficiency*; see e.g., Lemaire (1991).

particular, it is satisfied by the Euler allocation rules, including the ES-based Euler allocation in (2). Nevertheless, in risk management practice, the total capital requirement is not necessarily calculated by any specific risk measure of a portfolio loss, but could indeed be exogenous to the allocation problem as in the settings of Zaks et al. (2006), Dhaene et al. (2012) and Centrone and Rosazza Gianin (2018); see also Remark 2.4 of Asimit et al. (2019). We do not take top-down consistency as granted in this paper.

Shrinking independence (iii) reflects the requirement that decreases in the exposure to one line of business do not lead to changes in the capital allocated to another line, and it may require some more explanation. In this context, we may assume that each business line operates separately, and they are pooled together by the overarching structure of the firm. Shrinking independence has a clear organizational rationale, as a stability property of capital allocations. If shrinking independence does not hold, the manager of a line of business may see their allocated capital change (even increase), in response to portfolio changes outside their control. Indeed, this is a reason why insurance practitioners are often reluctant to operationalize Euler allocations, as mentioned by the respondents of Cabantous and Tsanakas (2019). At the same time, this property is very restrictive. Hence, two relaxations of shrinking independence, one replacing the equality by an inequality and one requiring the property only for positive dependence, are discussed in Sections 4.1 and 4.2. To simplify exposition, we will illustrate our main result using the stronger formulation (iii).

Shrinking independence is satisfied by any allocation rule induced by an invariant stressing mechanism of the type introduced by Millossovich et al. (2021), including the mixture-stress allocation in (3); see also the ones in Appendix A. Indeed, the mixture-stress allocation rule satisfies the stronger property:

(iv) *Strong independence*: for each i, $\Lambda_i(X_1,\dots,X_{j-1},g(X_j),X_{j+1},\dots,X_d)=\Lambda_i(\mathbf{X})$ for $j\neq i$ and a strictly increasing function g.

Strong independence thus ensures robustness of allocated risk capital not only to reductions in exposure, but also to more general monotonic risk reductions, e.g., by the purchasing of reinsurance. See also the discussions and a real-data example in Millossovich et al. (2021).[5]

Shrinking independence does not conflict with the idea of diversification and risk reduction, since it takes dependence into consideration. Indeed, the allocation rule (3) reflects diversification by penalizing positive dependence and rewarding negative dependence. For instance, for a fixed

[5] A much weaker stability requirement is implicit in the approach of Major (2018), whose capital allocation reflects fully sensitivity to portfolio weights, but only partially to reinsurance parameters, as the latter are not included in the Radon-Nikodym derivative that generates the risk functional considered.

$i = 1, \ldots, d$, $\mathrm{d}Q_{\mathbf{X}}/\mathrm{d}\mathbb{P}$ puts more weights on large value of X_i if $X_1, \ldots, X_n$ are comonotonic as compared to the case that $X_1, \ldots, X_n$ are independent.

To understand the role of dependence of $\mathbf{X}$ for the allocation rules, we briefly discuss another property based on copulas, which are a powerful tool to model the dependence structure of a random vector separately from its marginal distributions.[6]

(v) *Copula decomposition*: for each i, $\Lambda_i(\mathbf{X})$ is determined by the distribution of X_i and the (possibly non-unique) copulas of $\mathbf{X}$.

Since copulas are invariant under strictly increasing transforms, copula decomposition (v) implies strong independence (iv). Therefore, strong independence can reflect the intuitive idea of modeling individual business lines and their dependence structure separately. The mixture-stress allocation rule (3) satisfies copula decomposition.

Remark 1. Properties (iii), (iv) and (v) are also satisfied in the case that the risk capital of each business line is individually computed by a risk measure; that is, each $\Lambda_i(X_i)$ solely depends on X_i. Such individual allocations are not of further interest to us, as they ignore aggregation or diversification effects.

Remark 2. We do not impose continuity of Λ in $\mathcal{X}^d$. In this way we can include capital allocation rules such as those based on invariant stressing mechanisms in Millossovich et al. (2021), which are not necessarily continuous when handling discrete risk factors. For instance, discontinuity may arise in (3) when a sequence of continuous risk vectors $(\mathbf{X}_n)_{n\in\mathbb{N}}$ converges to a risk vector $\mathbf{X}$ with some discrete components. Instead, we only require the vanishing continuity, a much weaker requirement.

3 An impossibility theorem

We establish an impossibility theorem to show that shrinking independence and top-down consistency conflict in the sense that, together with vanishing continuity, they jointly force the allocation rule to be the trivial one based on the mean. This result is an impossibility theorem because in practice, the total capital requirement cannot be computed using the mean.

Theorem 1. *An allocation rule Λ satisfies properties (i)-(iii) if and only if $\Lambda(\mathbf{X}) = \mathbb{E}[\mathbf{X}]$ for all $\mathbf{X} \in \mathcal{X}^d$.*

[6] A d-copula is a joint distribution function on $\mathbb{R}^d$ with standard uniform marginals. Sklar's theorem implies that the joint distribution F of any random vector $\mathbf{X}$ can be expressed by a copula C of $\mathbf{X}$ through $F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))$ where $F_1, \ldots, F_d$ are the marginals of F. The copula C is unique if $F_1, \ldots, F_d$ are continuous. See Joe (2014) for a general treatment of copulas.

