A theory of multivariate stress testing

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Abstract

We present a theoretical framework for stressing multivariate stochastic models. We consider a stress to be a change of measure, placing a higher weight on multivariate scenarios of interest. In particular, a *stressing mechanism* is a mapping from random vectors to Radon-Nikodym densities. We postulate desirable properties for stressing mechanisms addressing alternative objectives. Consistently with our focus on dependence, we require throughout invariance to monotonic transformations of risk factors. We study in detail the properties of two families of stressing mechanisms, based respectively on mixtures of univariate stresses and on transformations of statistics we call Spearman and Kendall's cores. Furthermore, we characterize the aggregation properties of those stressing mechanisms, which motivate their use in deriving new capital allocation methods, with properties different to those typically found in the literature. The proposed methods are applied to stress testing and capital allocation, using the simulation model of a UK-based non-life insurer.

Keywords: Dependence, probability distortion, risk measure, sensitivity analysis, stress testing, systemic risk.

1 Introduction

Stress testing quantifies the response of a risk model to changes in assumptions, which may reflect shifts in the environment, occurrence of adverse events or movements in parameter values. This can be an internal exercise for a firm, e.g. for performing sensitivity analysis (Broadie and Glasserman, 1996, Hong and Liu, 2009, Borgonovo and Plischke, 2016) and model validation (Pesenti et al., 2019), or for allocating capital (Dhaene et al., 2012, Asimit et al., 2019) and measuring performance (Bauer and Zanjani, 2016). Stress testing can also apply across a market, by considering its connectedness and, thus, systemic risk (Brechmann et al., 2013, Gandy and Veraart, 2017). As stress testing allows the monitoring of financial institutions' exposures and of system vulnerabilities, it serves as a cornerstone of financial risk management and regulation (Duffie, 2018, Prudential Regulatory Authority, 2019, Financial Policy Committee, 2019); for more on stress test design see Rebonato (2010), Orlov et al. (2021), and Parlatore and Philippon (2022).

When performing stress tests, analysts change the statistical properties of risk factors that represent an institution's exposure or drive a system's behaviour. But there can be many different ways to effect such a change. A natural question – which is precisely the gap in the academic literature we aim to address – arises: *How should one choose a well-justified stress test, with properties appropriate to a specific application?* In the decision-theoretic (Gilboa, 2009, Wakker, 2010) and risk management (Artzner et al., 1999, Föllmer and Schied, 2011) literatures, such questions are typically answered using axiomatic approaches, which specify desirable (as well as undesirable) properties, in order to identify useful forms of decision criteria, such as utilities and risk measures. In our context of stress testing, we also need to identify suitable properties of stress tests and mechanisms for generating them. In order to engage in systematic study of such *stressing mechanisms*, we first need to introduce a theoretical framework with the associated mathematical formalism. The main contributions of the paper then lie in the formulation of such a new framework, in the justification of technical properties within it, and in theoretical results guiding the design of stressing mechanisms for various applications.

It is well understood that the aggregate risk in a system, be that a financial portfolio or a whole market, is profoundly affected by the dependence between uncertain inputs or *risk factors* – see the extensive treatment of McNeil et al. (2015) and references therein. This motivates the need for carefully and explicitly integrating stochastic dependence considerations into the design of stress tests. Nonetheless, to our knowledge, a theoretical framework that performs this integration task is currently missing from the academic literature. In this paper, we address this gap in the literature, by systematically studying multivariate stressing mechanisms, formally understood as Radon-Nikodym densities depending on random vectors of risk factors. Our stressing methods are endogenous by design, in which the risk factors themselves generate stress scenarios, different from settings with exogenously specified stress scenarios (e.g. Cambou and Filipović, 2017). Our approach is also distinct from the literature on systemic risk, which considers dependencies primarily from the perspective of network connections, and sees external shocks (instead of distortions of the probabilistic model) as sources of stress (Eisenberg and Noe, 2001).

In Section 2, we define stressing mechanisms via changes of probability measure. Such an approach focuses on how to stress a given model, rather than, e.g., estimating or approximating an unknown quantity of interest, and is common in the financial risk management (Breuer et al., 2012) and sensitivity analysis literatures (Pesenti et al., 2019). Defining stresses via measure changes affords numerical benefits, as it does not require repeated simulation under alternate model assumptions. However, these literatures do not explicitly integrate dependence considerations into the design of stress tests and the few papers that take a genuinely multivariate perspective, e.g., McNeil and Smith (2012), Pesenti et al. (2021), deploy rather different frameworks.

We address this issue in Section 3, where we postulate desirable properties, or axioms, for multivariate stressing mechanisms. While alternative properties, e.g., on the way stressing should impact the multivariate risk factor distribution, may be justified in different contexts, we consistently require an invariance property, which means that a stressing mechanism does not change when risk factors are subject to increasing transformations. Hence, stressing mechanisms are directly related to the dependence structure of risk factors. The invariance property has three interrelated implications: stressing mechanisms do not depend on arbitrary (non-linear) changes of scale; the stressed distribution of any risk factor depends only on its baseline distribution and the copula of the risk factors; stressing mechanisms can be formulated on the space of risk factors, without reference to specific portfolio structures.

On the technical side, our framework is related to the theory of risk measures, as developed by numerous authors, indicatively Artzner et al. (1999), Rockafellar et al. (2000), Szegö (2005), Pflug and Romisch (2007), Föllmer and Schied (2011); for a broad and complementary perspective, see Aven (2016). In fact, the stressing mechanisms introduced here can be directly used to construct new multivariate risk measures, e.g., as risk factors' expectations under a change of measure. Nevertheless, our framework has a different technical foundation than the classical theory of risk measures, be it univariate or multivariate – this also makes axiomatic characterizations more challenging. Risk measures are functions that map a random variable (or random vector) to a *real number* (or vector). In contrast, our stressing mechanisms are mappings from a random vector to a *Radon-Nikodym derivative*, and thus have a more complex mathematical structure. Our approach is more flexible than the distributional transforms studied by Liu et al. (2021), which map univariate to univariate distributions. Our framework is also quite different from dynamic or systemic risk measures studied by e.g., Riedel (2004) and Biagini et al. (2019), which map a random variable to one in a subspace via operations such as conditional expectation or optimization, or from multivariate risk measures, e.g., Prékopa (2012), Farkas et al. (2015), Prékopa and Lee (2018).

We introduce and study in depth two classes of invariant stressing mechanisms in Section 4. First, we consider mixtures of univariate stressing mechanisms. We prove a representation result which shows that an invariant stressing mechanism belongs to this class if and only if it satisfies a technical property, which roughly means that only minimal information is used to generate the stressing mechanism. Second, we introduce Spearman and Kendall stressing mechanisms, whose construction is directly inspired by the stochastic quantities underlying (multivariate) rank correlation coefficients. Besides other desirable properties, such stressing mechanisms preserve independence between risk factors. Furthermore, we study the aggregation properties of those multivariate stressing mechanisms by stochastically comparing their impact on marginal distributions, to that obtained by stressing one risk factor at a time. We show that mixture stresses produces diversification credits, while (dual) Spearman stresses produce aggregation penalties.

Detailed examples of stressing mechanisms, addressing different design criteria, are presented throughout Section 4. Through these examples we sometimes extend given classes of stressing mechanisms. In particular, in Section 4.4 we show how to strengthen (or indeed induce) dependence between risk factors, in a Multivariate Pareto, Gaussian, and independent setting. Furthermore, in Section 4.5, we modify our stressing mechanisms in order to effect changes in the volatility, rather than the size of individual risk factors.

The versatility of the proposed stressing mechanisms, and their distinct aggregation properties, allow us to deploy them in different contexts. We demonstrate this in Section 5, where we apply our methods to a real-life economic capital model, provided by a UK-based insurer. One of the key tasks of sensitivity analysis is assessing the comparative importance of uncertain model inputs (Borgonovo and Plischke, 2016). For such an application, mixture stresses are shown to be effective tools. By comparing consistently designed multivariate and univariate stresses, we assess variable importance in a new way that combines conceptual coherence with computational efficiency. Furthermore, capital allocation methods are often understood via measure changes (e.g. Furman and Zitikis, 2008). We show that capital allocations derived via (dual) Spearman stressing mechanisms address two known practical problems with standard Euler-type allocations (Tasche, 2004): instability of the allocation to local risk mitigations and excessive diversification credits for small uncorrelated risks. Thus, with our framework, we contribute new capital allocations with distinct properties, addressing issues that have often prevented the operationalization of extant methods. Our real-life example thus demonstrates the effectiveness of the proposed framework in addressing two key applications of stress testing.

2 Stressing mechanisms

2.1 Notation and terminology

Fix an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We call $\mathbb{P}$ the *baseline probability measure*. Let $\mathcal{X}$ be the set of all random variables in this probability space and $\mathbf{X} = (X_1, \dots, X_d) \in \mathcal{X}^d$ be a random vector of interest. We consider that $\mathbf{X}$ represents a vector of *risk factors*. $F_{\mathbf{X}}$ is the joint cumulative distribution function of $\mathbf{X}$ and F_{X_i} is its i-th margin, $i = 1, \dots, d$. Let $\bar{F}_{\mathbf{X}}(\mathbf{x}) = \mathbb{P}(\mathbf{X} > \mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$, with corresponding margins $\bar{F}_{X_i}$. Throughout the paper, all inequalities on vectors are component-wise; furthermore, terms such as "increasing" and "decreasing" are understood in the non-strict sense.

We denote by $\mathcal{U} \subset \mathcal{X}$ the set of all standard uniform random variables. If the random vector $\mathbf{X}$ has continuous margins, we define $U_i = F_{X_i}(X_i)$, $\bar{U}_i = \bar{F}_{X_i}(X_i)$, $i = 1, \dots, d$, such that $\mathbf{U} = (U_1, \dots, U_d) \in \mathcal{U}^d$. Thus, U_i is the uniform transform of the random variable X_i such that $F_{X_i}^{-1}(U_i) = X_i$ almost surely. For the existence of uniform transforms, without requiring continuity, see Föllmer and Schied (2011, Lemma A.28).

The joint distribution $C_{\mathbf{X}}$ of $\mathbf{U}$ is the *copula* of $\mathbf{X}$, meaning that we can write $F_{\mathbf{X}}(\mathbf{x}) = C_{\mathbf{X}}(F_1(x_1), \dots, F_d(x_d))$; the joint distribution $\bar{C}_{\mathbf{X}}$ of $\bar{\mathbf{U}} = (\bar{U}_1, \dots, \bar{U}_d) \in \mathcal{U}^d$, is the *survival copula* of $\mathbf{X}$ (Denuit et al., 2006, Sec. 4.4.1). In the case that distributions have discontinuities, the vector $\mathbf{X}$ generally admits more than one copula. We show in Appendix A how to uniquely construct from $\mathbf{X}$ a vector of uniform transforms $\mathbf{U} \in \mathcal{U}^d$ and identify $C_{\mathbf{X}}$ as the joint distribution of this $\mathbf{U}$. For simplicity of exposition we will assume throughout the paper that – unless otherwise specified – marginal distributions of risk factors are continuous; with the understanding that our arguments easily generalise to the discontinuous case, following Appendix A.

Let $\mathcal{R} \subset \mathcal{X}$ be the set of non-negative integrable random variables with expectation 1. Each element

$Z \in \mathcal{R}$ is a Radon-Nikodym density of a probability measure $\mathbb{Q}^Z$, that is, $Z = \mathrm{d}\mathbb{Q}^Z/\mathrm{d}\mathbb{P}$. Let $\mathcal{M}_d$ be the set of distributions on $\mathbb{R}^d$ and $\mathbb{F}_d$ be the set of measurable functions mapping $\mathbb{R}^d$ to $\mathbb{R}$. Further, denote by $\mathcal{M} = \mathcal{M}_1$ and $\mathbb{F} = \mathbb{F}_1$.

Some notions of dependence will be essential throughout the paper, since our setting is multivariate in nature. A random vector $X \in \mathcal{X}^d$ is comonotonic if there are increasing functions $f_1, f_2, \dots, f_d \in \mathbb{F}$ and some random variable $Z \in \mathcal{X}$ such that $X_i = f_i(Z)$. The pairs (X_i, U_i), as defined above, are comonotonic. Conversely, V, W are *countermonotonic*, if we can write $V = f_1(Z),\ W = f_2(Z)$, for an increasing function f_1, a decreasing function f_2, and some random variable Z. The pairs $(X_i, \bar{U}_i)$ are countermonotonic.

For distributions $G, H \in \mathcal{M}_d$, we say that H *stochastically dominates* G and write $G \preceq_{\mathrm{st}} H$, if for any $\mathbf{X} \sim G,\ \mathbf{Y} \sim H$ and any increasing function $f \in \mathbb{F}_d$, it holds that $\mathbb{E}[f(\mathbf{X})] \leqslant \mathbb{E}[f(\mathbf{Y})]$; with slight abuse of notation, we will also write $\mathbf{X} \preceq_{\mathrm{st}} \mathbf{Y}$. We say that $\mathbf{X} \in \mathcal{X}^d$ is *stochastically increasing* in $W \in \mathcal{X}$, if for $w_1 \leqslant w_2$ it holds that $\mathbb{P}(\mathbf{X} \leqslant \cdot \mid W = w_1) \preceq_{\mathrm{st}} \mathbb{P}(\mathbf{X} \leqslant \cdot \mid W = w_2)$. For distributions $G, H \in \mathcal{M}$, we say that H dominates G in *increasing convex order* and write $G \preceq_{\mathrm{icx}} H$, if for any $X \sim G,\ Y \sim H$ and any increasing convex function $f \in \mathbb{F}$, it holds that $\mathbb{E}[f(X)] \leqslant \mathbb{E}[f(Y)]$; again, we will also write $X \preceq_{\mathrm{icx}} Y$.

2.2 Definition of stressing mechanisms

Given the vector of risk factors $\mathbf{X}$, we are interested in stressing its distribution. We do this by the means of a Radon-Nikodym density, such that the stressing of the distribution of $\mathbf{X}$ arises through a change of measure.

Definition 1. A *stressing mechanism* is a mapping $\eta : \mathcal{X}^d \to \mathcal{R}$ satisfying the following properties:

(i) *Relevance.* For all $\mathbf{X} \in \mathcal{X}^d$, $\eta(\mathbf{X})$ is $\sigma(\mathbf{X})$-measurable, i.e. the realized value of $\eta(\mathbf{X})$ is determined by the realized value of $\mathbf{X}$.

(ii) *Law-invariance.* For all $\mathbf{X}$, $\mathbf{Y} \in \mathcal{X}^d$, $(\eta(\mathbf{X}), \mathbf{X}) \overset{\mathrm{d}}{=} (\eta(\mathbf{Y}), \mathbf{Y})$ if $\mathbf{X} \overset{\mathrm{d}}{=} \mathbf{Y}$.

A stressing mechanism can therefore be understood as a reweighting of outcomes. The property of relevance implies that $\mathbf{X}$ summarizes all information necessary for stressing. Note that the relevance property forces $\eta(\mathbf{c}) = 1$ for all constant vectors $\mathbf{c} \in \mathbb{R}^d$, such that, if risk factors have zero variance, there can be no stressing. Law invariance requires that vectors of risk factors with the same distribution will be stressed in the same way. We will also allow a stressing mechanism to be only defined on a subset of $\mathcal{X}^d$.

Stressing mechanisms can be represented as functions of the risk factors $\mathbf{X}$ and their distribution $F_{\mathbf{X}}$.

Proposition 1. *A mapping $\eta : \mathcal{X}^d \to \mathcal{R}$ is a stressing mechanism if and only if there exists $\Phi : \mathcal{M}_d \to \mathbb{F}_d$ such that for all $\mathbf{X} \in \mathcal{X}^d$,*

$$\eta(\mathbf{X}) = \Phi[F_{\mathbf{X}}](\mathbf{X}) \ \textit{a.s.}$$

Proposition 1 suggests that one can directly use the form $\eta(\mathbf{X}) = \Phi[F_{\mathbf{X}}](\mathbf{X})$. We shall call Φ in the above relation the *generator* of η. Consider a stressing mechanism η with generator Φ. For a random vector

$\mathbf{X}$, we call the distribution of $\mathbf{X}$ under $\mathbb{Q}^{\eta(\mathbf{X})}$ the *post-stress distribution* of $\mathbf{X}$, and we denote this by $F^{\eta}_{\mathbf{X}}$. In other words, for $\mathbf{x} \in \mathbb{R}^d$,

$$F^{\eta}_{\mathbf{X}}(\mathbf{x}) = \mathbb{Q}^{\eta(\mathbf{X})}(\mathbf{X} \leqslant \mathbf{x}) = \mathbb{E}[\eta(\mathbf{X})\mathbb{1}_{\{\mathbf{X}\leqslant\mathbf{x}\}}] = \int_{\mathbf{y}\leqslant\mathbf{x}} \Phi[F_{\mathbf{X}}](\mathbf{y})\mathrm{d}F_{\mathbf{X}}(\mathbf{y}). \tag{1}$$

In general we denote the i-th margin of $F^{\eta}_{\mathbf{X}}$ as $[F^{\eta}_{\mathbf{X}}]_i$ and expectations under $\mathbb{Q}^{\eta(\mathbf{X})}$ by $\mathbb{E}^{\eta(\mathbf{X})}$. As in most of the paper we keep $\mathbf{X}$ fixed, when there is no scope for misunderstanding, we will in the sequel simplify those notations to $F^{\eta}_{X_i}$, $\mathbb{Q}^{\eta}$, and $\mathbb{E}^{\eta}$.

2.3 Applications of stressing mechanisms

We now briefly introduce two particular applications of stressing mechanisms that we discuss extensively in the paper.