Proof. The "if" statement is straightforward, and we only show the "only if" statement. Take an arbitrary $\mathbf{X} = (X_1, \dots, X_d) \in \mathcal{X}^d$. For $\varepsilon \in (0,1)$ and $i = 1, \dots, d$, let

$$\mathbf{X}_{-i}^{\varepsilon} = (\varepsilon X_1, \dots, \varepsilon X_{i-1}, X_i, \varepsilon X_{i+1}, \dots, \varepsilon X_d),$$

that is, the risk vector $\mathbf{X}$ multiplied by ε except for the i-th component. Applying shrinking invariance (iii) repeatedly leads to

$$\Lambda_i(\mathbf{X}_{-i}^{\varepsilon}) = \Lambda_i(\mathbf{X}) \text{ for each } i. \tag{4}$$

Top-down consistency (ii) implies

$$\sum_{j \neq i} \Lambda_j(\mathbf{X}_{-i}^{\varepsilon}) + \Lambda_i(\mathbf{X}_{-i}^{\varepsilon}) = \sum_{j=1}^{d} \Lambda_j(\mathbf{X}_{-i}^{\varepsilon}) = \rho\left(\varepsilon \sum_{j \neq i} X_j + X_i\right) \text{ for each } i. \tag{5}$$

Putting (4) and (5) together, we have

$$\Lambda_i(\mathbf{X}) = \rho\left(\varepsilon \sum_{j \neq i} X_j + X_i\right) - \sum_{j \neq i} \Lambda_j(\mathbf{X}_{-i}^{\varepsilon}) \text{ for each } i. \tag{6}$$

Moreover, (iii) also implies that for $j \neq i$, we have $\Lambda_j(\mathbf{X}_{-i}^{\varepsilon}) = \Lambda_j(\varepsilon \mathbf{X})$. By using vanishing continuity (i) and continuity of ρ,

$$\sum_{j \neq i} \Lambda_j(\mathbf{X}_{-i}^{\varepsilon}) = \sum_{j \neq i} \Lambda_j(\varepsilon \mathbf{X}) \to 0 \quad \text{and} \quad \rho\left(\varepsilon \sum_{j \neq i} X_j + X_i\right) \to \rho(X_i) \quad \text{as } \varepsilon \downarrow 0. \tag{7}$$

Therefore, (6) and (7) lead to $\Lambda_i(\mathbf{X}) \to \rho(X_i)$. Noting that $\Lambda_i(\mathbf{X})$ does not depend on ε, we have $\Lambda_i(\mathbf{X}) = \rho(X_i)$ i.e., the allocation $\Lambda_i(\mathbf{X})$ depends only on the individual loss X_i. Moreover, using (ii) again,

$$\sum_{i=1}^{d} \rho(X_i) = \sum_{i=1}^{d} \Lambda_i(\mathbf{X}) = \rho\left(\sum_{i=1}^{d} X_i\right),$$

i.e., ρ is additive. Since ρ is continuous, additive and law invariant with $\rho(1) = 1$, we get from Lemma 1 below that $\rho(X) = \mathbb{E}[X]$ for all $X \in \mathcal{X}$. Hence, the allocation rule Λ has to be the mean. $\square$

Remark 3. While somewhat troubling, it is not in itself surprising that different potentially useful allocation properties may be in conflict; for an impossibility result in the context of cooperative game theory, see Csóka and Pintér (2016). Furthermore, Mohammed et al. (2021) characterized multivariate distributions of risk vectors, for which Euler allocations based on ES collapse to expected values.

The following lemma contains a known result used in the proof of Theorem 1, although we did not find an explicit statement; such a result appeared in, for instance, the proof of Lemma A.1 of Wang and Zitikis (2021). We provide a simple proof for the reader familiar with techniques in the theory of risk measures.

Lemma 1. *A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is continuous, additive, and law invariant if and only if $\rho(X) = \rho(1)\mathbb{E}[X]$ for all $X \in \mathcal{X}$.*

Proof. The proof is adapted from that of Lemma A.1 of Wang and Zitikis (2021). The "if" part is trivial to check, and we thus only prove the "only if" part. Let $\lambda = \rho(1)$. Continuity and additivity gives $\rho(c) = \lambda c$ for $c \in \mathbb{R}$. Continuity and additivity also imply that $\rho(aX) = a\rho(X)$ for $a > 0$, which further implies convexity of ρ. Hence, ρ is a finite coherent risk measure multiplied by λ on L^q; the arguments below show that ρ is Fatou continuous (Definition 7.23 of Rüschendorf (2013)).

1. If $q \in [1, \infty)$, then, ρ is a finite convex risk measure (multiplied by λ), which is Fatou continuous by Rüschendorf (2013, Theorem 7.24).

2. If $q = \infty$, then law invariance of ρ implies Fatou continuity by Theorem 30 of Delbaen (2012).

In both cases, Fatou continuity holds for ρ, and hence it admits a representation

$$\rho(X) = \int X \mathrm{d}Q \tag{8}$$

for a measure Q on $(\Omega, \mathcal{F})$; see e.g., Rüschendorf (2013, Theorem 7.20) and Föllmer and Schied (2016, Exercise 4.2.1). Since ρ is law invariant, Q has to be equal to $\mathbb{P}$ multiplied by a constant. □

Among other consequences of Theorem 1, we notice that, assuming vanishing continuity and top-down consistency, an allocation rule Λ that satisfies copula decomposition (v), must ignore dependence. This is because the copula decomposition property is stronger than shrinking independence and the trivial expectation-based allocation in Theorem 1 does not involve the dependence structure of $\mathbf{X}$. Hence, while shrinking independence is not intrinsically inconsistent with diversification, it becomes so when top-down consistency is assumed.

4 Relaxations of the properties

In this section we discuss three possible relaxations of properties (ii) and (iii). The main message is that with reasonable relaxations and some other additional assumptions, we arrive at the same conclusion of the impossibility theorem.