2.3.1 Stress testing of financial portfolios

We consider stress testing from an internal company perspective. For a given financial institution, e.g.,an insurer, let the function $f \in \mathbb{F}_d$ represent its portfolio structure, such that $Y := f(\mathbf{X})$ is the portfolio loss, see Figure 1 (left). The triple $(\mathbf{X}, f, \mathbb{P})$ can be understood as an *internal model*, used for risk and performance management purposes. The insurer is interested in the behaviour of the model output Y under alternative specifications of the risk factor distribution $F_{\mathbf{X}}$. This motivates stressing mechanisms of the form

$$\eta(\mathbf{X}) = \frac{\zeta(f(\mathbf{X}))}{\mathbb{E}[\zeta(f(\mathbf{X}))]}, \qquad \Phi[F_{\mathbf{X}}](\mathbf{x}) = \frac{\zeta(f(\mathbf{x}))}{\mathbb{E}[\zeta(f(\mathbf{X}))]}, \tag{2}$$

for some ζ such that $\zeta(f(X))$ is integrable, so that the Radon-Nikodym derivative depends on outcomes of the portfolio loss. Cambou and Filipović (2017), motivated by model uncertainty, propose choices of the function ζ using ϕ-divergence minimization, given constraints on the probabilities of specified events under $\mathbb{Q}^{\eta}$. In a related approach framed in the terms of sensitivity analysis, Pesenti et al. (2019) use entropy minimization arguments, given constraints on risk measures of the portfolio loss $Y = f(\mathbf{X})$. However, these approaches do not address the question of how to directly stress the (possibly numerous) risk factors of an internal model, in a consistent and parsimonious way that explicitly reflects their dependence structure and is not specific to a particular portfolio structure.

A further step is to apply a stressing mechanism across institutions in a financial market, aiming to capture systemically important events. This process is illustrated in Figure 1 (right), which depicts a market with three participants, with respective aggregation functions $f^{(1)}, f^{(2)}$ and $f^{(3)}$, each of which maps the shared risk factors $\mathbf{X}$ to its portfolio loss. A change of measure specified appropriately for all institutions would allow each of them to recalculate their portfolio loss distribution under stressed conditions and report e.g. their increased capital needs to the regulator. If stress testing is performed in such a manner, one has to consider that each of the participating companies will have a different internal model, which implies,

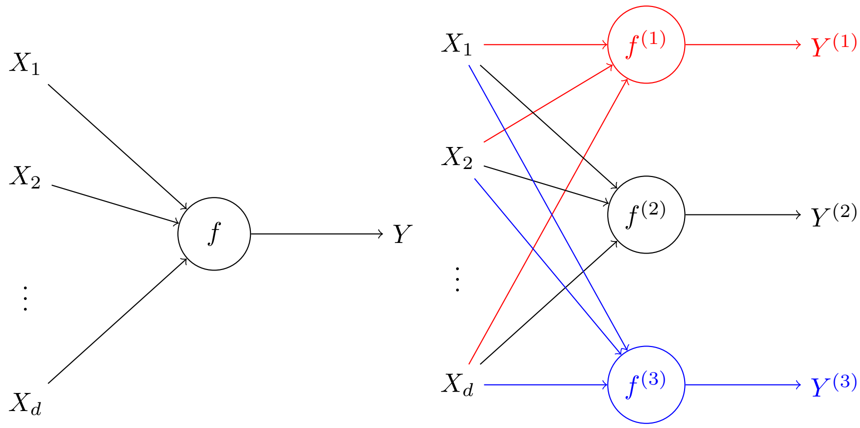

Figure 1: Left: Stylized representation of risk aggregation in an internal model; right: internal models across a market.

in an extension of our setting, their use of different baseline probability measures. Then a challenge for a regulator is to specify a stressing mechanism η, which can produce stressed models for different institutions in a consistent way. Such a stressing mechanism should be easy to specify and implement on very different models and not depend on a given portfolio structure, or choices on the marginal distributions F_i.

2.3.2 Capital allocation

A special case of risk aggregation occurs within linear portfolios, $Y = X_1 + \cdots + X_d$, where X_i represents the loss from the i-th line of business (or asset position). Then, for a stressing mechanism of the form $\varphi(Y) = \zeta(Y)/\mathbb{E}[\zeta(Y)]$, we can interpret the quantities

$$\mathbb{E}^{\varphi(Y)}[Y] = \mathbb{E}\left[Y \frac{\zeta(Y)}{\mathbb{E}[\zeta(Y)]}\right], \qquad \mathbb{E}^{\varphi(Y)}[X_i] = \mathbb{E}\left[X_i \frac{\zeta(Y)}{\mathbb{E}[\zeta(Y)]}\right]$$

as, respectively, the total capital requirement for the portfolio and the capital allocated to the i-th line of business (Furman and Zitikis, 2008, Dhaene et al., 2012). In the case of distortion risk measures (Wang et al., 1997, Acerbi and Tasche, 2002), the choice $\zeta(y) = \xi(F_Y(y))$, $y \in \mathbb{R}$ is made (for continuous F_Y), where ξ is a density on $[0, 1)$.

Following such an approach, modifications to one line of business change the stressing mechanism and, thus, the allocated capital to other lines of business. This presents practical challenges in industry applications, as large risk exposures end up dominating the aggregate capital. For example, consider a portfolio with additional exposure to X_i, that is, $(X_1, \ldots, (1 + w)X_i, \ldots, X_d)$ for some $w > 0$. Then the aggregate loss is $Y_w := Y + wX_i$ and the corresponding stressing mechanism $\varphi(Y_w) = \zeta(Y_w)/\mathbb{E}[\zeta(Y_w)]$. Let $\mathbb{E}[\mathbf{X}] = \mathbf{0}$ and $X_j \perp X_i$ for all $i \neq j,\ i, j \in \{1, \ldots, n\}$. Then, as $w \to \infty$, we have that $\mathbb{E}^{\varphi(Y_w)}[X_j] \to 0$, such that small well diversified positions do not attract a risk load, which can create perverse incentives for line managers. Furthermore, there is a conflict between basing an allocation method on a portfolio risk measure and satisfying reasonable criteria for stability of the allocation to local risk mitigation (Guan et al., 2023).

In the two applications discussed above, the need has arisen for mechanisms that (i) are functions of the whole vector $\mathbf{X}$ rather than only a function of them, e.g., a portfolio loss $f(\mathbf{X})$; and (ii) are invariant to increasing transforms of individual elements of $\mathbf{X}$, while reflecting its dependence structure.

3 Properties of stressing mechanisms

Naturally, one can design a variety of meaningful stressing mechanisms. For the purposes of this paper, we list a number of potentially desirable properties for a given stressing mechanism η. We emphasize that we do not consider all properties below attractive in all circumstances – instead we see them as a menu of possibilities that can be chosen from, depending on the problem context.

(a) **Invariance**. For all $\mathbf{X} \in \mathcal{X}^d$ and all strictly increasing functions $f_1, \dots, f_d$ on $\mathbb{R}$, $\eta\big((f_1(X_1), \dots, f_d(X_d))\big) = \eta(\mathbf{X})$ holds.

(b) **Joint stressing**. For all $\mathbf{X} \in \mathcal{X}^d$, $F_{\mathbf{X}} \preceq_{\mathrm{st}} F^{\eta}_{\mathbf{X}}$.

(c) **Marginal increasingness** (with respect to a given partial order $\preceq$ on $\mathcal{M}$). For all $\mathbf{X}, \mathbf{Y} \in \mathcal{X}^d$ and each $i = 1, \dots, d$, $F_{X_i} \preceq F_{Y_i}$ implies $F^{\eta}_{X_i} \preceq F^{\eta}_{Y_i}$.

(d) **Independence preserving**. If $\mathbf{X} \in \mathcal{X}^d$ has independent components under $\mathbb{P}$, then it does so under $\mathbb{Q}^{\eta(\mathbf{X})}$.

(e) **Symmetry**. For all $\mathbf{X} \in \mathcal{X}^d$ and all permutations σ of $\{1, \dots, d\}$, $\eta\big((X_{\sigma(1)}, \dots, X_{\sigma(d)})\big) = \eta(\mathbf{X})$ holds.

(f) **Constancy**. For all $1 < k < d$, $\mathbf{X} \in \mathcal{X}^k$ and $\mathbf{c} \in \mathbb{R}^{d-k}$, it holds that $\eta\big((\mathbf{X}, \mathbf{c})\big) = \eta\big((\mathbf{X}, 0, \dots, 0)\big)$.

(g) **Directness**. η is invariant and there exists $f : [0,1]^d \to \mathbb{R}_+$ measurable such that $\eta(\mathbf{U}) = f(\mathbf{U})$ holds for all $\mathbf{U} \in \mathcal{U}^d$.

Invariance to increasing transformations (a) represents a requirement that the stressing mechanism does not change when strictly increasing transformations are applied to the risk factors. This property ensures that stressing is not contingent on the particular scale that any given risk factor is expressed in. For example, in financial risk modeling, asset returns can be expressed as either linear returns or log-returns, depending on the convention in a particular context. These two choices are technically equivalent and should not lead to different stress scenarios and results. The invariance property is key to our paper, as it addresses the different but related issues identified in the discussion of Section 2.3. Note that for an invariant stressing mechanism η, we can write $\eta(\mathbf{X}) = \eta(\mathbf{U})$, such that we use as input the vector of uniform transforms $\mathbf{U} \in \mathcal{U}^d$, as constructed in Section 2.1 and Appendix A. As the stressing mechanism does not depend on the marginal distributions, the focus is placed on the dependence structure of $\mathbf{X}$, a critical concern for multivariate stress testing. At the same time, a careful selection of an invariant stressing mechanism can also have desired effects on marginal distributions, e.g.,by making their tails heavier, as discussed in Example 1.

Joint stressing (b) implies that risk factors become larger, in the usual stochastic order, under the post-stress distribution. In particular, the joint probability of risk factors concurrently exceeding a high threshold becomes higher. If one understands such joint exceedances as adverse events, this indicates that stressing increases portfolio risk. It is apparent that joint stressing implies that marginal distributions of risk factors increase in stochastic dominance, that is, $F_{X_i} \preceq_{\mathrm{st}} F^{\eta}_{X_i}$, $i = 1, \ldots, d$. Of course, one may not necessarily consider high joint values of $\mathbf{X}$ as adverse, e.g., in highly non-linear models with complex interactions. Nonetheless, the joint stressing property allows monitoring the movement of portfolio positions when risk factors are stochastically increased or indeed decreased by stressing decreasing functions of risk factors; in the context of an invariant stressing mechanism this just means substituting $\bar{U}_i = 1 - U_i$ for U_i.

Furthermore, sometimes the aim of stress testing is to modify risk factors' volatility, rather than to stochastically increase or decrease them. In such settings the joint-stressing property is generally not desirable. We defer a more systematic discussion of this point to Section 4.5, where we introduce stressing mechanisms that satisfy a specific variability stressing property that is well suited to the context of invariant stressing mechanisms. Furthermore, in Example 1 we introduce a two-tailed stress and in Example 7 a stress on risk factors' covariance structure.

Marginal increasingness (c) is defined with respect to a specific partial stochastic order on $\mathcal{M}$, e.g., $\preceq_{\mathrm{st}}$. It means that if we compare two models and their margins are ordered under $\mathbb{P}$, they should be ordered similarly under $\mathbb{Q}^{\eta}$, such that stressing both models preserves the ordering structure. For a related argument in the context of distributional regression, see Henzi et al. (2021). We consider marginal increasingness a generally desirable property. It is particularly useful when concurrently and consistently stressing different models, as in the regulatory context of Section 2.3.1.

The independence preserving property (d) reflects situations where risk factors are independent under the baseline model and a decision maker does not want to artificially introduce dependence via the stressing mechanism. Dependence may be implausible for some risk factors, e.g.,in an insurance context, between the California Earthquake and UK Windstorm & Flood scenarios specified by the UK's regulator (Prudential Regulatory Authority, 2019). Hence, in such situations, stressing only impacts the marginal distributions of the risk factors. Independence preserving will not be desirable in those situations where stress testing is meant to exacerbate positive correlation between risk factors, e.g.,in the case of a systemic risk stress. Examples 5, 6 and 7 in Section 4.4 show how one can design stressing mechanisms that increase dependence within particular models (or even induce such dependence, when $\mathbf{X}$ is independent under $\mathbb{P}$).

Symmetry (e) implies that the order of risk factors in the random vector $\mathbf{X}$ has no impact on the stressing mechanism. Such a property means that all risk factors, are, in some sense, stressed in the same way; the stressing mechanism does not *ex ante* consider some risk factors as more important or relevant than others. Thus symmetry is characteristic of a bottom-up approach to stress testing, where risk factors are stressed in a consistent way, and their comparative impacts (e.g. on the portfolio position) are monitored. Symmetry is not always desirable: for example, if stress testing happens as part of model risk management, in order to evaluate the worst plausible distribution for individual risk factors, it makes sense to stress risk factors differently, depending on the extent that their distribution is subject to uncertainty. While we often focus on symmetric stressing mechanisms, this is not a limitation of our framework: even symmetric stressing mechanisms such as those of Section 4.2 can be easily extended. Furthermore, in Section 5.2 we show how the comparison of asymmetric and symmetric stresses provides a means for identifying the main risk drivers in a portfolio.

Constancy (f) means that the particular values of risk factors that are constant have no impact on the stressing mechanisms. This makes sure that a risk factor being volatile is a precondition for its realization having an effect on stress testing.

Directness (g) is a technical property, strengthening invariance (a). A direct stressing mechanism depends on the realized value of $\mathbf{U}$ only, and not on its distribution. Direct stressing mechanisms can be applied without explicit knowledge of the copula $C_{\mathbf{X}}$. Directness is a quite a strong property, and we will see in Theorem 1 that it imposes some specific forms of η.

Finally, we note that it is important to consider the way that stressing mechanisms reflect risk diversification and aggregation – in particular, the ways that stressing the full vector of risk factors has different implications to stressing a single risk factor. This discussion parallels and is connected to the subadditivity property of risk measures (Artzner et al., 1999). However, given that we do not assume any (e.g., linear) portfolio structure, subadditivity does not directly transfer to our setting. In Section 4.3 we introduce a diversification/aggregation property, explain how it applies to broad classes of stressing mechanisms, and discuss its relation to subadditivity in more detail.

Any of the above properties may hold just on a subset of $\mathcal{X}^d$. This permits focusing attention on random vectors that satisfy particular properties. A vector $\mathbf{X} \in \mathcal{X}^d$ is *associated* if we have that $\mathbb{E}[g(\mathbf{X})h(\mathbf{X})] \geqslant \mathbb{E}[g(\mathbf{X})]\mathbb{E}[h(\mathbf{X})]$, for all increasing functions $g, h \in \mathbb{F}_d$ such that the expectations exist. Association is a general positive dependence property, encompassing cases such as independence, comonotonicity, and implying positive quadrant dependence (Denuit et al., 2006, Sec. 7.2.3). We then denote by $\mathcal{X}^d_+$ the set of associated random vectors in $\mathcal{X}^d$. Furthermore, we denote by $\mathcal{X}^d_\perp \subset \mathcal{X}^d_+$ the set of independent random vectors in $\mathcal{X}$ and by $\mathcal{X}^d(C) := \{\mathbf{X} : C_{\mathbf{X}} = C\}$ the set of vectors that share a given copula C. We now show how some of the stipulated properties hold on subsets of $\mathcal{X}^d$.

Proposition 2. *For a given stressing mechanism η, the following hold.*

i) Let Φ be the generator of η. If, for all $\mathbf{X} \in \mathcal{X}^d_+$, the function $\mathbf{x} \mapsto \Phi[F_{\mathbf{X}}](\mathbf{x})$ is increasing, then η satisfies joint stressing on $\mathcal{X}^d_+$.

ii) For a given copula C, any invariant stressing mechanism is marginally increasing with respect to $\preceq_{\mathrm{st}}$ on $\mathcal{X}^d(C)$.

iii) Any invariant stressing mechanism satisfies constancy.

It can be noted that, unless in some very special cases, the converse of Proposition 2(i) does not hold.

4 Classes of invariant stressing mechanisms

4.1 Mixture stressing mechanisms

The first class of stressing mechanisms we consider is based on mixtures. For stressing mechanisms $\eta_1, \dots, \eta_k$ it is apparent that $\eta = \sum_{i=1}^k \lambda_i \eta_i$, where $(\lambda_1, \dots, \lambda_k) \in \Delta_k = \{\boldsymbol{\lambda} \in [0,1]^k : \sum_{i=1}^k \lambda_i = 1\}$ the standard simplex in $\mathbb{R}^k$, is again a stressing mechanism. Here, we focus on mixtures of univariate stressing mechanisms.

We first explain the simple case of univariate stressing mechanisms, which will be the basis for a characterization result. Assuming invariance, a univariate stressing mechanism on $\mathcal{X}$ can be represented by $\eta : \mathcal{U} \to \mathcal{R}$. Since we know that U is uniform, η will have the form $\eta(U) = g(U)$, for some $g \in \mathcal{G}$, where $\mathcal{G}$ is the set of probability density functions over $[0,1]$. Hence, univariate invariant stressing mechanisms satisfy the directness property discussed in Section 3. Let now $\widehat{g}(u) = \int_0^u g(v)\mathrm{d}v$, $u \in [0,1]$, the cumulative distribution function of g. For $x \in [0,1]$ and any continuously distributed X with distribution F_X, we have for such η that

$$F_X^\eta(x) = \mathbb{E}[g(F_X(X))\mathbb{1}_{\{X \leqslant x\}}] = \int_0^{F_X(x)} g(u)\mathrm{d}u = \widehat{g} \circ F_X(x), \quad x \in \mathbb{R}.$$

(Again we refer to Appendix A for the case that distributions are not continuous – all our arguments in this section still hold under the generalized definition of U.) Hence, the post-stress distribution F_X^η is a probability distortion of F_X. Probability distortions are characterized by Liu et al. (2021) among distributional transforms via a property similar to the invariance. It is clear that $F_X \preceq_{\mathrm{st}} F_X^\eta$ if and only if $\widehat{g}(t) \leqslant t$ for all $t \in [0,1]$; this condition is weaker than increasingness of g. Let $\mathcal{G}^* \subset \mathcal{G}$ be the set of functions $g \in \mathcal{G}$ satisfying $\widehat{g}(t) \leqslant t$ for all $t \in [0,1]$.

Direct stressing mechanisms are precisely represented by mixtures of univariate stressing mechanisms.

Theorem 1. *An invariant stressing mechanism η is direct if and only if it is a mixture of univariate stressing mechanisms, i.e., there exist functions $g_1, \dots, g_d \in \mathcal{G}$ and $\boldsymbol{\lambda} \in \Delta_d$ such that*

$$\eta(\mathbf{U}) = \sum_{i=1}^d \lambda_i g_i(U_i), \qquad a.s., \tag{3}$$

where $\mathbf{U} = (U_1, \dots, U_d) \in \mathcal{U}^d$. Moreover, assuming (3) holds,

(i) η is jointly stressing on $\mathcal{X}_\perp^d$ if and only if $g_i \in \mathcal{G}^*$ for each i with $\lambda_i > 0$;

(ii) η is symmetric if and only if $g_1 = \dots = g_d$ and $\lambda_1 = \dots = \lambda_d = 1/d$;

(iii) η is independence preserving if and only if it is univariate, i.e., at most one of $\lambda_1 g_1, \dots, \lambda_d g_d$ is not a constant.

(iv) η is marginally increasing with respect to $\preceq_{\mathrm{icx}}$ on $\mathcal{X}_{\perp}^{d}$, if and only if g_i is increasing for each i with $\lambda_i > 0$.

We showed in Proposition 2 that all stressing mechanisms satisfy marginal increasingness with respect to the stochastic order $\preceq_{\mathrm{st}}$, on vectors sharing a copula. As Theorem 1(iv) shows, for the mixture stressing mechanisms considered here and the case of independent risk factors, we can also satisfy such a property with respect to the increasing convex order, if the functions g_i are increasing.

Example 1 (**Univariate stresses**)**.** A simple special case of a stressing mechanism is one that only depends on a single risk factor; for example consider the (direct and invariant) stressing mechanisms of the form:

$$\eta_i(X_i) = (1-\theta)\bar{U}_i^{-\theta}, \; \theta \in (0,1).$$

It is easy to show that, if we use such a stressing mechanism, the tail of the marginal distribution of X_i becomes heavier, in the sense that

$$\bar{F}_{X_i}^{\eta_i(X_i)}(x) = \int_x^{\infty} (1-\theta)\bar{F}_{X_i}(t)^{-\theta} \mathrm{d}F_{X_i}(t) = \bar{F}_{X_i}(x)^{1-\theta},$$

which also demonstrates how the post-stress marginal distribution stochastically dominates F_{X_i}.

If we additionally assume that X_i is Pareto distributed, $X_i \sim \mathrm{Par}(\alpha, b)$, then under Q^{η_i} the distribution of X_i remains Pareto, but with a reduced tail index $(1-\theta)\alpha$, indicating a heavier tail. Alternatively, if X_i is exponentially distributed with rate parameter β, its post-stress distribution is also exponential, with rate parameter $(1-\theta)\beta$. Note that Pareto and Exponential tails are canonical modelling tools for the excesses of random variables above high thresholds, see e.g., McNeil et al. (2015, Sec. 5.2). Furthermore, we can also stress both tails of the marginal distributions, by a simple adjustment to the univariate stresses considered so far,

$$\tilde{\eta}_i(X_i) = 2^{-\theta}(1-\theta)\left(U_i^{-\theta}\mathbb{1}_{\{U_i<0.5\}} + \bar{U}_i^{-\theta}\mathbb{1}_{\{U_i\geqslant 0.5\}}\right), \quad \theta \in (0,1),$$

which can be used as a building block for more complex stresses. The tail impacts are similar to those discussed earlier, but now apply to both tails. For example, if F_i is a Laplace (double exponential) distribution, simple calculation shows that $F_i^{\tilde{\eta}_i}$ is again Laplace, but with higher scale parameter. Naturally, asymmetry can also be introduced by assigning different exponents to the two terms of $\tilde{\eta}_i$.

If we assume independence of $\mathbf{X}$, the distribution of the remaining risk factors $\mathbf{X}_{-i}$ is unaffected by the use of the stresing mechanism $\eta_i(X_i)$. Let us consider the case that the random vector $\mathbf{X}$ is not independent. How does the stressing mechanism η_i impact the distribution of the whole vector $\mathbf{X}$? One way to characterize the post-stress distribution of $\mathbf{X}$ is via *inverse Rosenblatt transforms*, as considered by Rüschendorf and de Valk (1993) and Pesenti et al. (2021). We can always represent the risk factors by

$$\mathbf{X} = \boldsymbol{\psi}(X_i, \boldsymbol{V}) = \left(\psi^{(1)}(X_i, \boldsymbol{V}), \ldots, \psi^{(n)}(X_i, \boldsymbol{V})\right) \text{ a.s.},$$

where $\boldsymbol{\psi} : \mathbb{R}^d \to \mathbb{R}^d$ and $\mathbf{V} \sim \mathcal{U}^{d-1}$. Then, under $\mathbb{Q}^{\eta_i(X_i)}$, the vector $(X_i, \mathbf{V})$ remains independent and $\mathbf{V}$ remain uniform. This, together with the representation $\mathbf{X} = \boldsymbol{\psi}(X_i, \boldsymbol{V})$ also allows easy simulation of $\mathbf{X}$ under $\mathbb{Q}^{\eta_i(X_i)}$.

Example 2 (Mixtures of univariate stresses). Now consider stressing mechanisms of the form:

$$\eta(\mathbf{X}) = \sum_{i=1}^{d} \lambda_i (1-\theta) \bar{U}_i^{-\theta}, \quad \text{for} \quad (\lambda_1, \dots, \lambda_d) \in \Delta_d, \ \theta \in (0,1),$$

noting that this directly generalizes Example 1 above. Denoting $g(u) = (1-\theta)(1-u)^{-\theta}$, we can write for any $f \in \mathbb{F}_d$

$$\mathbb{E}^{\eta}\left[f(\mathbf{X})\right] = \sum_{i=1}^{d} \lambda_i \mathbb{E}\left[f(\mathbf{X}) g(U_i)\right] = \mathbb{E}\left[\sum_{i=1}^{d} \mathbb{1}_{A_i} f(\mathbf{X}) g(U_i)\right],$$

where A_i are events independent of $\mathbf{X}$, with $\mathbb{P}(A_i) = \lambda_i, \ i = 1, \dots, d$. The first equality shows how the mixture stressing mechanism can be understood as a weighted average of the 'cascade' stresses discussed in Example 1, each starting at a different X_i. The second equality shows how one can evaluate the expectations under $\mathbb{Q}^{\eta}$, by choosing at random, within each simulated scenario, with respect to which marginal to stress the model.

The case where $\mathbf{X}$ is independent yields a simple form for the post-stress marginal distributions:

$$F^{\eta}_{X_i} = \lambda_i \hat{g} \circ F_{X_i} + (1-\lambda_i) F_{X_i},$$

where $\hat{g}$ is the cumulative distribution function corresponding to g. In particular, for our choice of univariate stress with $g(u) = (1-\theta)(1-u)^{-\theta}$, we have that

$$\bar{F}^{\eta}_{X_i}(x) = \lambda_i \bar{F}_{X_i}(x)^{1-\theta} + (1-\lambda_i) \bar{F}_{X_i}(x).$$

Hence, the marginal post-stress survival functions are expressed as mixtures of the baseline distributions and their (heavier-tailed) transformed ones. In this way, stressing can be seen to represent a contamination of the marginal distributions with respect to heavier tailed ones, as is often done in the study of model uncertainty (e.g. Cont et al., 2010, Pesenti et al., 2021).

4.2 Spearman and Kendall stressing mechanisms

The second class of stressing mechanisms we study are based on quantities we term *Spearman's* and *Kendall's cores*. These are defined below.

Definition 2. For any $\mathbf{X} \in \mathcal{X}^d$, with $\mathbf{U}$ and $\bar{\mathbf{U}}$ the associated vectors of uniforms as defined in Section 2.1 we define the following quantities:

i) The random variables $S(\mathbf{X}) = U_1 \cdot \ldots \cdot U_d$ and $\bar{S}(\mathbf{X}) = \bar{U}_1 \cdot \ldots \cdot \bar{U}_d$ are called respectively *Spearman's core* and *Spearman's dual core*.

ii) The random variables $K(\mathbf{X}) = C_{\mathbf{X}}(\mathbf{U})$ and $\bar{K}(\mathbf{X}) = \bar{C}_{\mathbf{X}}(\bar{\mathbf{U}})$ are called respectively *Kendall's core* and *Kendall's dual core*.

Spearman's and Kendall's cores play a key role in the construction of dependence measures. For $d = 2$ the Spearman and Kendall rank correlation coefficients are defined respectively by $r_S(X_1, X_2) = 12\mathbb{E}[S(\mathbf{X})] - 3$ and $r_K(X_1, X_2) = 4\mathbb{E}[K(\mathbf{X})] - 1$. For $d > 2$, these variables can be understood as summaries of multivariate dependence; for example the distribution of $K(\mathbf{X})$ is intrinsically linked with Archimedean copulas (Genest and Rivest, 1993). We find the Spearman and Kendall's (dual) cores attractive building blocks for stressing mechanisms, as they give suitable summaries of the multivariate behaviour of $\mathbf{X}$, without reference to the marginal distributions or a particular portfolio structure.

Some elementary properties of Spearman's and Kendall's cores are stated below.

Proposition 3. *The following properties of (dual) Spearman's and Kendall's cores hold:*

i) If $\mathbf{X}$ is independent, then $K(\mathbf{X}) = S(\mathbf{X})$ and $\bar{K}(\mathbf{X}) = \bar{S}(\mathbf{X})$.

ii) If $\mathbf{X}$ is comonotonic, $S(\mathbf{X}) = U_i^d$, $\bar{S}(\mathbf{X}) = \bar{U}_i^d$, $K(\mathbf{X}) = U_i$, and $\bar{K}(\mathbf{X}) = \bar{U}_i$, for any $i = 1, \ldots, d$.

iii) If, for some i, j, the pair (X_i, X_j) is countermonotonic, then $K(\mathbf{X}) = \bar{K}(\mathbf{X}) = 0$.

iv) $A(\mathbf{X}) \preceq_{\mathrm{st}} V$, for any $V \in \mathcal{U}$ and $A \in \{S, \bar{S}, K, \bar{K}\}$.

We focus here on a class of stressing mechanisms that are defined as powers of (dual) Spearman's and Kendall's cores. Specifically, we consider stressing mechanisms of the form

$$
\begin{aligned}
\eta(\mathbf{X}) &= \frac{A(\mathbf{X})^{\theta}}{\mathbb{E}\left[A(\mathbf{X})^{\theta}\right]}, \quad A \in \{S, K\}, \quad \theta > 0, \\
\eta(\mathbf{X}) &= \frac{A(\mathbf{X})^{-\theta}}{\mathbb{E}\left[A(\mathbf{X})^{-\theta}\right]}, \quad A \in \{\bar{S}, \bar{K}\}, \quad 0 < \theta < 1,
\end{aligned}
\tag{4}
$$

when these are well-defined. Such stressing mechanisms satisfy a number of the properties we formulated in Section 3.

Proposition 4. *Stressing mechanisms of the form (4) satisfy, on their domain, the following properties:*

i) Invariance, independence preserving, and symmetry.

ii) Joint stressing on $\mathcal{X}_+^d$.

iii) Marginal increasingness with respect to the order $\preceq_{\mathrm{icx}}$ on $\mathcal{X}_\perp^d$.

Finally, we note that it is easy in this framework to produce stressing mechanisms that are not symmetric e.g.,by $\eta(\mathbf{X}) = U_1^{\theta_1} \cdot \ldots \cdot U_d^{\theta_d} / \mathbb{E}[U_1^{\theta_1} \cdot \ldots \cdot U_d^{\theta_d}]$, for some $\theta_1, \ldots, \theta_d > 0$. We do not pursue this route further in the current paper, as we do not consider *a priori* reasons for stressing one particular risk factor more than another. Furthermore, similarly to the additive mixtures of univariate stressing mechanisms in (3), we may define a class of multiplicative mixtures via $\eta(\mathbf{X}) = g_1(U_1) \cdot \ldots \cdot g_d(U_d)/\mathbb{E}[g_1(U_1) \cdot \ldots \cdot g_d(U_d)]$ for some suitably chosen positive univariate functions $g_1, \ldots, g_d$. The Spearman stressing mechanisms belong to this class via the specification $g_1(u) = \cdots = g_d(u) = u^\theta$. For concision, we omit a thorough discussion of this broader class of stressing mechanisms here. Such more flexible constructions can give rise to stressing mechanisms designed to impact on the variability of risk factors.

The following two examples present applications of the dual Spearman and the Kendall stressing mechanisms.

Example 3 (**Dual Spearman and independence**)**.** Once again we building upon the univariate construction of Example 1. Assume that the vector of risk factors $\mathbf{X}$ is independent. A decision maker wants to stress the marginal distributions of $\mathbf{X}$ but not induce artificially dependence between its elements. Consider Spearman's dual stressing mechanism $\eta(\mathbf{X}) = \bar{S}(\mathbf{X})^{-\theta}/\mathbb{E}\left[\bar{S}(\mathbf{X})^{-\theta}\right]$, $\theta \in (0,1)$, and recall that, by independence of $\mathbf{X}$, we have $\bar{S}(\mathbf{X}) = \bar{K}(\mathbf{X}) = \bar{U}_1 \cdot \ldots \cdot \bar{U}_d$. Then, by Proposition 4, we have that $\mathbf{X}$ remains independent under $\mathbb{Q}^\eta$. Furthermore, for the marginals we have

$$\bar{F}^\eta_{X_i}(x) = \bar{F}_{X_i}(x)^{1-\theta} > \bar{F}_{X_i}(x), \quad i = 1, \ldots, d,$$

which demonstrates how the post-stress marginal distribution stochastically dominates F_{X_i}. Note that, in contrast to the univariate stress of Example 1, this transformation holds for all risk factors, rather than just a single one. Furthermore, expectations of the risk factors X_i under $\mathbb{Q}^\eta$ can be directly interpreted as distortion risk measures – specifically, $\mathbb{E}^\eta[X_i]$ corresponds to the *proportional hazards transform* of Wang (1996).

A fuller discussion of the case of (dual) Spearman stressing mechanisms under independence of $\mathbf{X}$ is given in Appendix C.

Example 4 (**Kendall and benchmark risk factors**)**.** Here show how a stressing mechanism based on Kendall's core can be constructed via the comparison of $\mathbf{X}$ to a suitably defined benchmark. Define $\mathbf{X}^{(i)}$, $i = 1, \ldots, n$, as independent copies of $\mathbf{X}$ and denote

$$\mathbf{W} := \bigvee_{i=1}^{n} \mathbf{X}^{(i)} = \Big(\max\left(X_1^{(1)}, \ldots, X_1^{(n)}\right), \ldots, \max\left(X_d^{(1)}, \ldots, X_d^{(n)}\right)\Big),$$

the component-wise maximum of $\mathbf{X}^{(i)}$, $i = 1, \ldots, n$. Consider the stressing mechanism based on Kendall's core $\eta(\mathbf{X}) = K(\mathbf{X})^n/\mathbb{E}\left[K(\mathbf{X})^n\right]$. Then, for any function $g \in \mathbb{F}_d$, we have:

$$\begin{aligned}
\mathbb{E}^{\eta}[g(\mathbf{X})] &= \frac{1}{\mathbb{E}[K(\mathbf{X})^n]}\mathbb{E}\left[g(\mathbf{X}) \cdot F_{\mathbf{X}}(\mathbf{X})^n\right] \\
&= \frac{1}{\mathbb{E}[K(\mathbf{X})^n]}\mathbb{E}\left[g(\mathbf{X}) \cdot \mathbb{P}\left(\mathbf{X} \geqslant \mathbf{W} \middle| \mathbf{X}\right)\right] \\
&= \frac{1}{\mathbb{E}[K(\mathbf{X})^n]}\mathbb{E}\left[g(\mathbf{X}) \cdot \mathbb{1}_{\{\mathbf{X} \geqslant \mathbf{W}\}}\right]
\end{aligned}$$

Hence, the post-stress joint density of $\mathbf{X}$, $f^{\eta}_{\mathbf{X}}(\mathbf{x})$, is proportional to the quantity $\mathbb{E}[f_{\mathbf{X}}(\mathbf{x})\mathbb{1}_{\{\mathbf{x} \geqslant \mathbf{W}\}}]$. To interpret this relation, let first $n = 1$ and view $\mathbf{X} = \mathbf{X}^{(1)}$ a benchmark set of risk factors. Then, the stressing mechanism places a non-zero weight on only those states where the risk factors dominate the benchmark $\mathbf{X}$. For $n > 1$, we have the stricter requirement that $\mathbf{X}$ must dominate the component-wise maximum of the benchmark vectors $\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(n)}$.

4.3 Diversification, aggregation and capital allocation

In the theory of risk measurement, considerations of *subadditivity* are fundamental (Artzner et al., 1999). Subadditivity requires that the pooling of risk exposures results in a portfolio with a lower risk measurement than the sum of its parts, reflecting diversification benefits. A less used alternative is *superadditivity* (e.g., Tsanakas, 2009, Wang et al., 2015), which postulates that (under certain dependence assumptions) the risk of pooled positions would be higher than the sum of its parts, leading to an aggregation penalty. The sub/super-additivity properties do not transfer naturally to our current context of stress testing, since we are generally not dealing with linear portfolios. First, as we make no assumption that a portfolio loss is linear in $\mathbf{X}$, the comparison of a stressed portfolio with a sum of risk assessments (e.g., post-stress expectations) pertaining to elements of X_i is not meaningful. Second, we may not be able to naturally interpret stressing mechanisms applied to sums of vectors of risk factors, since if we have two non-linear portfolios $f(\mathbf{X})$ and $h(\mathbf{Y})$, it does not generally follow that $\mathbf{X} + \mathbf{Y}$ are risk factors of a meaningful portfolio (think for example a situation where X_i represents an interest rate or an inflation index, rather than an asset value).