4.1 Relaxing shrinking independence

Shrinking independence (iii) may be seen as quite strong, as it requires that the allocated capital to line i with risk X_i remains unchanged, when reducing another line with risk X_j for $j \neq i$. A natural relaxation of this property is

(iii') *Weak shrinking independence*: $\Lambda_i(X_1, \dots, X_{j-1}, aX_j, X_{j+1}, \dots, X_d) \leqslant \Lambda_i(\mathbf{X})$ for all $j \neq i$ and $a \in (0, 1)$.

Weak shrinking independence (iii') means that if another line of business reduces their risk exposure, then the allocated capital for an unchanged business line does not increase (but may decrease). This property is much weaker than (iii), and it is arguably quite natural in an insurance context. As a reduction in exposure to X_j would be expected to reduce the risk of the portfolio, (iii') only requires that this change has no adverse impact on other lines of business $i \neq j$.

The property (iii') in place of shrinking independence (iii) is too weak to establish the impossibility theorem.[7] In the literature of capital allocation, one often considers a coherent risk measure which calculates the total capital; see e.g., Kalkbrener (2005).[8] In such a setting, we can strengthen (ii) to:

(ii') *Top-down consistency with a subadditive risk measure*: $\sum_{i=1}^n \Lambda_i(\mathbf{X}) = \rho(\sum_{i=1}^n X_i)$ for some subadditive risk measure ρ with $\rho(1) = 1$.

We also modify the continuity in (i), which says that if the exposure in one business line vanishes, then so is its allocated capital. This natural assumption is technically slightly stronger than (i).

(i') *Component-wise vanishing continuity*: $\Lambda(\varepsilon\mathbf{X}) \to \mathbf{0}$ and $\Lambda_j(X_1, \dots, X_{j-1}, \varepsilon X_j, X_{j+1}, \dots, X_d) \to 0$ for each $j = 1, \dots, d$ as $\varepsilon \downarrow 0$.

The next result states that the impossibility theorem holds with the above modifications.

[7] Although we do not have a concrete counter-example, getting an impossibility theorem for (i), (ii) and (iii') does not seem to be possible at least with the proof techniques in this paper.

[8] A coherent risk measure of Artzner et al. (1999) is defined to satisfy four properties: monotonicity, translation invariance, positive homogeneity, and subadditivity. The only property we need here is subadditivity: $\rho(X+Y) \leqslant \rho(X) + \rho(Y)$ for $X, Y \in \mathcal{X}$.

Theorem 2. *An allocation rule Λ satisfies properties (i'), (ii') and (iii') if and only if $\Lambda(\mathbf{X}) = \mathbb{E}[\mathbf{X}]$ for all $\mathbf{X} \in \mathcal{X}^d$.*

Proof. We follow the same logic and the same notation as in the proof of Theorem 1, and it is clear that we only need to show the "only if" statement. By continuity of ρ, we have, as in (7),

$$\rho\left(\varepsilon \sum_{j \neq i} X_j + X_i\right) \to \rho(X_i) \quad \text{as } \varepsilon \downarrow 0. \tag{9}$$

With weak shrinking invariance (iii') replacing (iii), the equality in (6) becomes an inequality, giving rise to

$$\Lambda_i(\mathbf{X}) \geqslant \rho\left(\varepsilon \sum_{j \neq i} X_j + X_i\right) - \sum_{j \neq i} \Lambda_j(\mathbf{X}^{\varepsilon}_{-i}) \text{ for each } i. \tag{10}$$

Next, we verify

$$\lim_{\varepsilon \downarrow 0} \Lambda_j(\mathbf{X}^{\varepsilon}_{-i}) \to 0 \ \text{ for } j \neq i. \tag{11}$$

For a fixed $\delta > 0$ and $\varepsilon \in (0, \delta)$, we have, by using (iii'),

$$\Lambda_j(\mathbf{X}^{\varepsilon}_{-i}) \leqslant \Lambda_j(\varepsilon_1 X_1, \dots, \varepsilon_{i-1} X_{i-1}, X_i, \varepsilon_{i+1} X_{i+1}, \dots, \varepsilon_d X_d) \tag{12}$$

where $\varepsilon_k = \delta$ for $k \notin \{i, j\}$ and $\varepsilon_j = \varepsilon$. For fixed δ, the right-hand side of (12) goes to 0 by (i'). Hence, $\limsup_{\varepsilon \downarrow 0} \Lambda_j(\mathbf{X}^{\varepsilon}_{-i}) \leqslant 0$. Moreover, using (iii') again, we have

$$\sum_{j \neq i} \Lambda_j(\mathbf{X}^{\varepsilon}_{-i}) \geqslant \sum_{j \neq i} \Lambda_j(\varepsilon \mathbf{X}) \to 0 \geqslant \limsup_{\varepsilon \downarrow 0} \sum_{j \neq i} \Lambda_j(\mathbf{X}^{\varepsilon}_{-i}).$$

This gives (11). Putting (9), (10) and (11) together leads to $\Lambda_i(\mathbf{X}) \geqslant \rho(X_i)$. This inequality together with (ii') and subadditivity of ρ gives

$$\sum_{i=1}^d \rho(X_i) \leqslant \sum_{i=1}^d \Lambda_i(\mathbf{X}) = \rho\left(\sum_{i=1}^d X_i\right) \leqslant \sum_{i=1}^d \rho(X_i).$$

Thus, ρ is additive and $\Lambda_i(\mathbf{X}) = \rho(X_i)$. Using Lemma 1 we know that ρ is the mean. □