Nonetheless, it is important to consider the ways that risk aggregation is treated in our context of multivariate stressing, albeit under a somewhat different conceptualisation. When using stressing mechanisms (3) and (4), individual risk factors are stressed according to their dependence with all other elements of $\mathbf{X}$. Here we consider how post-stress distributions of risk factors compare to the situation when risk factors are stressed one-by-one in a 'stand-alone' manner. This reasoning leads to the property:

- **Diversification/aggregation**. Consider the multivariate stressing mechanism $\eta : \mathcal{X}^d \mapsto \mathcal{R}$ and the univariate stressing mechanism $\eta_i : \mathcal{X} \mapsto \mathcal{R}$. For a given vector $\mathbf{X} \in \mathcal{X}^d$, the pair (η, η_i) satisfies diversification, if $F^{\eta(\mathbf{X})}_{X_i} \preceq_{\text{st}} F^{\eta_i(X_i)}_{X_i}$; conversely, the pair (η, η_i) satisfies aggregation, if $F^{\eta_i(X_i)}_{X_i} \preceq_{\text{st}} F^{\eta(\mathbf{X})}_{X_i}$. Furthermore, the pair (η, η_i) is diversification neutral if $F^{\eta(\mathbf{X})}_{X_i} = F^{\eta_i(X_i)}_{X_i}$

The diversification/aggregation property codifies the requirement that a risk factor stochastically increases or decreases, when we move from univariately stressing it (via $\eta_i(X_i)$) to multivariately stressing it (via $\eta(\mathbf{X})$). However, to make a meaningful comparison, we need to consider a natural way of reducing the dimension of a stressing mechanism's input vector, from d to 1. In particular, for symmetric stressing mechanisms of form (3) (hence with $\lambda_i = 1/d$, $g_i = g$ for all i), we define for each $i = 1, \dots, d$, $\eta_i : \mathcal{X} \to \mathcal{R}$ by

$$\eta_i(X_i) = g(U_i). \tag{5}$$

Similarly, for stressing mechanisms of form (4), focusing here on Spearman's core, we define for $A = S$ and $A = \bar{S}$ respectively, for each $i = 1, \dots, d$,

$$\begin{aligned} \eta_i(X_i) &= (1+\theta)U_i^{\theta}, \quad \theta > 0, \\ \eta_i(X_i) &= (1-\theta)\bar{U}_i^{-\theta}, \quad \theta \in (0,1). \end{aligned} \tag{6}$$

We then obtain the following result.

Proposition 5.

i) Let the stressing mechanisms η and η_i have respectively the form (3) and (5), with $\lambda_1 = \dots = \lambda_d = 1/d$, $g_1 = \dots = g_d = g$, and g increasing. Then, for each $i = 1, \dots, d$, the pair (η, η_i) satisfies diversification for all $\mathbf{X} \in \mathcal{X}^d$. In particular, (η, η_i) is diversification neutral if $\mathbf{X}$ is comonotonic.

ii) Let the stressing mechanisms η and η_i have respectively the form (4) and (6), with $A \in \{S, \bar{S}\}$. Then, the pair (η, η_i) satisfies aggregation for each $i = 1, \dots, d$ such that $\mathbf{X}_{-i}$ is stochastically increasing in X_i. In particular, (η, η_i) is diversification neutral if $\mathbf{X}$ is independent.

Proposition 5 demonstrates the different aggregation behaviours of the stressing mechanisms (3) and (4). Part i) shows that symmetric and increasing mixture-based stressing mechanisms induce a *diversification credit* with respect to stand-alone stresses on risk factors. Part ii) shows that Spearman-based stressing mechanisms induce, under positive dependence, an *aggregation penalty*.

We may now examine the connection between the diversification property and subadditivity, in the special case of a linear portfolio $Z = \sum_{i=1}^d X_i$. Denote the stressing mechanism $\zeta(Z) = g(U_Z)$, where g is as in Proposition 5i). Then we may interpret $\rho(Z) := \mathbb{E}^{\zeta(Z)}[Z]$ as a portfolio risk measure and correspondingly denote by $\rho(X_i) = \mathbb{E}^{\eta_i(X_i)}[X_i]$ the stand-alone risk measure for position X_i – in fact ρ is a distortion risk measure (Wang et al., 1997). A sufficient condition for subadditivity of the risk measure ρ is $\mathbb{E}^{\zeta(Z)}[X_i] \leqslant \mathbb{E}^{\eta_i(X_i)}[X_i]$ for all i. If that condition holds, as it does for increasing g, we have

$$\rho(Z) = \sum_{i=1}^d \mathbb{E}^{\zeta(Z)}[X_i] \leqslant \sum_{i=1}^d \mathbb{E}^{\eta_i(X_i)}[X_i] = \sum_{i=1}^d \rho(X_i).$$

Notice now that the diversification/aggregation property and Proposition 5 invite comparisons between $\mathbb{E}^{\eta(\mathbf{X})}[X_i]$ and $\mathbb{E}^{\eta_i(X_i)}[X_i]$. Thus, switching from a risk-measure- to a stressing-mechanism-view relies on substituting for $\zeta(Z)$ the stressing mechanism $\eta(\mathbf{X})$, which depends on the random vector $\mathbf{X}$ and does not presume any specific portfolio structure, linear or otherwise; in particular, Proposition 5i) gives us $\mathbb{E}^{\eta(\mathbf{X})}[X_i] \leqslant \mathbb{E}^{\eta_i(X_i)}[X_i]$ as an analogue to the subadditivity condition. Using similar arguments, the stressing mechanisms of Proposition 5ii) can be understood as having an aggregation property that is analogous to superadditivity under positive dependence.

The preceding arguments take a particular interpretation in the context of capital allocation. In the discussion of Section 2.3.2, we argued that with standard capital allocation approaches, the scale of positions has a disproportionate effect on allocated capital amounts. The invariant stressing mechanisms of equations (3) and (4) do not, by construction, suffer from such effects. Furthermore, Proposition 5 allows us to elaborate on how such stressing mechanisms reward diversification or penalize aggregation in portfolios.

Let each risk factor represent the loss from a line of business, such that $Z = \sum_{i=1}^{d} X_i$ is the portfolio loss. For a stressing mechanism η, we interpret $\mathbb{E}^{\eta(\mathbf{X})}[Z]$ as the portfolio's capital requirement and by $\mathbb{E}^{\eta(\mathbf{X})}[X_i]$ the capital allocated to the i-th line of business. The stand-alone capital for the i-th line of business is given by $\mathbb{E}^{\eta_i(X_i)}[X_i]$, which, for η_i as in (5) and (6), is a distortion risk measure of X_i (Wang et al., 1997). A recurring concern in the literature is that the capital allocated to any X_i be no more than the stand-alone capital of the same line, were it to leave the portfolio, as this would create incentives for portfolio fragmentation (Denault, 2001, Tsanakas, 2009).

The stochastic ordering relations in Proposition 5 translate directly to ordering of the expectations of (or allocated capitals to) X_i under different stressing mechanisms. Thus the stressing mechanisms (3) are consistent with standard game theoretic criteria, given the implication that the allocated capital $\mathbb{E}^{\eta(\mathbf{X})}[X_i]$ is generally less than that stand-alone capital $\mathbb{E}^{\eta_i(X_i)}[X_i]$. On the other hand, the mechanisms (4), penalize risk aggregation in a way that is typically not considered in the capital allocation literature. Here, if the losses are independent, each is allocated a stand-alone level of capital $\mathbb{E}^{\eta_i(X_i)}[X_i]$ given by a distortion risk measure – hence individual risks are not 'diversified away'. Furthermore, in the case of positive dependence, an aggregation penalty is applied. We will investigate this further via the numerical example of Section 5.3.

4.4 Stressing risk factor dependence

In Examples 1–4, we introduced stressing mechanisms that impact the joint distribution of $\mathbf{X}$. Nonetheless, the construction of these stresses does not focus on the dependence of $\mathbf{X}$; in particular, the Spearman and Kendall stresses preserve risk factor Independence. Here we introduce via examples stressing mechanisms that are specifically designed to impact on the dependence structure of $\mathbf{X}$. Examples 5 and 7 show how to stress within a multivariate Pareto or Gaussian (copula) context, while Example 6 presents a method for inducing an Archimedean copula between risk factors, when $\mathbf{X}$ is independent under the baseline model.

Example 5 (**Dual Kendall and Multivariate Pareto).** We consider the case that $\mathbf{X}$ follows a standard multivariate Pareto model, that is, $\mathbf{X} \sim \mathrm{MPar}_d(\alpha)$ with joint survival function (e.g. Denuit et al., 2006, Sec.

7.2.4.1):

$$\bar{F}_{\mathbf{X}}(\mathbf{x}) = \left(\sum_{i=1}^{d} x_i + 1 \right)^{-\alpha}, \quad \mathbf{x} > \mathbf{0},\ \alpha > 0.$$

For this distribution, any pair of variables has Kendall's rank correlation equal to $r_K(X_i, X_j) = \frac{1}{1+2\alpha}$. Such a model may be used if $\mathbf{X}$ represent losses from different lines of business. Then, in a stress testing exercise, we may be interested in finding out how the overall risk profile changes, if both the heavy-tailedness of margins and the dependence between elements of $\mathbf{X}$ increases.

Consider the stressing mechanism based on the dual Kendall's core, $\eta(\mathbf{X}) = \bar{K}(\mathbf{X})^{-\theta}/\mathbb{E}\left[\bar{K}(\mathbf{X})^{-\theta}\right], \theta \in (0,1)$. Then, direct calculation (see Appendix C) leads to the post-stress distribution:

$$\bar{F}^{\eta}_{\mathbf{X}}(\mathbf{x}) = \left(\sum_{i=1}^{d} x_i + 1 \right)^{-(1-\theta)\alpha}, \quad \mathbf{x} > \mathbf{0},$$

which is in the same family, with a modified tail index. The bivariate Kendall's rank correlation coefficient becomes $\frac{1}{1+2(1-\theta)\alpha}$, demonstrating a strengthening of the dependence between elements of $\mathbf{X}$.

Given the invariance of Kendall's dual stressing mechanism, the same process as in Example 5 can be followed, not only in the case of a multivariate Pareto distribution, but more generally in the case when $\mathbf{X}$ has a Clayton survival copula, which is precisely the copula of the multivariate Pareto distribution. Before proceeding, we recall some basic facts about Archimedean copulas, see e.g., McNeil et al. (2015, Sec. 7.4). A function ϕ is completely monotonic on an interval $[a, b]$, if it satisfies $(-1)^k \frac{\partial^k}{dt^k}\phi(t) \geqslant 0$, for all $k \in \mathbb{N},\ t \in (a, b)$. Given a completely monotonic function $\phi : [0, \infty) \to [0, 1]$ with $\phi(0) = 1$ and $\lim_{t\to\infty} \phi(t) = 0$, an *Archimedean copula* with generator ϕ is defined by $C_\phi(\mathbf{u}) = \phi\left(\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_d)\right)$. Archimedean copulas satisfy a number of positive dependence properties, including association, as well as the somewhat stronger property of MTP2; for details see Müller and Scarsini (2005). The Clayton copula discussed above, is a special case of an Archimedean copula with generator $\phi(t) = (1 + t)^{-1/\lambda},\ \lambda > 0$.

Example 6 (**Inducing dependence via Achimedean copulas**)**.** In an alternative setting, we may start with a vector of risk factors $\mathbf{X}$ that is independent, and seek a stressing mechanism under which $\mathbf{X}$ becomes dependent. This can be achieved in the general case of Archimedean copulas.

When starting from independent $\mathbf{X}$, we can always design a stressing mechanism such that the post-stress copula of $\mathbf{X}$ belongs to an Archimedean family. Specifically, this is achieved by introducing a stressing mechanism of the form

$$\eta(\mathbf{X}) = \int_0^\infty t^d S(\mathbf{X})^{t-1} dG(t),$$

where G is a distribution on $\mathbb{R}_+$. The choice G is associated with the Archimedean copula one wants to achieve (e.g., for the Clayton copula above G is a $\Gamma(1/\lambda)$ distribution), see Appendix C for more details. Note that the stressing mechanism η we use here is a (continuous) mixture over t of stressing mechanisms of the form $S(\mathbf{X})^{t-1}$. Insofar, it is conceptually related, but distinct, from the mixture stressing mechanisms of

Section 4.1.

Finally, in previous examples, stressing generally aimed at making the vector of risk factors stochastically larger. Such stresses are obviously suitable in situations where large outcomes of risk factors are associated with adverse events. However this is not a universally applicable setting, as it may be desirable to examine the impact of increasing specifically the volatility (and more broadly, covariance) of risk factors, e.g., when modelling multivariate asset returns. We do this in Example 7 below.

Example 7 (**Covariance stresses for Multivariate Normal risk factors**)**.** Here, we diverge from previous forms of stressing mechanisms, to discuss stressing mechanisms of the type

$$\eta(\mathbf{X}) = c \cdot \exp\left(-\frac{1}{2}\mathbf{Z}^\top \mathbf{A}\mathbf{Z}\right), \tag{7}$$

where $\mathbf{Z} = (Z_1, \ldots, Z_d)^\top$ with $Z_i = \Phi^{-1}(U_i),\ i = 1, \ldots, d$, $\mathbf{A} \in \mathbb{R}^{d\times d}$ is symmetric and positive definite, and c is a normalization constant. If $\mathbf{X}$ is has a Gaussian copula, then $\mathbf{Z}$ is multivariate normally distributed, with standard margins and correlation matrix $\mathbf{R}$, $\mathbf{Z} \sim \mathrm{N}_d(\mathbf{0}, \mathbf{R})$, then it is easily shown that

$$\mathbf{Z} \overset{\mathbb{Q}^{\eta(\mathbf{X})}}{\sim} \mathrm{N}_d\left(\mathbf{0}, \mathbf{\Sigma} = (\mathbf{R}^{-1} + \mathbf{A})^{-1}\right),$$

since, by the properties of $\mathbf{A}$, the matrix $\mathbf{\Sigma}$ is a covariance matrix. Given the construction of the random vector $\mathbf{Z}$ via $\mathbf{U}$, the stressing mechanism (7) is invariant – hence selecting a matrix $\mathbf{A}$ with a target $\mathbf{\Sigma}$ in mind does not depend on the marginal distribution specification. The stressing mechanism (7) impacts the correlation parameter of the Gaussian copula of $\mathbf{X}$ as well as the marginal distributions. As $\mathbf{\Sigma} = \{\sigma_{i,j}\}_{i,j=1,\ldots,d}$ is a covariance matrix, the copula of $\mathbf{X}$ under $\mathbb{Q}^{\eta(\mathbf{X})}$ is parameterised by the corresponding correlation matrix $\mathbf{R}^* = \{r^*_{i,j}\}_{i,j=1,\ldots,d},\ r^*_{i,j} = \sigma_{i,j}/\sqrt{\sigma_{i,i}\sigma_{j,j}}$. Furthermore, the marginal distribution of X_i becomes $F_i^{\eta(\mathbf{X})}(x) = \Phi\left(\frac{1}{\sqrt{\sigma_{i,i}}}\Phi^{-1}(F_i(x))\right)$. In the special case where $\mathbf{X}$ is multivariate normally distributed (i.e., it has both a Gaussian copula and normal margins), the stressed distribution is again multivariate normal, with margins that have unchanged mean and standard deviation scaled by the volatility stress factor $\sqrt{\sigma_{i,i}}$. (We note that a stressing mechanism related to (7) was proposed by Wang (2007), who was interested in stressing means rather than volatilities and used a linear rather than a quadratic form in the exponent.)

These transformations become more transparent in a simple example where $d = 2$ and $\mathbf{X}$ is independent, such that $\mathbf{R} = \mathbf{I}_d$ is the identity matrix. Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \quad a > |b|.$$

Then the stressed covariance matrix of $\mathbf{Z}$ is

$$\mathbf{\Sigma} = (\mathbf{A} + \mathbf{I}_d)^{-1} = \begin{bmatrix} \frac{a+1}{(a+1)^2-b^2} & -\frac{b}{(a+1)^2-b^2} \\ -\frac{b}{(a+1)^2-b^2} & \frac{a+1}{(a+1)^2-b^2}. \end{bmatrix}$$

Simple manipulations show that the correlation and variance induced by stressing mechanism are

$$r^*_{1,2} = -\frac{b}{a+1}, \qquad \sigma_{i,i} = \frac{1}{(a+1)\big(1-(r^*_{1,2})^2\big)}, \quad i = 1, 2.$$

Hence, to induce a positive correlation through stressing, it is enough to set $b < 0$, while to induce a volatility stress greater than 1, one needs to choose $a < \frac{1}{1-(r^*_{1,2})^2} - 1$; taking $a < 0$ would be sufficient for this.

4.5 Stressing risk factor variability

In much of the discussion of previous sections it was implicitly or explicitly assumed that a stress is meant to make a risk factor stochastically larger – this is reflected by the joint stressing property that is satisfied by the stressing mechanisms we examined most closely, that is, the mixture and Spearman stressing mechanisms of (3) and (4) respectively. (Furthermore, the risk factor X_i can be made stochastically smaller by swapping U_i and $\bar{U}_i$.) Nonetheless, such a setting is not universally applicable. In many cases a risk analyst may want to consider non-monotonic transformations of risk factors, particularly with a view to stressing their variability. We have already considered some stressing mechanisms that focus on making risk factors more volatile rather than stochastically larger, namely, the two-tailed stress $\tilde{\eta}_i(X_i)$ of Example 1 and the Gaussian/covariance stress of Example 7.

Here we take a more systematic perspective and show how one can extend the mixture and Spearman stressing mechanisms of (3) and (4) in order to impact the variability of risk factors. We note that in our context of invariant stressing mechanisms, many variability concepts like the variance or, more generally, the convex order cannot be meaningfully applied, as they depend explicitly on statistics that are not invariant to monotonic transformations (e.g., risk factor means). For that reason, we consider as an alternative the *quantile spread order* (Townsend and Colonius, 2005, Bellini et al., 2022), which is defined as follows. The quantile spread of a distribution $G \in \mathcal{M}_1$ is the function given by $\mathrm{QS}_G(p) := G^{-1}(p) - G^{-1}(1-p)$, for $p \in (1/2, 1)$. Then, for distributions $G, H \in \mathcal{M}_1$, we say that H dominates G in quantile spread order, and write $G \preceq_{\mathrm{QS}} H$, if for any $p \in (1/2, 1)$ it holds that $\mathrm{QS}_G(p) \leqslant \mathrm{QS}_H(p)$. The quantile spread order is a symmetric version of the dispersive order in Shaked and Shanthikumar (2007, Section 3.B). We can now formulate the following variability property of stressing mechanisms.

- **Quantile-spreading.** The stressing mechanism $\eta(\mathbf{X})$ is quantile-spreading for X_i if $F_{X_i} \preceq_{\mathrm{QS}} F^{\eta(\mathbf{X})}_{X_i}$.

Hence, if $\eta(\mathbf{X})$ satisfies this property for X_i, the spread between, e.g., the 95th and 5th quantiles of X_i increases after stressing the risk factors.

Now we can proceed by defining stressing mechanisms that satisfy this property (we will discuss stressing mechanisms that instead reduce risk factor variability at the end of this section). We start by considering univariate stressing mechanisms of the form:

$$\eta_i(X_i) = \frac{U_i^{-\theta}\bar{U}_i^{-\theta}}{B(1-\theta, 1-\theta)}, \quad 0 < \theta < 1, \tag{8}$$

where $B(\cdot,\cdot)$ is the Beta function providing here the normalization constant; see also Wirch and Hardy (1999) for a related approach to risk measurement. The stressing mechanisms (8) include components that are both increasing and decreasing in the risk factor X_i. The impact on the distribution of X_i is characterised below.

Proposition 6. *The stressing mechanism (8) is quantile-stressing for X_i. Furthermore, $F_{X_i}^{\eta_i(X_i)}(1/2) = F_{X_i}(1/2)$.*

Hence the univariate stressing mechanisms (8) increase the variability of a risk factor in the sense of quantile spreading, while keeping the median unchanged. We can now use $\eta_i(X_i)$ as building block for multivariate stressing mechanisms that are analogous to mixture and (dual) Spearman stresses:

$$\eta_M(\mathbf{X}) = \sum_{i=1}^{d} \lambda_i \eta_i(X_i), \quad (\lambda_1, \ldots, \lambda_d) \in \Delta_d \tag{9}$$

$$\eta_S(/\mathbf{X}) = \frac{\prod_{i=1}^{d} \eta_i(X_i)}{\mathbb{E}\left[\prod_{i=1}^{d} \eta_i(X_i)\right]} = \frac{S(\mathbf{X})^{-\theta} \bar{S}(\mathbf{X})^{-\theta}}{\mathbb{E}\left[S(\mathbf{X})^{-\theta} \bar{S}(\mathbf{X})^{-\theta}\right]}. \tag{10}$$

The properties of those multivariate stresses are now characterised as follows.

Proposition 7. *For the stressing mechanisms (8), (9) and (10) the following hold:*

i) If $\mathbf{X}$ is independent, then for each $i = 1, \ldots, d$,

$$F_{X_i} \preceq_{\mathrm{QS}} F_{X_i}^{\eta_M(\mathbf{X})} \preceq_{\mathrm{QS}} F_{X_i}^{\eta_i(X_i)} = F_{X_i}^{\eta_S(\mathbf{X})}.$$

ii) If $\mathbf{X}$ is comonotonic, then for each $i = 1, \ldots, d$,

$$F_{X_i} \preceq_{\mathrm{QS}} F_{X_i}^{\eta_M(\mathbf{X})} = F_{X_i}^{\eta_i(X_i)} \preceq_{\mathrm{QS}} F_{X_i}^{\eta_S(\mathbf{X})},$$

where for the last ordering relation we assume $d\theta < 1$.

Through Proposition 7, we see once more how Spearman-type stressing mechanisms induce more penal multivariate stresses, compared to those constructed by mixtures of univariate ones.

Finally, we note that there are applications where an adverse scenario corresponds to reducing (rather than increasing) risk factor variability, as is the case in options with convex pay-offs, like straddles and strangles. This can be easily addressed in our framework. As an alternative to (8), consider $\eta_i(X_i) = U_i^{\theta} \bar{U}_i^{\theta} / B(1+\theta, 1+\theta)$, $\theta > 1$. An argument analogous to the proof of Proposition 6 shows that adopting such a stressing mechanism induces *quantile shrinking*, that is, $F_{X_i} \succeq_{\mathrm{QS}} F_{X_i}^{\eta(\mathbf{X})}$.

5 Real-data application

5.1 Data

Here we illustrate the use of the stressing mechanisms introduced in previous sections, in two applications: stress testing of a simulation model and capital allocation. The applications are based on a dataset provided by a UK-based non-life insurer, including $n = 10^5$ simulated scenarios from a number of random variables in the insurer's economic capital model. The variables that we will consider here are:

- $X_i,\ i = 1, \dots, 16$: losses from $d = 16$ lines of business in \$m, gross of reinsurance (i.e. not taking into account the losses recovered from reinsurance contracts).

- Y: Net Portfolio Loss in \$m. This includes all assets held and reinsurance recoveries, as well as losses from different sources of risk, such as market, operational, and credit risks.

As is not untypical when dealing with complex computational models (e.g. Pesenti et al., 2021), this model is largely a black box to us. We do not have a parametric form for the distributions of X_i, which are themselves outputs of sub-models. Furthermore we do not have access to the relationship between gross losses $\mathbf{X} = (X_1, \dots, X_{16})$ and the Net Portfolio Loss Y; in general it holds that $Y = g(\mathbf{X}, \mathbf{V})$ for some non-linear function g and additional sources of uncertainty $\mathbf{V}$.

A summary of the statistical behaviour of $\mathbf{X}$ is given in Figure 2, where we show box plots of $X_i,\ i = 1, \dots, 16$, and a heatmap of their Spearman rank correlation matrix. It can be observed that the marginal distributions tend to be very skewed, while the correlations are positive, with mostly low values, but with some pairs in the higher range of 0.4-0.6.

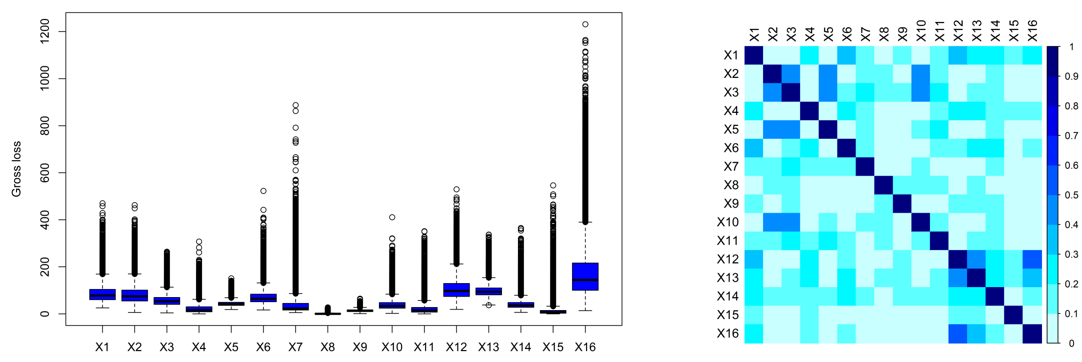

Figure 2: Distributional characteristics of gross losses $\mathbf{X}$. (left: box plot; right: rank correlation matrix)

The framework developed in previous sections is standing on the assumption that an existing multivariate model is available to the user, either through distributional specification or – as in this section – through simulation of the underlying risk factors. In Appendix D we show that empirical versions of our stressing mechanisms are easily derived and produce stressed multivariate distributions that converge to those obtained under a fully specified model and this is the approach we follow in the present analysis. Note that numerical evaluation relies on re-weighting given simulated scenarios and hence there is no need to proceed with expensive additional evaluations of model functions – or indeed direct access to the data generating mechanism (Pesenti et al., 2019).

5.2 Stress testing gross losses

We begin by monitoring how the model output Y responds to stressing model outputs X_i, following the particular forms of marginal and mixture stressing mechanisms discussed in Examples 1 and 2. In particular, to stress individual lines of business and the portfolio loss, we use

$$\eta_i(X_i) = (1-\theta)\bar{U}_i^{-\theta}, \quad i = 1, \dots, 16$$
$$\eta(\mathbf{X}) = \frac{1-\theta}{d}\sum_{j=1}^{d}\bar{U}_j^{-\theta}, \quad \theta = 0.5.$$

The choice of $\theta = 0.5$ is made on order to effect a substantial change in the distribution of $\mathbf{X}$ in a specific way: if F_{X_i} has a Pareto-type right tail, $F_{X_i}^{\eta_i(X_i)}$ will then also have Pareto-type tail with the tail index halved. The use of the same value of θ ensures that lines of business and the portfolio are stressed in a consistent manner. Alternative approaches to defining the stress parameter could be based on the overall plausibility of such as stress, measured by the (e.g. Kullback-Leibler) divergence between the baseline and stressed models, see Breuer and Csiszár (2013).

These changes can be observed in Figures 3a) and b), where we show the quantile functions of two particular lines of business: Cargo (X_1) and Treaty (X_{16}). In dashed blue lines we plot the baseline quantile function; in solid red the quantile functions under the mixture stress η; in solid green the quantile functions under the marginal stresses $\eta_1(X_1)$ and $\eta_{16}(X_{16})$. It is seen how both the mixture and marginal stresses produce an increase in stochastic dominance to the marginal distributions of X_1 and X_{16}. This effect is much more pronounced for the marginal stresses focusing on the individual line of business, consistently with Proposition 5i).

In Figure 3c) and d) we plot, under the same stressing mechanisms, the quantile function of the net portfolio loss Y. The only difference between those two plots is the positioning of the green line. We see that the impact on Y of stressing X_1 is approximately the same as that of stressing the whole vector $\mathbf{X}$ via the mixture stress $\eta(\mathbf{X})$. On the other hand, we observe that applying the stressing mechanism $\eta_{16}(X_{16})$ impacts the portfolio Y more than the mixture stress, which indicates the higher importance of X_{16} in the portfolio.

Following these observations, we use all stress testing mechanisms $\eta_1(X_1), \dots, \eta_{16}(X_{16})$, to investigate the relative importance of different lines of business to the portfolio loss. In Figure 4a) we show the impact

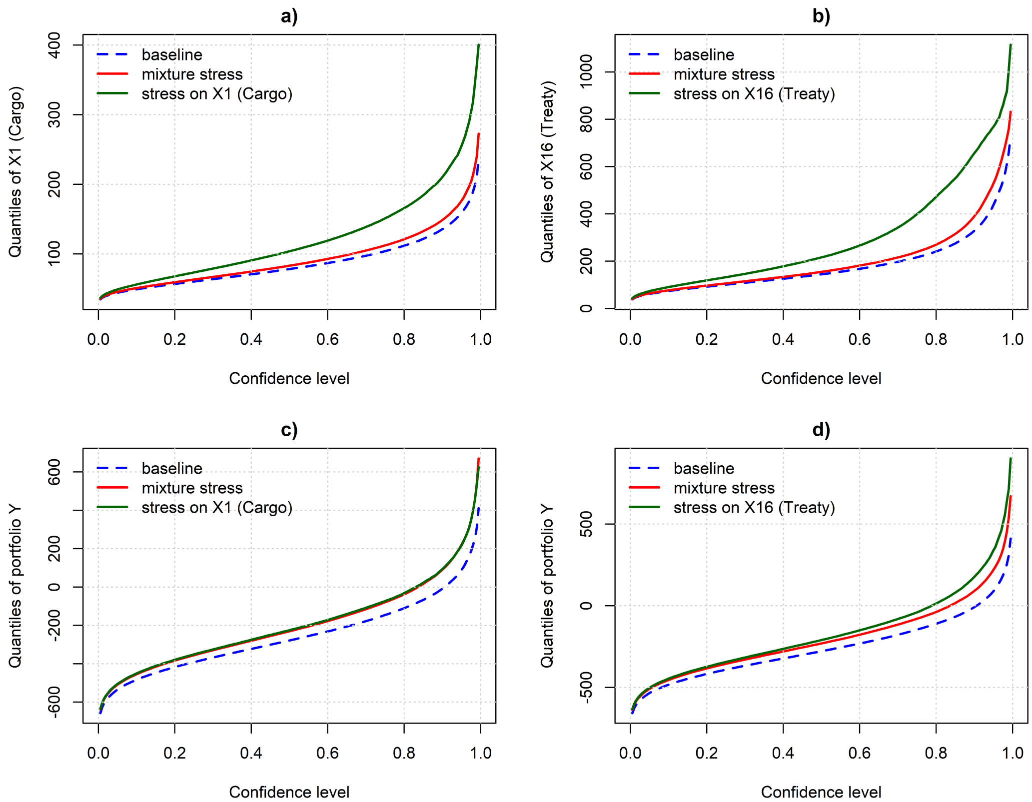

Figure 3: a), b) Quantiles of X_1 and X_{16}; c), d) quantiles of Y. Dashed blue: baseline model; red: model under mixture stress $\eta(X)$; green: model under marginal stresses $\eta_1(X_1)$ (a, c) and $\eta_{16}(X_{16})$ (b, d).

of stressing on the marginal tail properties of $\mathbf{X}$. Specifically we plot the *mean excess ratios*

$$\mathbb{E}^{\eta(\mathbf{X})}\left[X_i/t_i - 1 | X_i > t_i\right], \qquad \mathbb{E}^{\eta_i(X_i)}\left[X_i/t_i - 1 | X_i > t_i\right], \quad i = 1, \ldots, 16,$$

for thresholds $t_i = F_{X_i}^{-1}(0.99)$. These quantities are directly linked to the tail properties of marginal distributions (McNeil et al., 2015, Sec. 5.2.3). Consistently with previous arguments, the marginal stresses have a higher impact on tails of marginal distributions F_{X_i} compared to the mixture stress.

In Figure 4b), the relative importance of different lines of business is illustrated, by depicting (in green bars) the 99th quantile of Y under stresses $\eta_1(X_1), \ldots, \eta_{16}(X_{16})$. All those stresses produce an increase to $F_Y^{-1}(0.99)$, compared to the baseline (blue dashed line). However these impacts are not homogeneous, with some marginal stresses moving the portfolio loss quantile more. We can consider $X_3, X_7, X_{12}, X_{13}, X_{16}$ as the most important lines of business in the portfolio, since the respective marginal stresses produce an impact on the 99th portfolio quantile that is more than the benchmark given by the mixture stress (red line).

We next consider the way that stressing mechanisms impact upon the dependence dependence structure of $\mathbf{X}$. In Figure 5a), the values of the pairwise (Spearman) rank correlations of $\mathbf{X}$ under the mixture stress $\eta(\mathbf{X})$ are plotted, against the baseline model. It is seen that there is no substantial impact of $\eta(\mathbf{X})$ on correlations, with some very modest positive effect observed for higher values.

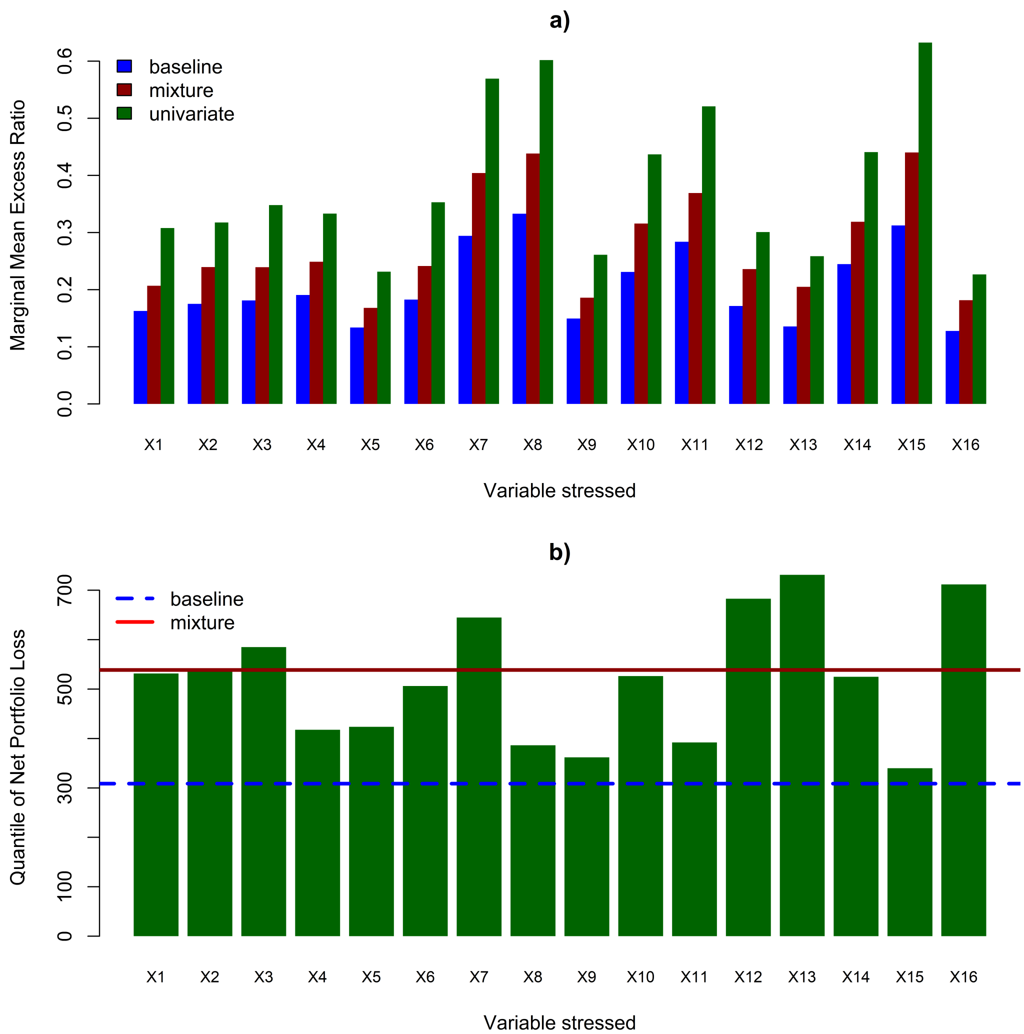

Figure 4: a) Mean excess ratios of $X_1, \ldots, X_{16}$; b) 99th quantile of the Net Portfolio Loss Y under baseline and stressed models. Blue corresponds to the baseline model, red to the mixture stress $\eta(\mathbf{X})$; green to marginal stresses $\eta_i(X_i),\ i = 1, \ldots, d$.