Remark 4. Assume that top-down consistency (ii) holds. In this setting, Kalkbrener (2005) further imposed a property called *diversification*, which implies that $\Lambda_i(\mathbf{X}) \leqslant \rho(X_i)$ for each i, meaning that the risk capital for the sub-portfolio X_i of portfolio $\mathbf{X}$ does not exceed the risk capital if X_i is considered as a stand-alone portfolio.[9] It is clear that this property, together with top-down

[9] Otherwise the business line i may be disadvantaged by being part of the portfolio $\mathbf{X}$. There is certainly a

consistency, implies subadditivity and thus is stronger than requiring ρ to be subadditive. Hence, such an enhancement of (ii) in place of (ii'), requiring the diversification property, is also sufficient for Theorem 2. This also reveals that weak shrinking independence property conflicts not just with subadditivity, but also with the implications of such a property for rational behaviour.[10]

4.2 Shrinking independence for only positive dependence

In case the risk vector $\mathbf{X} = (X_1, \ldots, X_d)$ has some hedging effect among its components, shrinking independence or weak shrinking independence may not be appealing. In such a case, shrinking the exposure of one business line may lead to a reduction in the hedging effect to another business line. Then, it could be reasonable that the capital allocated to the business line that is now less hedged indeed faces an increase.

Nevertheless, weak shrinking independence would be natural if the risk vector is strongly positively dependent in some sense, as there is no hedging effect in such a situation. In view of this, we will discuss a much weaker version of shrinking independence, where the property is only imposed on very positively dependent risk vectors. For this relaxation, we first define what we mean by positive dependence. Let R be the Spearman rank correlation of a bivariate vector, defined as

$$R(X, Y) = \mathrm{Corr}(F(X), G(Y)), \qquad X, Y \in \mathcal{X},$$

where $\mathrm{Corr}(\cdot)$ is Pearson's correlation coefficient, and F and G are the distribution functions of X and Y, respectively. If one of X and Y is degenerate, then we set $R(X, Y) = 1$.[11] We define positive dependence in the following sense: for $r \in [-1, 1]$, we say that the random vector $\mathbf{X}$ is *r-positively dependent*, if $R(X_i, X_j) \geqslant r$ for all $i \neq j$. Clearly, if $\mathbf{X}$ has continuous marginals, then 1-positive dependence is equivalent to comonotonicity.[12] Hence, the property of r-positive

game-theoretic flavour to it, relating to the concept of the *core* of co-operative games and ideas of *individual rationality*; see e.g., Lemaire (1991); Denault (2001). At the same time, note that the desirability of such properties is context-dependent; for example, Kim and Hardy (2009) challenge the diversification property from a solvency-option perspective.

[10] On the other hand, if top-down consistency is not assumed, a property similar to diversification could be obtained for the mixture stress allocation; see Proposition 5 of Millossovich et al. (2021). Top-down consistency is also relaxed by Centrone and Rosazza Gianin (2018), in the context of (non-coherent) convex and quasi-convex risk measures, as the price to pay for requiring the diversification property without subadditivity; see also Canna et al. (2021).

[11] This convention does not affect our discussion or results. If one of X and Y is degenerate, then X and Y are comonotonic. As comonotonicity is the strongest form of positive dependence, it is natural to set the rank correlation to 1.

[12] Two random variables X and Y are comonotonic if $X = f(X+Y)$ and $Y = g(X+Y)$ a.s. for increasing functions f and g. Comonotonicity of a random vector means pair-wise comonotonicity. For more on comonotonicity and other

dependence gets stronger as r increases, and it forms a continuum from arbitrary dependence ($r = -1$) to comonotonicity ($r = 1$). We denote by $\mathcal{X}_r^d$ the set of all r-positively dependent random vectors with continuous marginals.

Remark 5. The dependence measure R may be replaced by another measure of concordance (see McNeil et al. (2015)), which may either be a joint concordance measure or a bivariate concordance measure, such as (joint or bivariate) Kendall's tau. From the proof of Theorem 3 below, it will be clear that such a choice is irrelevant to our discussion.

We are now ready to further relax weak shrinking independence to the following version.

(iii')$_r$ *Weak shrinking independence under r-positive dependence*:
$\Lambda_i(X_1, \dots, X_{j-1}, aX_j, X_{j+1}, \dots, X_d) \leqslant \Lambda_i(\mathbf{X})$ for all $j \neq i$, $a \in (0,1)$ and $\mathbf{X} \in \mathcal{X}_r^d$.

Theorem 3. *Suppose that an allocation rule Λ satisfies properties (i') and (ii'). For $r \in (0,1)$, Λ satisfies (iii')$_r$ if and only if $\Lambda(\mathbf{X}) = \mathbb{E}[\mathbf{X}]$ for all $\mathbf{X} \in \mathcal{X}_r^d$. Moreover, if (iii')$_r$ holds, then the risk measure ρ used to calculate the total capital must be the mean for any risk vector in $\mathcal{X}^d$.*

Proof. The "if" statement is straightforward, and we will only show the "only if" statement below. Take $\mathbf{X} \in \mathcal{X}_r^d$. Note that all random vectors like $\mathbf{X}_{-i}^{\varepsilon}$ and $\mathbf{X}$ share the same pair-wise values of the dependence measure R. The same argument as in the proof of Theorem 2 carries through and leads to

$$\Lambda_i(\mathbf{X}) = \rho(X_i) \text{ for each } i \quad \text{and} \quad \rho\left(\sum_{i=1}^d X_i\right) = \sum_{i=1}^d \rho(X_i);$$

that is, ρ is additive for components of a risk vector in $\mathcal{X}_r^d$. Lemma 2 below guarantees that ρ has to be the mean for all random variables in $\mathcal{X}$. □

By taking $r < 1$ close to 1 in Theorem 3, we know that the allocation rule is, once again, the trivial one if weak shrinking independence holds for very positive dependence. We note that the statement in Theorem 3 is made only for $\mathbf{X} \in \mathcal{X}_r^d$ because (iii')$_r$ is also only formulated for $\mathbf{X} \in \mathcal{X}_r^d$. As such, Theorem 3 does not say anything about the form of $\Lambda(\mathbf{X})$ for $\mathbf{X}$ outside $\mathcal{X}_r^d$. Nevertheless, the total capital has to be the mean of the aggregate position, that is, $\rho(\sum_{i=1}^d X_i) = \mathbb{E}[\sum_{i=1}^d X_i]$ for all $\mathbf{X} \in \mathcal{X}^d$ (not only those in $\mathcal{X}_r^d$), and this contradicts risk management practice; so the interpretation of the impossibility theorem remains valid.