We now show how correlation can be induced between risk factors, following the approach of Example 6. We focus our attention on the random variables X_7 and X_{16}, representing gross losses from, respectively, a Marine Liability and a Treaty line. According to Figure 4b), both these variables are important drivers of portfolio loss; at the same time, from Figure 2 we note that their pairwise sample rank correlation is quite low (Spearman and Kendall measures of rank correlation are $\hat{r}_S(X_7, X_{16}) = 0.079$, $\hat{r}_K(X_7, X_{16}) = 0.053$). It is then of interest to monitor the impact of a bivariate stress on (X_7, X_{16}), which also increases the dependence strength between those two variables.

For this purpose we use a stressing mechanism of the form

$$\xi(X_7, X_{16}) = \int_0^\infty t^2 (\bar{U}_7 \cdot \bar{U}_{16})^{t-1} dG(t),$$

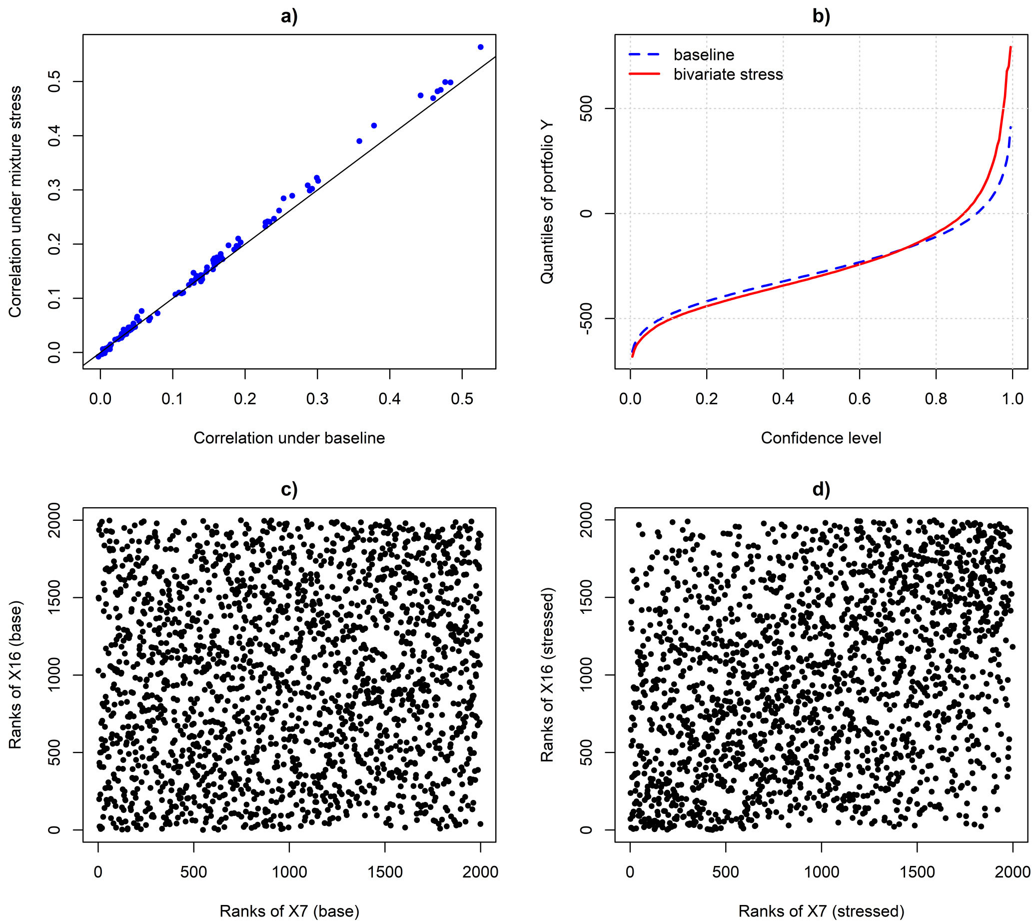

Figure 5: a) Pairwise rank correlations under the mixture stress $\eta(\mathbf{X})$ against baseline model; b) quantiles of the portfolio loss Y under the bivariate stress $\xi(X_7, X_{16})$; c) scatter plot of sample ranks of X_7, X_{16}; d) scatter plot of sample ranks of X_7, X_{16}, re-sampled with respect to the stressing mechanism $\xi(X_7, X_{16})$.

where G is a Gamma distribution with shape parameter $a = 2$ and scale parameter equal to $b = \frac{1-(1-p)^{-1/a}}{\log(1-p)}$, where $p = 0.75$. If the variables X_7, X_{16} were independent under $\mathbb{P}$, they would have a bivariate Clayton survival copula under $\mathbb{Q}^\xi$, with Kendall's rank correlation equal to $r_K(X_7, X_{16}) = \frac{1}{1+2a} = 0.2$. The choice of the scale parameter is such that the p-th quantile of the post-stress marginal distributions is the same as that of the baseline marginals. While (X_7, X_{16}) are likely not independent in the given model, we use this stressing mechanism in the expectation that, due to the low sample correlation, we will reach meaningful results.

Following this process, we find that the post-stress bivariate distribution of (X_7, X_{16}) has sample rank correlation $\hat{r}_K^\xi(X_7, X_{16}) = 0.238$, which is very close to the target value. In Figures 5 c) and d) we show scatter plots of the sample ranks of those two variables, under the baseline model and under (i.e., re-sampled from) $\mathbb{Q}^\xi$. The way that the stressing mechanism $\xi(X_7, X_{16})$ has induced dependence is clearly visible. In Figure 5b) the quantiles of Y are plotted. It is seen that the bivariate stress has a profound impact on the portfolio risk, given that it impacts the joint tail of (X_7, X_{16}).

5.3 Capital allocation

Here we use stressing mechanisms of the type (4) in order to derive a capital allocation mechanism, which, consistently with the discussion in Example 2.3.2 and Section 4.3, provides an alternative to standard capital allocation approaches.

We carry out the allocation exercise on the portfolio of gross losses. Define the total gross loss as $Z = \sum_{i=1}^{16} X_i$. We assume that the total capital for the portfolio of gross losses is given by an *Expected Shortfall* or *Conditional Value-at-Risk* risk measure (McNeil et al., 2015, Sec. 2.3.4) at confidence level $p = 0.975$, that is, (assuming continuity) the capital to be allocated is equal to $\mathrm{ES}_p(Z) = \mathbb{E}\left[Z \mid Z > F_Z^{-1}(p)\right] = 1740.7$, where the quantile threshold used is the Value-at-Risk of Z, that is, $\mathrm{VaR}_p(Z) = F_Z^{-1}(p)$. Following the standard Euler approach (Tasche, 2004), the capital allocated to X_i is $d_i^{eu} := \mathbb{E}\left[X_i \mid Z > F_Z^{-1}(p)\right]$.

We now formulate an alternative. Let $\eta(\mathbf{X}) = \bar{S}(\mathbf{X})^{-\theta}/\mathbb{E}\left[\bar{S}(\mathbf{X})^{-\theta}\right]$, where we calibrate $\theta = 0.202$, such that $\mathbb{E}^{\eta(\mathbf{X})}[Z] = \mathrm{ES}_p(Z)$. Consequently, we define the (dual) *Spearman allocation*, $d_i^{sp} := \mathbb{E}^{\eta(\mathbf{X})}[X_i]$. Figure 6 shows the Euler and Spearman allocations, in parts a) (dark red) and b) (dark green) respectively; the allocations are plotted against each other in Figure 6. It is seen how the allocations produced are broadly consistent, with the difference that the Euler allocation seems to penalise more severely X_{16}.

We now consider the case that a modification in the portfolio takes place. We assume that a non-linear reinsurance product is bought by the company to protect against X_{16}. Specifically, this reduces the tail risk of the 16-th line of business, as its loss now becomes $\tilde{X}_{16} = X_{16} - 0.9\left(X_{16} - F_{X_{16}}^{-1}(0.8)\right)_+$. In Figure 6, parts a), b), the Euler and Spearman allocations for the modified portfolio are shown in light red and light green respectively and are plotted against each other in part d). For both methods, we see that the capital allocated to the 16-th line of business drops substantially, reflecting the protection by the reinsurance bought. For the Spearman allocation it is clear that this modification impacts only the capital allocated to the X_{16}, with other lines of business unaffected – this is an implication of invariance of the stressing mechanism η. However, for the Euler allocation all lines of business are affected. Some of those changes would in practice be unwelcome. For example, we see that in response to a *reduction* in the risk of X_{16} (and therefore to the risk of the portfolio), the capital allocated to X_6 and X_7 actually *increases*. As discussed in Section 2.3.2 this is an organizationally unwelcome situation, which would in practice be untenable.

Following Section 4.3, in the presence of positive dependence, the Spearman allocation produces allocated capital amounts that dominate stand-alone risk. In our case, given the positive dependence seen from Figure 2, it is reasonable to expect that

$$d_i^{sp} \geqslant \rho_\theta(X_i) := \int_0^1 F_{X_i}^{-1}(u)(1-\theta)(1-u)^{1-\theta} du,$$

where ρ_θ is the distortion risk measure derived from the proportional hazards transform (Wang, 1996). Hence, in the presence of positive dependence, we can view the stand-alone risk $\rho_\theta(X_i)$ as a lower bound for d_i^{sp}, which is attained in the case of independence. Given that $\rho_\theta(X_i) > \mathbb{E}[X_i]$, we can say that in the Spearman allocation the risk of X_i is not 'diversified away'.

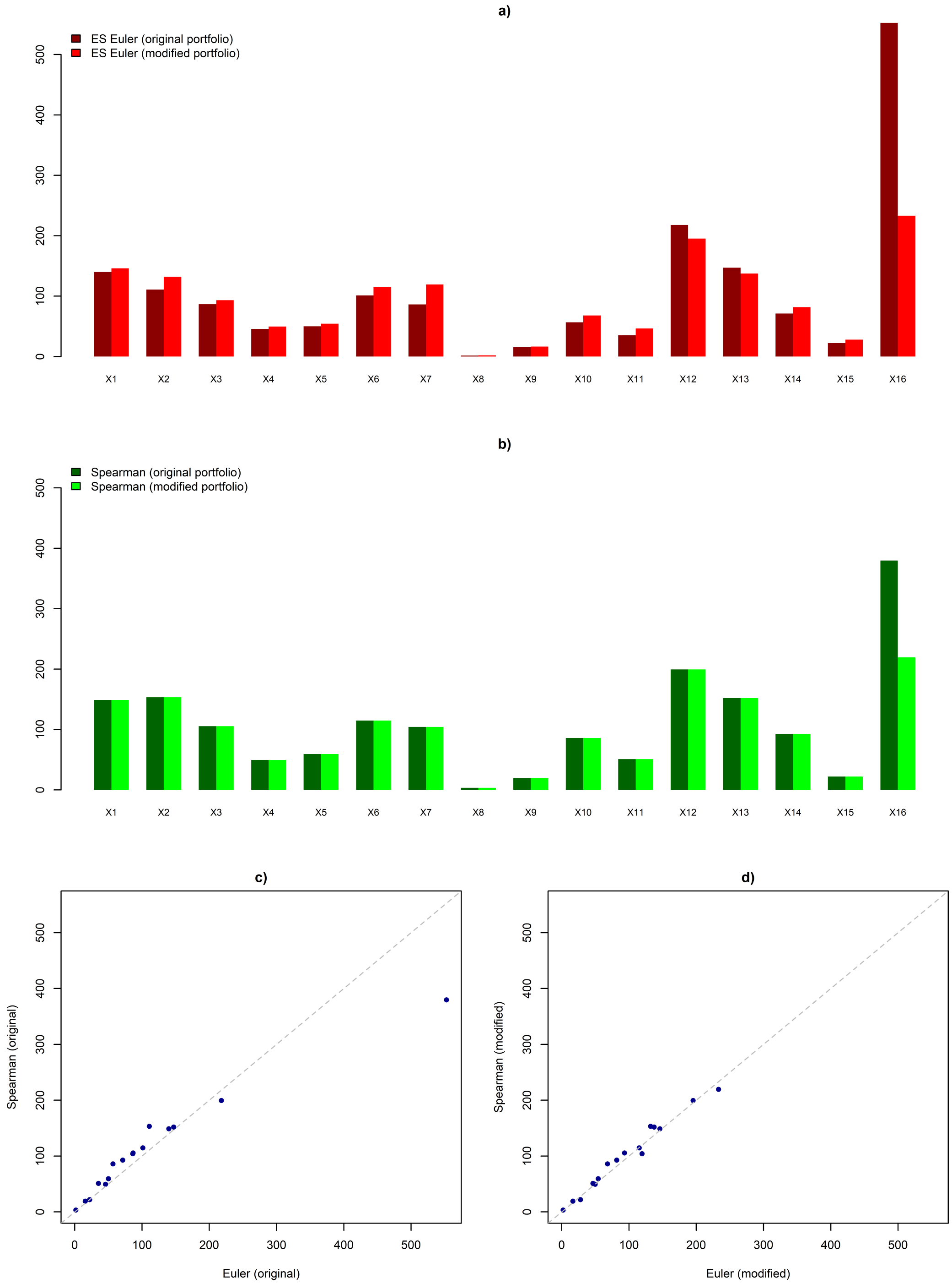

Figure 6: a) Euler allocation for the original (dark red) and modified (light red) portfolio; b) Spearman allocation for the original (dark green) and modified (light green) portfolio; c)-d) Spearman vs Euler allocations.

In Figure 7a) (returning to the original portfolio) we show that the stand-alone risk $\rho_\theta(X_i)$ captures the volatility of gross losses. Specifically, we plot the excess risk over the mean $\rho_\theta(X_i)/\mathbb{E}[X_i] - 1$ against coefficients of variation for the gross losses, against the losses' coefficients of variation. The essentially linear relationship confirms that the stand-alone risk captures loss volatility.

Using the Spearman allocation in the presence of positive dependence induces an aggregation penalty $d_i^{sp} - \rho_\theta(X_i)$. This should reflect the extent to which X_i is positively dependent to the other variables in $\mathbf{X}$. To capture this effect (if somewhat crudely), we plot in Figure 7b) the aggregation penalty, normalized by the standard deviation of X_i, against the average rank correlation of X_i to other risks, that is, the quantity $r_i := \frac{1}{d-1}\sum_{j\neq i}^{16} r_S(X_i, X_j)$. The clear positive relationship confirms our intuition that the allocation method appropriately penalizes positive dependence to other lines of business.

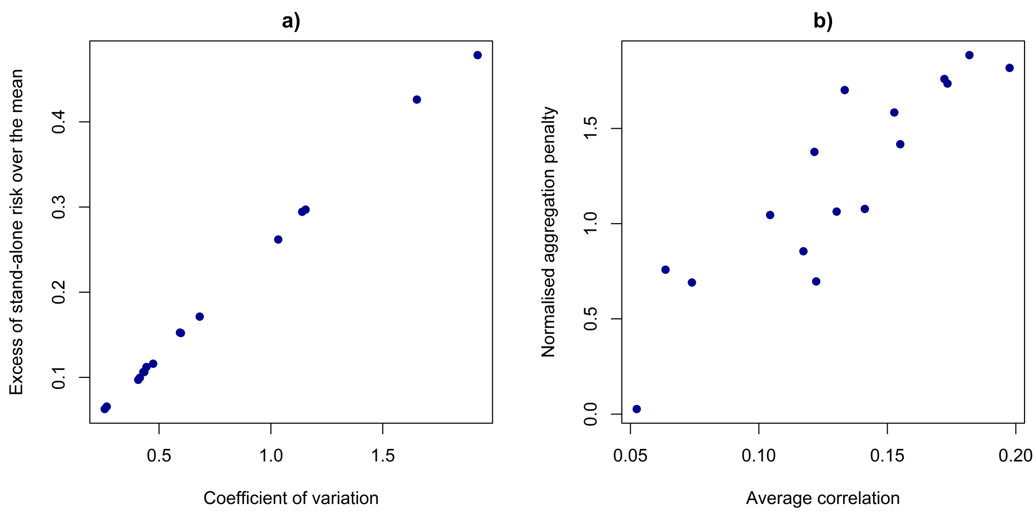

Figure 7: a) Excess stand-alone risk $\rho_\theta(X_i)/\mathbb{E}[X_i] - 1$ against coefficients of variation for the gross losses; b) normalized aggregation penalty $(d_i^{es} - \rho_\theta(X_i))/\sigma(X_i)$ against average rank correlation r_i.