The proof of Theorem 3 uses the following lemma which characterizes risk measures with additivity for random vectors in $\mathcal{X}_r^2$. As far as we are aware, this lemma is the first of its sort in

dependence concepts, see Dhaene et al. (2002) and Puccetti and Wang (2015).

the literature, and it may be of independent interest.[13] We remark that a key step in the proof of Lemma 2 is that the risk measure needs to be additive for comonotonic risks. This means that the proof would not hold if we considered r-positive dependence with respect to the usual linear correlation, since linear correlation equal to 1 does not guarantee comonotonicity, given the impact of marginal distributions.

Lemma 2. *Fix $r \in (0,1)$. A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is continuous, subadditive, additive for $(X, Y) \in \mathcal{X}_r^2$, and law invariant if and only if $\rho(X) = \rho(1)\mathbb{E}[X]$ for all $X \in \mathcal{X}$.*

Proof. Without loss of generality, assume $\rho(1) = 1$.

First, we will consider the case of $\mathcal{X} = L^q$ for $q \in [1, \infty)$. Since any comonotonic random vector with continuous marginals is in $\mathcal{X}_r^2$, and ρ is continuous, we know that ρ is additive for any pair of comonotonic random variables in $\mathcal{X}$, thus comonotonic-additive. A comonotonic-additive and continuous risk measure is always positively homogeneous, meaning that $\rho(\alpha X) = \alpha\rho(X)$ for $\alpha \in (0, \infty)$ and $X \in \mathcal{X}$. This can be checked from the equality $\rho(nX) = n\rho(X)$ for any natural number n and $X \in \mathcal{X}$, together with continuity. Using $\rho(1) = 1$, comonotonic-additivity and positive homogeneity, we further get $\rho(X + m) = \rho(X) + m$ for all $m \in \mathbb{R}$ and $X \in \mathbf{X}$. Denote by $\tilde{\rho}(X) = \rho(X) - \mathbb{E}[X]$, $X \in \mathcal{X}$. It follows from properties of ρ that $\tilde{\rho}$ is subadditive, comonotonic-additive, law invariant, positively homogeneous, and satisfying $\tilde{\rho}(X + m) = \tilde{\rho}(X)$ for all $m \in \mathbb{R}$ and $X \in \mathcal{X}$. Therefore, $\tilde{\rho}$ is a comonotonic-additive coherent measure of variability in the sense of Furman et al. (2017), and by the representation result in Theorem 2.1 of Furman et al. (2017), we can write

$$\tilde{\rho}(X) = \int_0^1 F_X^{-1}(t)\mathrm{d}h(t), \quad X \in \mathcal{X},$$

where $h : [0,1] \to \mathbb{R}$ is left-continuous, convex, and satisfies $h(0) = h(1) = 0$, and F_X^{-1} is the left quantile function of X. Right-continuity of h is shown in the proof of Theorem 2.1 of Furman et al. (2017). Hence, h is continuous since $\tilde{\rho}$ is finite on L^q.

Next, we show that h has to be the constant 0. Suppose for the purpose of contradiction that h is not always 0. Since h is convex and $h(0) = h(1) = 0$, it must be non-positive, and there exists a smallest minimal point $s = \min(\arg\min_{t\in[0,1]} h(t))$ with $h(s) < 0$. Note that h is convex and nonlinear in any neighbourhood of s. Hence, for any $\varepsilon > 0$ with $[s - \varepsilon, s + \varepsilon] \subseteq [0,1]$,

$$\int_{s-\varepsilon}^{s+\varepsilon} (t - s)\mathrm{d}h(t) = \varepsilon h(s + \varepsilon) + \varepsilon h(s - \varepsilon) - \int_{s-\varepsilon}^{s+\varepsilon} h(t)\mathrm{d}t > 0. \tag{13}$$

[13] This lemma may be seen as a special result on functionals that collapse to the mean; see e.g., Bellini et al. (2021) for other results on such functionals.

Denote by $A = [s - \varepsilon, s + \varepsilon]$. Let U be a uniform random variable on $[0, 1]$, and

$$V = (2s - U)\mathbb{1}_{\{U \in A\}} + U\mathbb{1}_{\{U \notin A\}}.$$

We take $\varepsilon > 0$ small enough so that $R(U, V) > r$. This is possible since $R(U, V) \to 1$ as $\varepsilon \downarrow 0$. We can easily check that $U + V = 2s\mathbb{1}_{\{U \in A\}} + 2U\mathbb{1}_{\{U \notin A\}}$. Using (13), we have

$$\rho(2U) - \rho(U + V) = \int_0^1 2t\mathrm{d}h(t) - \int_0^1 \left(2t\mathbb{1}_{\{t \notin A\}} + 2s\mathbb{1}_{\{t \in A\}}\right) \mathrm{d}h(t) = \int_{s-\varepsilon}^{s+\varepsilon} (2t - 2s)\mathrm{d}h(t) > 0. \quad (14)$$

On the other hand, note that $\rho(U) = \rho(V)$ because V is uniformly distributed on $[0, 1]$, and ρ is law invariant. Since ρ is additive on $\mathcal{X}_r^2$, we have

$$\rho(U + V) = \rho(U) + \rho(V) = \rho(U) + \rho(U) = \rho(2U).$$

This leads to a contradiction. Hence, h is a constant 0, which implies that $\tilde{\rho}$ is also 0. Therefore, $\rho(X) = \mathbb{E}[X]$ for all $X \in \mathcal{X}$.