6 Concluding remarks

Our paper is the first systematic study of multivariate stressing mechanisms. We presented a novel framework and, as such, there are still many questions and directions to explore.

First, there may be other useful properties that are relevant in specific contexts, in addition to the ones considered in Section 3. The desirability of these theoretical properties, as well as their technical soundness, needs thorough future investigation, given the additional complexity of multivariate stressing mechanisms compared to their univariate counterparts. Furthermore, the stressing mechanisms we introduced can be directly used to construct new multivariate risk measures e.g. as mappings $\mathbf{X} \mapsto \mathbb{E}^{\eta(\mathbf{X})}[\mathbf{X}]$, thus contributing to the literature on vector-valued risk measures; see e.g., Jouini et al. (2004), Embrechts and Puccetti (2006), Maume-Deschamps et al. (2017) and the references therein. The systematic study of the properties of stressing mechanisms is then closely related to the study of the corresponding risk measures. Future work can develop these aspects further, with results in the vein of Proposition 5.

A keystone of our paper is the property of invariance. This allows us to formulate stressing mechanisms that are applicable across portfolios with differing characteristics. To us, this is an important and desirable feature of a stress testing framework. Nonetheless, at the same time it is also a limitation. By not allowing stressing mechanisms to depend on variables such as a portfolio loss, the link of the stressing mechanism to, for instance, the regulatory capital is severed – hence we sharply diverge from reverse stress testing procedures (Pesenti et al., 2019). This trade-off points to a deeper conceptual issue, which is partially explored by Guan et al. (2023), who prove that it is impossible to design capital allocation mechanisms that both reproduce regulatory portfolio capital and prevent risk reductions in one line from increasing the capital of others. The latter is of course precisely the situation that our invariant stressing mechanisms addressed in Section 5.3.

Second, the practical deployment of multivariate stressing mechanisms will depend on the specifics of different contexts. We do not anticipate one particular type of multivariate stressing to become accepted as universally best. In the examples of Section 4 we showcased the versatility of our proposed framework, accommodating different design criteria. In Sections 2.3 and 5 we focused on two applications: stress testing of loss variables and capital allocation. For those specific contexts, we can offer the following recommendations:

- Overall, we find stressing mechanisms involving terms of the type $\bar{U}_i^{-\theta}$ effective, since they transform (joint) tails in a coherent way, making them heavier, while preserving their Paretian or exponential features. Furthermore, the relationship to distortion risk measures can be exploited for the purpose of interpretation.

- The calibration of the stress parameter θ follows from the desired impact on distributions' tail properties. (Stressing volatilities and correlations in a Gaussian – rather than Paretian – context can be done by the methods of Example 7.) Alternative calibrations can follow from statistical arguments, by setting the maximum plausible divergence between $F_{\mathbf{X}}$ and $F_{\mathbf{X}}^{\eta}$ (Breuer and Csiszár, 2013).

- When evaluating the relative importance of risk factors, it is useful to compare univariate to the relevant mixture stresses, as this comparison reveals those risk factors with dominant idiosyncratic effects.

- In capital allocation, the family of stressing mechanisms to use depends on preferences: mixture-based allocation produces diversification credits (thus removing incentives for fragmentation), while Spearman-based allocations produce aggregation penalties (thus preventing uncorrelated risks from being diversified away). Whichever of those is a priority will depend on the specific organizational context – for example, the use of Spearman allocations may be preferred for calculating line managers' remuneration, as it removes perverse incentives for taking on poorly compensated uncorrelated risks.

Multivariate stressing mechanisms naturally appear in many other areas of application in addition to the ones we discussed in this paper. One such application is importance sampling, commonly used to increase the accuracy of simulation-based estimates. Applications of importance sampling appear naturally in various areas including statistical and financial studies; see e.g., Glasserman and Tayur (1995) and Glasserman and Li (2005). Most expositions of importance sampling start with specifying an alternative density for $\mathbf{X}$, while we start by explicitly specifying a Radon-Nikodym derivative. We believe that this formulation, combined with advanced importance sampling approaches (e.g., Owen and Zhou, 2000), can produce general-purpose importance sampling schemes useful in situations where the multivariate densities are not given in explicit form, as in some hierarchically built portfolio models.

A different application is the monitoring of systemic risk across a financial market. A regulator can specify a multivariate stress test according to our methods, by focusing on a set of relevant risk factors and their dependence structure. This can be applied separately by participating firms, allowing them to e.g. recalculate their portfolio loss distribution under stressed conditions and report their increased capital needs. The invariance property ensures that such an application will be meaningful, even when implemented on models with different distributional assumptions and portfolio structures. While we alluded to this idea in Section 2.3.1, a full development of such an approach remains a topic for future research. Hence, multivariate stressing can become a useful complement to approaches based on CoVaR and CoES (Adrian and Brunnermeier, 2016, Banulescu-Radu et al., 2021), which can themselves be seen as expectations under some (slightly more general) stressing mechanisms, or other measures of systemic risk (Chen et al., 2013).

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References

Acerbi, C. and Tasche, D. (2002). Expected Shortfall: a natural coherent alternative to Value at Risk. *Economic Notes*, 31(2):379–388.

Adrian, T. and Brunnermeier, M. K. (2016). Covar. *American Economic Review*, 106(7):1705–1741.

Artzner, P., Delbaen, F., Eber, J.-M., and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, 9(3):203–228.

Asimit, V., Peng, L., Wang, R., and Yu, A. (2019). An efficient approach to quantile capital allocation and sensitivity analysis. *Mathematical Finance*, 29(4):1131–1156.

Aven, T. (2016). Risk assessment and risk management: Review of recent advances on their foundation. *European Journal of Operational Research*, 253(1):1–13.

Banulescu-Radu, D., Hurlin, C., Leymarie, J., and Scaillet, O. (2021). Backtesting marginal expected shortfall and related systemic risk measures. *Management Science*, 67(9):5730–5754.

Bauer, D. and Zanjani, G. (2016). The marginal cost of risk, risk measures, and capital allocation. *Management Science*, 62(5):1431–1457.

Bellini, F., Fadina, T., Wang, R., and Wei, Y. (2022). Parametric measures of variability induced by risk measures. *Insurance: Mathematics and Economics*, 106:270–284.

Biagini, F., Fouque, J.-P., Frittelli, M., and Meyer-Brandis, T. (2019). A unified approach to systemic risk measures via acceptance sets. *Mathematical Finance*, 29(1):329–367.

Borgonovo, E. and Plischke, E. (2016). Sensitivity analysis: a review of recent advances. *European Journal of Operational Research*, 248(3):869–887.

Brechmann, E. C., Hendrich, K., and Czado, C. (2013). Conditional copula simulation for systemic risk stress testing. *Insurance: Mathematics and Economics*, 53(3):722–732.

Breuer, T. and Csiszár, I. (2013). Systematic stress tests with entropic plausibility constraints. *Journal of Banking & Finance*, 37(5):1552–1559.

Breuer, T., Jandačka, M., Mencía, J., and Summer, M. (2012). A systematic approach to multi-period stress testing of portfolio credit risk. *Journal of Banking & Finance*, 36(2):332–340.

Broadie, M. and Glasserman, P. (1996). Estimating security price derivatives using simulation. *Management science*, 42(2):269–285.

Cambou, M. and Filipović, D. (2017). Model uncertainty and scenario aggregation. *Mathematical Finance*, 27(2):534–567.

Chen, C., Iyengar, G., and Moallemi, C. C. (2013). An axiomatic approach to systemic risk. *Management Science*, 59(6):1373–1388.

Cont, R., Deguest, R., and Scandolo, G. (2010). Robustness and sensitivity analysis of risk measurement procedures. *Quantitative Finance*, 10(6):593–606.

Denault, M. (2001). Coherent allocation of risk capital. *Journal of Risk*, 4:1–34.

Denuit, M., Dhaene, J., Goovaerts, M., and Kaas, R. (2006). *Actuarial Theory for Dependent Risks: Measures, Orders and Models*. John Wiley & Sons.

Dhaene, J., Tsanakas, A., Valdez, E. A., and Vanduffel, S. (2012). Optimal capital allocation principles. *Journal of Risk and Insurance*, 79(1):1–28.

Duffie, D. (2018). Financial regulatory reform after the crisis: An assessment. *Management Science*, 64(10):4835–4857.

Eisenberg, L. and Noe, T. H. (2001). Systemic risk in financial systems. *Management Science*, 47(2):236–249.

Embrechts, P. and Puccetti, G. (2006). Bounds for functions of multivariate risks. *Journal of Multivariate Analysis*, 97(2):526–547.

Farkas, W., Koch-Medina, P., and Munari, C. (2015). Measuring risk with multiple eligible assets. *Mathematics and Financial Economics*, 9(1):3–27.

Financial Policy Committee (2019). Financial Stability Report. Bank of England.

Föllmer, H. and Schied, A. (2011). *Stochastic Finance: An introduction in discrete time*. Walter de Gruyte, 3rd edition.

Furman, E. and Zitikis, R. (2008). Weighted risk capital allocations. *Insurance: Mathematics and Economics*, 43(2):263–269.

Gandy, A. and Veraart, L. A. (2017). A Bayesian methodology for systemic risk assessment in financial networks. *Management Science*, 63(12):4428–4446.

Genest, C. and Rivest, L.-P. (1993). Statistical inference procedures for bivariate archimedean copulas. *Journal of the American Statistical Association*, 88(423):1034–1043.

Gilboa, I. (2009). *Theory of decision under uncertainty*, volume 45. Cambridge university press.

Glasserman, P. and Li, J. (2005). Importance sampling for portfolio credit risk. *Management science*, 51(11):1643–1656.

Glasserman, P. and Tayur, S. (1995). Sensitivity analysis for base-stock levels in multiechelon production-inventory systems. *Management Science*, 41(2):263–281.

Guan, Y., Tsanakas, A., and Wang, R. (2023). An impossibility theorem on capital allocation. *Scandinavian Actuarial Journal*, 2023:290–302.

Henzi, A., Ziegel, J. F., and Gneiting, T. (2021). Isotonic distributional regression. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 83:963–993.

Hong, L. J. and Liu, G. (2009). Simulating sensitivities of conditional value at risk. *Management Science*, 55(2):281–293.

Jouini, E., Meddeb, M., and Touzi, N. (2004). Vector-valued coherent risk measures. *Finance and Stochastics*, 8(4):531–552.

Liu, P., Schied, A., and Wang, R. (2021). Distributional transforms, probability distortions, and their applications. *Mathematics of Operations Research*, 46(4):1490–1512.

Marshall, A. W. and Olkin, I. (1988). Families of multivariate distributions. *Journal of the American Statistical Association*, 83(403):834–841.

Maume-Deschamps, V., Rullière, D., and Said, K. (2017). Multivariate extensions of expectiles risk measures. *Dependence Modeling*, 5(1):20–44.

McNeil, A. J., Frey, R., and Embrechts, P. (2015). *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton University Press, revised edition.

McNeil, A. J. and Smith, A. D. (2012). Multivariate stress scenarios and solvency. *Insurance: Mathematics and Economics*, 50(3):299–308.

Müller, A. and Scarsini, M. (2005). Archimedean copulae and positive dependence. *Journal of Multivariate Analysis*, 93(2):434–445.

Orlov, D., Zryumov, P., and Skrzypacz, A. (2021). Design of macro-prudential stress tests. Technical report, Simon Business School Working Paper No. FR 17-14, Stanford University.

Owen, A. and Zhou, Y. (2000). Safe and effective importance sampling. *Journal of the American Statistical Association*, 95(449):135–143.

Parlatore, C. and Philippon, T. (2022). Designing stress scenarios. Technical report, National Bureau of Economic Research.

Pesenti, S. M., Millossovich, P., and Tsanakas, A. (2019). Reverse sensitivity testing: What does it take to break the model? *European Journal of Operational Research*, 274(2):654–670.

Pesenti, S. M., Millossovich, P., and Tsanakas, A. (2021). Cascade sensitivity measures. *Risk Analysis*, forthcoming.

Pflug, G. C. and Romisch, W. (2007). *Modeling, measuring and managing risk*. World Scientific.

Prékopa, A. (2012). Multivariate value at risk and related topics. *Annals of Operations Research*, 193(1):49–

69.

Prékopa, A. and Lee, J. (2018). Risk tomography. *European Journal of Operational Research*, 265(1):149–168.

Prudential Regulatory Authority (2019). General insurance stress test 2019: Scenario specification, guidelines and instructions. Bank of England.

Rebonato, R. (2010). *Coherent stress testing: A Bayesian approach to the analysis of financial stress*. John Wiley & Sons.

Riedel, F. (2004). Dynamic coherent risk measures. *Stochastic processes and their applications*, 112(2):185–200.

Rockafellar, R. T., Uryasev, S., et al. (2000). Optimization of Conditional Value-at-Risk. *Journal of Risk*, 2:21–42.

Ruschendorf, L. (1976). Asymptotic distributions of multivariate rank order statistics. *The Annals of Statistics*, 4(5):912–923.

Rüschendorf, L. (2013). *Mathematical Risk Analysis. Dependence, Risk Bounds, Optimal Allocations and Portfolios*. Springer, Heidelberg.

Rüschendorf, L. and de Valk, V. (1993). On regression representations of stochastic processes. *Stochastic Processes and their Applications*, 46(2):183–198.

Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic Orders*. Springer.

Szegö, G. (2005). Measures of risk. *European Journal of Operational Research*, 163(1):5–19.

Tasche, D. (2004). Allocating portfolio economic capital to sub-portfolios. In A. Dev (ed.), Economic Capital: A Practitioner's Guide, Risk Books, pp. 275–302.

Townsend, J. T. and Colonius, H. (2005). Variability of the max and min statistic: A theory of the quantile spread as a function of sample size. *Psychometrika*, 70(4):759–772.

Tsanakas, A. (2009). To split or not to split: Capital allocation with convex risk measures. *Insurance: Mathematics and Economics*, 44(2):268–277.

Wakker, P. P. (2010). *Prospect theory: For risk and ambiguity*. Cambridge university press.

Wang, R., Bignozzi, V., and Tsanakas, A. (2015). How superadditive can a risk measure be? *SIAM Journal on Financial Mathematics*, 6(1):776–803.

Wang, S. (1996). Premium calculation by transforming the layer premium density. *ASTIN Bulletin*, 26(1):71–92.

Wang, S. (2007). Normalized exponential tilting: pricing and measuring multivariate risks. *North American Actuarial Journal*, 11(3):89–99.

Wang, S. S., Young, V. R., and Panjer, H. H. (1997). Axiomatic characterization of insurance prices. *Insurance: Mathematics and Economics*, 21(2):173–183.

Wirch, J. L. and Hardy, M. R. (1999). A synthesis of risk measures for capital adequacy. *Insurance: Mathematics and Economics*, 25(3):337–347.

Technical Appendices

A The vector U in the case of discontinuous marginals

When the marginals $\mathbf{X} \in \mathcal{X}^d$ are continuous, we define $U_i = F_{X_i}(X_i) \in \mathcal{U}$. If F_{X_i} has discontinuities, then $F_{X_i}(X_i)$ is no longer uniform. Furthermore, the copula of $\mathbf{X}$ is not uniquely defined. We address this issue here, following Rüschendorf and de Valk (1993). Let $\mathbf{V} \in \mathcal{U}^d$ be independent and also independent of $\mathbf{X}$. Then, we define

$$\begin{aligned}\tilde{F}_{X_i}(x, v) &:= F_{X_i}(x) + v(F_{X_i}(x) - F_{X_i}(x-)),\\ U_i &:= \tilde{F}_{X_i}(X_i, V_i).\end{aligned}$$

It then follows by Proposition 1.3 of Rüschendorf and de Valk (1993) that $U_i \in \mathcal{U}$ and that $F_{X_i}^{-1}(U_i) = X_i$ a.s. In the extreme case when X_i is degenerate, we have that $U_i = V_i$. Define $C_X(\mathbf{u}) := \mathbb{P}(\mathbf{U} \leqslant \mathbf{u})$. When in the paper we talk about the copula of $\mathbf{X}$, we avoid ambiguity by referring always to this $C_{\mathbf{X}}$, the uniquely defined distribution of $\mathbf{U}$ as constructed above.

B Proofs of results stated in Sections 2–4

Proof of Proposition 1

$\Leftarrow$: For each $\mathbf{X} \in \mathcal{X}^d$, the condition implies that $\eta(\mathbf{X})$ is $\sigma(\mathbf{X})$-measurable. Moreover, for $\mathbf{X}, \mathbf{Y} \in \mathcal{X}^d$ with $F_{\mathbf{X}} = F_{\mathbf{Y}}$, then letting $f = \Phi[F_{\mathbf{X}}] \in \mathbb{F}_d$, we have

$$(\eta(\mathbf{X}), \mathbf{X}) = (f(\mathbf{X}), \mathbf{X}) \overset{\mathrm{d}}{=} (f(\mathbf{Y}), \mathbf{Y}) = (\eta(\mathbf{Y}), \mathbf{Y}).$$

$\Rightarrow$: To get the stated condition, for $\mathbf{X} \in \mathcal{X}^d$, let $g_{\mathbf{X}}(\mathbf{x})$ be the point-mass $\eta(\mathbf{X})$ takes given $\mathbf{X} = \mathbf{x}$, which defines a function $g_{\mathbf{X}} \in \mathbb{F}_d$. Clearly, $\eta(\mathbf{X}) = g_{\mathbf{X}}(\mathbf{X})$. For $\mathbf{X}, \mathbf{Y} \in \mathcal{X}^d$ with $F_{\mathbf{X}} = F_{\mathbf{Y}}$, note that $(g_{\mathbf{X}}(\mathbf{X}), \mathbf{X}) = (\eta(\mathbf{X}), \mathbf{X}) \overset{\mathrm{d}}{=} (\eta(\mathbf{Y}), \mathbf{Y}) = (g_{\mathbf{Y}}(\mathbf{Y}), \mathbf{Y})$. As a consequence, the conditional distributions satisfy

$$(g_{\mathbf{X}}(\mathbf{X}), \mathbf{X})|_{\mathbf{X}=\mathbf{x}} \overset{\mathrm{d}}{=} (g_{\mathbf{Y}}(\mathbf{Y}), \mathbf{Y})|_{\mathbf{Y}=\mathbf{x}} \qquad \text{for } F_{\mathbf{X}}\text{-a.s. } \mathbf{x} \in \mathbb{R}^d.$$

Since $\eta(\mathbf{X})$ is $\sigma(X)$-measurable, the above conditional distributions are all point-masses, implying $g_{\mathbf{X}}(\mathbf{x}) = g_{\mathbf{Y}}(\mathbf{x})$ for $F_{\mathbf{X}}$-a.s. $\mathbf{x} \in \mathbb{R}^d$, so that the function $g_{\mathbf{X}}$ only depends on the distribution of $\mathbf{X}$. Letting $\Phi[F_{\mathbf{X}}] = g_{\mathbf{X}}$ concludes the argument.