The case of $\mathcal{X} = L^\infty$ is analogous. The only difference in the L^∞ case is that one needs to further argue that, if h jumps at 0, then equation (14) holds. If h has a jump at 0, then h is not a constant on the interval $[0, 2\varepsilon]$ for any $\varepsilon > 0$. Since h is not a constant on $[0, 2\varepsilon]$, and h is convex, we get that (14) holds with $s = \varepsilon$. This violates the additivity of ρ on $\mathcal{X}_r^2$. The case that h jumps at 1 is similar. Therefore, h has no jump at 0 or 1. The rest of the proof is the same as in the case of L^p for $p \in [1, \infty)$. □

Lemma 2 also reveals the reason why $r = 1$ is not included in Theorem 3. Indeed, assuming continuity, additivity on $\mathcal{X}_1^2$ is precisely comonotonic-additivity. This property is satisfied by distortion risk measure (Wang et al., 1997) – more broadly: for any signed Choquet integral, e.g., Wang et al. (2020) – not necessarily equal to the mean. For instance, the ES-based Euler allocation principle leads to the risk measure ρ being an ES, which is additive for comonotonic random variables.

4.3 Relaxing law invariance

Next, we relax law invariance in the assumption of the risk measure ρ appearing in top-down consistency. Absence of law invariance means that the total capital can be assessed not only based on the distribution of the total risk, but also on other characteristics, such as scenario-based analysis; see Wang and Ziegel (2021) for a theory of non-law invariant risk measures in risk management. In the next result, we will see that allowing for this extra flexibility in the risk assessment does not give rise to more choices of capital allocation rules; we return to the case of the mean, with respect to a probability measure possibly different from $\mathbb{P}$. Below, $Q \ll \mathbb{P}$ means that Q is absolutely continuous with respect to $\mathbb{P}$.

Theorem 4. *Let $\mathcal{X} = L^q$ for some $q \in [1, \infty)$, and do not assume that the risk measure ρ is necessarily law invariant. An allocation rule Λ satisfies properties (i)-(iii) if and only if $\Lambda(\mathbf{X}) = \mathbb{E}^Q[\mathbf{X}]$ on $\mathcal{X}^d$ for some probability measure $Q \ll \mathbb{P}$.*

The proof of Theorem 4 follows from similar arguments as in that of Theorem 1, and we only mention the differences. Law invariance appears in the proof of Theorem 1 through the application of Lemma 1. Lemma 3 below is a variant of Lemma 1 which does not rely on law invariance. Having Lemma 3 (a) in place of Lemma 1 leads to a proof of Theorem 4.

Lemma 3. *(a) A mapping $\rho : L^q \to \mathbb{R}$ where $q \in [1, \infty)$ is continuous and additive if and only if $\rho(X) = \rho(1)\mathbb{E}^Q[X]$ on L^q for some probability measure $Q \ll \mathbb{P}$.*

(b) A mapping $\rho : L^\infty \to \mathbb{R}$ is continuous and additive if and only if $\rho(X) = \rho(1)\mathbb{E}^Q[X]$ on L^∞ for some finitely additive measure $Q \ll \mathbb{P}$ with total mass 1.

Proof. The proof is identical to that of Lemma 1, with the exception that, continuity on L^∞ without law invariance is not sufficient to guarantee Fatou continuity. As a consequence, Q in (8) is not necessarily a probability measure; instead, Q is only finitely additive. Absolute continuity of Q with respect to $\mathbb{P}$ is obviously necessary; otherwise $\mathbb{E}^Q$ is not finite. □

Remark 6. In the statements of Theorem 4 and Lemma 3, there is an implicit requirement for Q that $\mathbb{E}^Q$ is finite on $\mathcal{X}$. This is because Λ and ρ in these results are assumed to take real values. Since finitely additive measures are not an easy object to work with, we did not include the case $q = \infty$ in Theorem 4, although it is clear that the result holds similarly, and the economic interpretation remains the same.

5 Concluding remarks

The main result in this short paper reveals a profound conflict between two operational considerations in capital allocation rules: top-down consistency and shrinking independence. Both properties are potentially useful in different contexts for the design of capital allocation rules. Unfortunately, as shown from our impossibility theorem, they do not live well together, and this result still holds true when we relax some of the conditions in the two properties.

We do not argue that either property is desirable or not, as desirability clearly depends on the context. Based on our impossibility theorem, if top-down consistency is required, then (weak) shrinking independence cannot be achieved, and vice versa. For researchers who take top-down consistency as granted (which is reasonable in some applications), our main result advises that hoping for shrinking independence is futile. If top-down consistency is not required, then shrinking independence can be used, while at the same time upholding diversification properties.

The two properties encode different organizational requirements. On the one hand, top-down consistency requires capital to be calculated by a centralized approach; the performance of each line of business is solely understood through its contribution to portfolio risk. On the other hand, shrinking independence relates to a bottom-up view of the capital allocation process, recognizing some autonomy to business lines – while diversification should still be reflected in allocated capital, the risk of individual lines should also be understood in its own right. This tension between top-down and bottom-up approaches to insurance operations is already foreshadowed in a premium calculation context by Bühlmann (1985).

Hence, our impossibility theorem adds evidence to the view that there are no universally good methods for capital allocation; one always needs to carefully consider context-specific priorities in given applications when designing allocation rules. This observation may also partially explain why capital allocation has remained an active field of study in finance and insurance, with rich theoretical and applied research findings; see e.g., the recent advances in Boonen et al. (2017), Centrone and Rosazza Gianin (2018), Boonen et al. (2020) and Bauer and Zanjani (2021).

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References

Artzner, P., Delbaen, F., Eber, J.-M. and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, **9**(3), 203–228.

Asimit, V., Peng, L., Wang, R. and Yu, A. (2019). An efficient approach to quantile capital allocation and sensitivity analysis. *Mathematical Finance*, **29**(4), 1131–1156.

Bauer, D. and Zanjani, G. (2021). Economic capital and RAROC in a dynamic model. *Journal of Banking and Finance* **125**, 106071.

Bellini, F., Koch-Medina, P., Munari, C. and Svindland, G. (2021). Law-invariant functionals that collapse to the mean. *Insurance: Mathematics and Economics*, **98**, 83–91.

Boonen, T. J., Guillen, M., Santolino, M. (2019). Forecasting compositional risk allocations. *Insurance: Mathematics and Economics*, **84**, 79-86.

Boonen, T. J., De Waegenaere, A., and Norde, H. (2020). A generalization of the Aumann–Shapley value for risk capital allocation problems. *European Journal of Operational Research*, **282**(1), 277-287.

Boonen, T. J., Tsanakas, A. and Wüthrich, M. V. (2017). Capital allocation for portfolios with non-linear risk aggregation. *Insurance: Mathematics and Economics*, **72**, 95–106.

Bühlmann, H. (1985). Premium calculation from top down. *ASTIN Bulletin*, **15**(2), 89–101.

Cabantous, L. and Tsanakas, A. (2019). Between representation and tool: Strategizing with quantitative models. *Academy of Management Proceedings*, **2019**(1), 17687.

Canna, G., Centrone, F., and Rosazza Gianin, E. (2021). Haezendonck-Goovaerts capital allocation rules. *Insurance: Mathematics and Economics*, **101**(B), 173–185.

Centrone, F. and Rosazza Gianin, E. (2018). Capital allocation à la Aumann–Shapley for non-differentiable risk measures. *European Journal of Operational Research*, **267**(2), 667–675.

Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M. and Montrucchio, L. (2011). Risk measures: Rationality and diversification. *Mathematical Finance*, **21**(4), 743–774.

Csóka, P. and Pintér, M. (2016). On the Impossibility of Fair Risk Allocation. *The B.E. Journal of Theoretical Economics*, **16**(1), 143–158.

Delbaen, F. (2012). *Monetary Utility Functions*. Osaka University Press, Osaka.

Denault, M. (2001). Coherent allocation of risk capital. *Journal of Risk*, **4**, 1–34.

Dhaene, J., Denuit, M., Goovaerts, M. J., Kaas, R. and Vynche, D. (2002). The concept of comonotonicity in actuarial science and finance: Theory. *Insurance: Mathematics and Economics* **31**(1), 3–33.

Dhaene, J., Tsanakas, A., Valdez, E. A. and Vanduffel, S. (2012). Optimal capital allocation principles. *Journal of Risk and Insurance*, **79**(1), 1–28.

Föllmer, H. and Schied, A. (2016). *Stochastic Finance. An Introduction in Discrete Time*. Fourth Edition. Walter de Gruyter, Berlin.

Furman, E., Wang, R. and Zitikis, R. (2017). Gini-type measures of risk and variability: Gini shortfall, capital allocation and heavy-tailed risks. *Journal of Banking and Finance*, **83**, 70–84.

Furman, E. and Zitikis, R. (2008). Weighted risk capital allocations. *Insurance: Mathematics and Economics*, **43**(2), 263–269.

Joe, H. (2014). *Dependence Modeling with Copulas*. London: Chapman & Hall.

Kalkbrener, M. (2005). An axiomatic approach to capital allocation. *Mathematical Finance*, **15**(3), 425–437.

Kim, J. H. T. and Hardy, M. (2009). A capital allocation based on a solvency exchange option. *Insurance: Mathematics and Economics*, **44**(3), 357–366.

Lemaire, J. (1991). Cooperative game theory and its insurance applications. *ASTIN Bulletin*, **21**(1), 17–40.

Major, J. A. (2018). Distortion measures and homogeneous financial derivatives. *Insurance: Mathematics and Economics*, **79**, 82–91.

Millossovich, P., Tsanakas, A. and Wang, R. (2021). A theory of multivariate stress testing. *SSRN*: 3966204.

McNeil, A. J., Frey, R. and Embrechts, P. (2015). *Quantitative Risk Management: Concepts, Techniques and Tools.* Revised Edition. Princeton, NJ: Princeton University Press.

Mohammed, N., Furman, E., and Su, J. (2021). Can a regulatory risk measure induce profit-maximizing risk capital allocations? The case of conditional tail expectation. *Insurance: Mathematics and Economics*, **101**(B), 425–436.

Puccetti, G. and Wang R. (2015). Extremal dependence concepts. *Statistical Science*, **30**(4), 485–517.

Rüschendorf, L. (2013). *Mathematical Risk Analysis. Dependence, Risk Bounds, Optimal Allocations and Portfolios.* Springer, Heidelberg.

Schmeidler, D. (1986). Integral representation without additivity. *Proceedings of the American Mathematical Society*, **97**(2), 255–261.

Scaillet, O. (2004). Nonparametric estimation and sensitivity analysis of Expected Shortfall. *Mathematical Finance*, **14**(1), 115–129.

Targino, R. S., Peters, G. W., and Shevchenko, P. V. (2015). Sequential Monte Carlo Samplers for capital allocation under copula-dependent risk models. *Insurance: Mathematics and Economics*, **61**, 206–226.

Tsanakas, A. (2009). To split or not to split: Capital allocation with convex risk measures. *Insurance: Mathematics and Economics*, **44**(2), 268–277.

Wang, R., Wei, Y. and Willmot, G. E. (2020). Characterization, robustness and aggregation of signed Choquet integrals. *Mathematics of Operations Research*, **45**(3), 993–1015.

Wang, R., and Ziegel, J. F. (2021). Scenario-based risk evaluation. *Finance and Stochastics*, **25**, 725–756.

Wang, R. and Zitikis, R. (2021). An axiomatic foundation for the Expected Shortfall. *Management Science*, **67**, 1413–1429.

Wang, S. S., Young, V. R., and Panjer, H. H. (1997). Axiomatic characterization of insurance prices. *Insurance: Mathematics and Economics*, **21**(2), 173–183.

Zaks, Y., Frostig, E., and Levikson, B. (2006). Optimal pricing of a heterogeneous portfolio for a given risk level. *ASTIN Bulletin*, **36**(1), 161–185.

A Two classes of capital allocation rules

In this appendix, we formally define two classes of capital allocation rules. The first class is based on Euler's principle. The Euler allocation rule is a top-down method, in which the aggregate capital is computed via a positively homogeneous risk measure ρ. For $\boldsymbol{\lambda} \in \mathbb{R}^d$, write $r_{\rho,\mathbf{X}}(\boldsymbol{\lambda}) = \rho(\boldsymbol{\lambda} \cdot \mathbf{X})$ and let $S = \sum_{i=1}^d X_i = \mathbf{1} \cdot \mathbf{X}$. The ρ-based Euler allocation rule is defined as

$$\Lambda_i(\mathbf{X}) = \frac{\partial r_{\rho,\mathbf{X}}}{\partial \lambda_i}(\mathbf{1}), \qquad i = 1, \dots, d.$$

The Euler allocation satisfies top-down consistency (ii), due to Euler's principle for positively homogeneous functions. More precisely, the function $r_{\rho,\mathbf{X}}$ is positively homogeneous, meaning that $r_{\rho,\mathbf{X}}(t\boldsymbol{\lambda}) = t r_{\rho,\mathbf{X}}(\boldsymbol{\lambda})$ for $t \geqslant 0$ and $\boldsymbol{\lambda} \in \mathbb{R}^d$. Euler's principle gives

$$\rho(S) = r_{\rho,\mathbf{X}}(\mathbf{1}) = \sum_{i=1}^{d} \frac{\partial r_{\rho,\mathbf{X}}}{\partial \lambda_i}(\mathbf{1}) = \sum_{i=1}^{d} \Lambda_i(\mathbf{X}).$$

For a positively homogeneous risk measure, the Aumann-Shapley capital allocation is equivalent to the Euler allocation; see e.g., Denault (2001). In case ρ is the standard deviation, the Euler allocation becomes the covariance principle, defined as

$$\Lambda_i(\mathbf{X}) = \frac{\mathrm{Cov}(X_i, S)}{\sqrt{\mathrm{Var}(S)}}, \qquad i = 1, \dots, d.$$

In case ρ is ES at level p, we arrive at the ES-based Euler allocation (2).

The second class of allocation rules is based on stress scenarios. The stress allocation rule of Millossovich et al. (2021) is defined as $\Lambda(\mathbf{X}) = \mathbb{E}^{Q_{\mathbf{X}}}[\mathbf{X}]$ in (1), under the assumption that the Radon-Nikodym density $\frac{\mathrm{d}Q_{\mathbf{X}}}{\mathrm{d}\mathbb{P}}$ is invariant under strictly increasing marginal transforms on $\mathbf{X}$. Invariance is a natural property in stress testing since the choice of the counting units or a transform e.g., from asset returns to log-returns, should not affect stress scenarios, as discussed by Millossovich et al. (2021). Clearly, any stress allocation rule satisfies shrinking independence (iii); indeed, strong independence (iv) holds.

Examples of stress allocation rules include the mixture-stress allocation in (3), the Spearman allocation, and the dual Spearman allocation. Assume that each X_i has a continuous distribution function F_i and write $U_i = F_i(X_i)$. The Spearman allocation is defined via the stress scenario in (1) as

$$\frac{\mathrm{d}Q_{\mathbf{X}}}{\mathrm{d}\mathbb{P}} = \frac{\prod_{i=1}^{d} U_i^{\theta}}{\mathbb{E}[\prod_{i=1}^{d} U_i^{\theta}]} \quad \text{for some } \theta > 0,$$

and the dual Spearman allocation is defined via the stress scenario in (1) as

$$\frac{\mathrm{d}Q_{\mathbf{X}}}{\mathrm{d}\mathbb{P}} = \frac{\prod_{i=1}^{d} (1-U_i)^{-\theta}}{\mathbb{E}[\prod_{i=1}^{d} (1-U_i)^{-\theta}]} \quad \text{for some } \theta \in (0,1),$$

The name "Spearman" comes from the fact that $\mathbb{E}[\prod_{i=1}^{d} U_i]$ is a linear transform of the multivariate Spearman's rank correlation of $\mathbf{X}$. The Spearman and dual Spearman allocation rules enjoy several useful properties, including an independence-preserving property, meaning that an independent vector $\mathbf{X}$ under $\mathbb{P}$ remains independent under $Q_{\mathbf{X}}$.