Proof of Proposition 2

i) Let $g \in \mathbb{F}_d$ be any increasing function and $\mathbf{X} \in \mathcal{X}_+^d$. By increasingness of $\Phi[F_{\mathbf{X}}](\mathbf{x})$ and association of $\mathbf{X}$, we have

$$\mathbb{E}^\eta[g(\mathbf{X})] = \mathbb{E}\left[g(\mathbf{X})\Phi[F_{\mathbf{X}}](\mathbf{X})\right] \geqslant \mathbb{E}\left[g(\mathbf{X})\right]\mathbb{E}\left[\Phi[F_{\mathbf{X}}](\mathbf{X})\right] = \mathbb{E}[g(\mathbf{X})].$$

ii) Consider $\mathbf{X}, \mathbf{Y} \in \mathcal{X}(C)$. Without loss of generality, we can assume that for each $i = 1, \dots, d$ the pair (X_i, Y_i) is comonotonic. Furthermore, by invariance, and the shared copula of $\mathbf{X}, \mathbf{Y}$ we have that $\eta(\mathbf{X}) = \eta(\mathbf{Y}) = \eta(\mathbf{U})$. Let $X_i \preceq_{\mathrm{st}} Y_i$ and consider any increasing function $f \in \mathbb{F}$. By stochastic dominance and comonotonicity it then follows that $Y_i \geqslant X_i$, and hence $f(Y_i) - f(X_i) \geqslant 0$ a.s. Consequently $\mathbb{E}[\big(f(Y_i) - f(X_i)\big)\eta(\mathbf{U})] \geqslant 0$, implying in turn that $\mathbb{E}^\eta[f(Y_i)] \geqslant \mathbb{E}^\eta[f(X_i)]$.

iii) Let $\mathbf{X} \in \mathcal{X}^d$, such that $\mathbb{P}(X_i = c_{i-k}),\ i = k+1, \dots, d$. By invariance we have that $\eta(\mathbf{X}) = \eta(\mathbf{U})$. As degenerate (constant) random variables are discontinuous, we need to refer to the definition of $\mathbf{U}$ in Appendix A. From that it follows that $U_{k+1}, \dots, U_d$ are standard uniforms independent of each other and of $(X_1, \dots, X_k)$. Hence $\mathbf{U}$ does not depend on the particular value of $c_{i-k},\ i = k+1, \dots, d$.

Proof of Theorem 1

It is straightforward to check that (3) defines a direct stressing mechanism, and hence the "if" direction of the main statement is trivial.

To show the "only if" direction, suppose that η is direct, and let $f : [0,1]^d \to \mathbb{R}_+$ be a measurable function such that $\eta(\mathbf{U}) = f(\mathbf{U})$. Since f is bounded from below, the duality result in Theorem 2.3 of Rüschendorf (2013) implies that

$$\sup_{\mathbf{U} \in \mathcal{U}^d} \mathbb{E}[f(\mathbf{U})] = \min\left\{\sum_{i=1}^d \int_0^1 f_i(u)\mathrm{d}u \mid f_i : [0,1] \to \mathbb{R} \text{ measurable for each } i \text{ and } \bigoplus_{i=1}^d f_i \geqslant f\right\},$$

where $\big(\bigoplus_{i=1}^d f_i\big)(u_1, \dots, u_d) = \sum_{i=1}^d f_i(u_i)$. Therefore, there exist $f_i : [0,1] \to \mathbb{R}$ measurable, $i = 1, \dots, d$, with $\bigoplus_{i=1}^d f_i \geqslant f$ and $\mathbf{U} \in \mathcal{U}^d$ such that

$$\mathbb{E}[f(\mathbf{U})] = \sum_{i=1}^d \int_0^1 f_i(u)\mathrm{d}u = \int_{[0,1]^d} \bigoplus_{i=1}^d f_i(\mathbf{u})\mathrm{d}\mathbf{u}, \quad \mathbf{u} = (u_1, \dots, u_d).$$

Recalling that $\mathbb{E}[f(\mathbf{U})] = 1$ for all $\mathbf{U} \in \mathcal{U}^d$, choosing an independent $\mathbf{U}$ gives

$$1 = \int_{[0,1]^d} f(\mathbf{u})\mathrm{d}\mathbf{u} = \int_{[0,1]^d} \bigoplus_{i=1}^d f_i(\mathbf{u})\mathrm{d}\mathbf{u},$$

which further gives $\bigoplus_{i=1}^d f_i = f$ almost everywhere as $\bigoplus_{i=1}^d f_i \geqslant f$.

Let $a_i := \inf f_i \in \mathbb{R}$ be the essential infimum of f_i on $[0,1]$ with respect to the Lebesgue measure and $a := \sum_{i=1}^d a_i \geqslant 0$ since f is non-negative. Letting $\hat{f}_i = f_i - a_i + a/d$, it follows that $f = \bigoplus_{i=1}^d \hat{f}_i$ and each of $\hat{f}_1, \dots, \hat{f}_d$ is non-negative. Finally, for each $i = 1, \dots, d$, let $\lambda_i = \int_0^1 \hat{f}_i(u)\mathrm{d}u$ and $g_i = \hat{f}_i/\lambda_i$, where g_i is set to 1 if $\lambda_i = 0$. It follows that $g_1, \dots, g_d \in \mathcal{G}$, and (3) holds.

Next, we show the four equivalence statements (i)-(iv).

(i) We first show the "if" direction. Suppose that $g_i \in \mathcal{G}^*$ for each i with $\lambda_i > 0$ and $(U_1, \dots, U_d)$ is independent. Let $A \subset [0,1]^d$ be an increasing set, and, for each $(u_2, \dots, u_d) \in [0,1]^{d-1}$, the section of A

$$A(u_2, \dots, u_d) = \{u_1 \in [0,1] : (u_1, \dots, u_d) \in A\},$$

which is an increasing subset of $[0,1]$. Assume without loss of generality that $\lambda_1 > 0$. As explained before, $g_1 \in \mathcal{G}^*$ implies $\mathbb{E}[g_1(U_1)\mathbb{1}_{\{U_1 \in B\}}] \geqslant \mathbb{P}(U_1 \in B)$ for any increasing subset B of $[0,1]$. We have

$$\begin{aligned}
\mathbb{E}[g_1(U_1)\mathbb{1}_{\{\mathbf{U} \in A\}}] &= \mathbb{E}[\mathbb{E}[g_1(U_1)\mathbb{1}_{\{\mathbf{U} \in A\}}|U_2, \dots, U_d]] \\
&= \mathbb{E}[\mathbb{E}[g_1(U_1)\mathbb{1}_{\{U_1 \in A(U_2, \dots, U_d)\}}|U_2, \dots, U_d]] \\
&\geqslant \mathbb{E}[\mathbb{P}(U_1 \in A(U_2, \dots, U_d)|U_2, \dots, U_d)] = \mathbb{P}(\mathbf{U} \in A).
\end{aligned}$$

Hence, the post-stress probability of $\mathbf{U} \in A$ is larger or equal to $\mathbb{P}(\mathbf{U} \in A)$ under a univariate stressing. Since η is a mixture of univariate stressing mechanisms with $g_i \in \mathcal{G}^*$, we know that η is jointly stressing.

To show the "only if" direction, note that joint stressing implies the marginal order $F_{U_i} \preceq_{\mathrm{st}} F^\eta_{U_i}$ for each $i = 1, \dots, d$. If $(U_1, \dots, U_d)$ is independent, the post-stress distribution of U_i is given by

$$\lambda_i \hat{g}_i \circ F_{U_i} + \sum_{j=1, j \neq i}^d \lambda_j F_{U_j} = \lambda_i \hat{g}_i + (1 - \lambda_i) F_U, \tag{B.1}$$

where F_U is the identity on $[0,1]$. Hence, the order $F_{U_i} \preceq_{\mathrm{st}} \hat{g}_i$ implies $g_i \in \mathcal{G}^*$ if $\lambda_i > 0$.

(ii) The "if" direction follows from the fact that f is symmetric. To show the "only if" direction, we use (B.1) again. Symmetry implies $\lambda_i \hat{g}_i = \lambda_j \hat{g}_j$ for $i, j = 1, \dots, d$, and hence $\lambda_1 = \cdots = \lambda_d$ and $g_1 = \cdots = g_d$.

(iii) We first show the "if" direction, and without loss of generality we assume $f(u_1, \dots, u_d) = g_1(u_1)$ by absorbing the constants $\lambda_j g_j$, $j > 1$ into g_1. Take an independent $(U_1, \dots, U_d) \in \mathcal{U}^d$. Note that, in this case, $F^\eta_{U_i} = F_{U_i}$ for $i = 2, \dots, d$. For $\mathbf{u} = (u_1, \dots, u_d) \in [0,1]^d$, we have

$$F^\eta_{\mathbf{U}}(\mathbf{u}) = \mathbb{E}[g_1(U_1)\mathbb{1}_{\{\mathbf{U} \leqslant \mathbf{u}\}}] = \mathbb{E}[g_1(U_1)\mathbb{1}_{\{U_1 \leqslant u_1\}}] \prod_{i=2}^d \mathbb{P}(U_i \leqslant u_i) = \prod_{i=1}^d F^\eta_{U_i}(u_i).$$

Hence, η is independence preserving.

Next, we show the "only if" direction. Suppose that $\lambda_1 g_1$ and $\lambda_2 g_2$ are both non-constant. We will focus on (U_1, U_2). For the distribution of (U_1, U_2), due to independence, g_j for $j > 2$ can be treated as constants. Similarly to the above argument, we can assume $f(u_1, \dots, u_d) = \lambda_1 g_1(u_1) + \lambda_2 g_2(u_2)$ by absorbing the constants into g_1 and g_2. For $u_1, u_2 \in [0, 1]$, we have

$$\begin{aligned}\mathbb{E}[f(\mathbf{U})\mathbb{1}_{\{U_1 \leqslant u_1\}}\mathbb{1}_{\{U_2 \leqslant u_2\}}] &= \lambda_1 \mathbb{E}[g_1(U_1)\mathbb{1}_{\{U_1 \leqslant u_1\}}]u_2 + \lambda_2 \mathbb{E}[g_2(U_2)\mathbb{1}_{\{U_2 \leqslant u_2\}}]u_1 \\ &= \lambda_1 \hat{g}_1(u_1)u_2 + \lambda_2 \hat{g}_2(u_2)u_1.\end{aligned}$$

Moreover,

$$\mathbb{E}[f(\mathbf{U})\mathbb{1}_{\{U_1 \leqslant u_1\}}] = \lambda_1 \mathbb{E}[g_1(U_1)\mathbb{1}_{\{U_1 \leqslant u_1\}}] + \lambda_2 u_1 = \lambda_1 \hat{g}_1(u_1) + \lambda_2 u_1,$$

and similarly, $\mathbb{E}[f(\mathbf{U})\mathbb{1}_{\{U_2 \leqslant u_2\}}] = \lambda_1 u_2 + \lambda_2 \hat{g}_2(u_2)$. Therefore,

$$\begin{aligned}&\mathbb{E}[f(\mathbf{U})\mathbb{1}_{\{U_1 \leqslant u_1\}}\mathbb{1}_{\{U_2 \leqslant u_2\}}] - \mathbb{E}[f(\mathbf{U})\mathbb{1}_{\{U_1 \leqslant u_1\}}]\mathbb{E}[f(\mathbf{U})\mathbb{1}_{\{U_2 \leqslant u_2\}}] \\ &= \lambda_1 \hat{g}_1(u_1)u_2 + \lambda_2 \hat{g}_2(u_2)u_1 - (\lambda_1 \hat{g}_1(u_1) + \lambda_2 u_1)(\lambda_1 u_2 + \lambda_2 \hat{g}_2(u_2)) \\ &= \lambda_1(1-\lambda_1)\hat{g}_1(u_1)u_2 + \lambda_2(1-\lambda_2)\hat{g}_2(u_2)u_1 - \lambda_1\lambda_2\hat{g}_1(u_1)\hat{g}_2(u_2) - \lambda_1\lambda_2 u_1 u_2 \\ &= \lambda_1\lambda_2(\hat{g}_1(u_1) - u_1)(u_2 - \hat{g}_2(u_2)). \qquad \text{(B.2)}\end{aligned}$$

Since $\hat{g}_1$ and $\hat{g}_2$ are both not the identity and $\lambda_1\lambda_2 > 0$, we know that (B.2) cannot be always 0. Hence, the post-stress distribution of (U_1, U_2) is not independent, a contradiction.

(iv) Using (B.1), we know that the post-stress distribution of U_i is given by $F^{\eta}_{U_i} = \lambda_i \hat{g}_i + (1-\lambda_i)F_{U_i}$. Using $F^{\eta}_{X_i} = F^{\eta}_{U_i} \circ F_{X_i}$, we obtain $F^{\eta}_{X_i} = \lambda_i \hat{g}_i \circ F_{X_i} + (1-\lambda_i)F_{X_i}$. Thus, we have that that $F^{\eta}_{X_i} = \tilde{g}_i \circ F_{X_i}$ where $\tilde{g}_i(t) = \lambda_i \hat{g}_i(t) + (1-\lambda_i)t$, $t \in [0, 1]$. The marginal increasing in $\preceq_{\mathrm{icx}}$ property is equivalent to, for each $i = 1, \dots, d$,

$$\tilde{g}_i \circ F_X \preceq_{\mathrm{icx}} \tilde{g}_i \circ F_Y \text{ for all } F_X \preceq_{\mathrm{icx}} F_Y. \qquad \text{(B.3)}$$

Proposition 2 of Liu et al. (2021) implies that (B.3) holds if and only if $\tilde{g}_i$ is convex, which means that $\lambda_i g_i$ is increasing. Therefore, marginal increasingness in $\preceq_{\mathrm{icx}}$ is equivalent to g_i increasing whenever $\lambda_i > 0$.

Proof of Proposition 3

i) It is immediate.

ii) It follows from the observation that, if $\mathbf{X}$ is comonotonic, $U_1 = \cdots = U_d := U$ and $C_{\mathbf{X}}(\mathbf{u}) = \min\{u_1, \dots, u_d\}$, such that $K(\mathbf{X}) = \min\{U, \dots, U\} = U$.

iii) Without loss of generality, let X_1, X_2 be countermonotonic, such that $U_2 = 1 - U_1$. Then,

$$\begin{aligned}
K(\mathbf{X}) &= \mathbb{P}(U_1 \leqslant u_1, U_2 \leqslant u_2, \ldots, U_d \leqslant u_d)|_{(u_1,u_2,\ldots,u_d)=(U_1,U_2,\ldots,U_d)} \\
&= \mathbb{P}(U_1 \leqslant u_1, 1-U_1 \leqslant u_2, \ldots, U_d \leqslant u_d)|_{(u_1,u_2,\ldots,u_d)=(U_1,1-U_1,\ldots,U_d)} \\
&= \mathbb{P}(U_1 \leqslant u_1, 1-U_1 \leqslant 1-u_1, \ldots, U_d \leqslant u_d)|_{(u_1,u_2,\ldots,u_d)=(U_1,1-U_1,\ldots,U_d)}.
\end{aligned}$$

Note now that $\mathbb{P}(U_1 \leqslant u_1, 1-U_1 \leqslant 1-u_1, \ldots, U_d \leqslant u_d) = 0$, since $\mathbb{P}(U_1 = u_1) = 0$. The proof for $\bar{K}$ is similar.

iv) We only prove this for K, S. For any $\mathbf{u}$, the upper Frechet bound (Denuit et al., 2006, Sec 1.9.2) gives $C_{\mathbf{X}}(\mathbf{u}) \leqslant \min\{u_1, \ldots, u_d\}$. Consequently,

$$\begin{aligned}
K(\mathbf{X}) &= C_{\mathbf{X}}(U_{\mathbf{X}}) \leqslant \min\{U_1, \ldots, U_d\} \preceq_{\text{st}} V \\
S(\mathbf{X}) &= U_1 \cdot \ldots \cdot U_d \leqslant \min\{U_1, \ldots, U_d\} \preceq_{\text{st}} V.
\end{aligned}$$

Proof of Proposition 4

i) Invariance and symmetry are immediate. Let $\mathbf{X}$ be independent and $\eta(\mathbf{X}) = S(\mathbf{X})^\theta / \mathbb{E}\left[S(\mathbf{X})^\theta\right]$. Then, for $f_1, \ldots, f_d \in \mathbb{F}$

$$\begin{aligned}
\mathbb{E}^\eta[f_i(X_i)] &= \mathbb{E}\left[f_i(X_i)(1+\theta)U_i^\theta\right] \prod_{j \neq i} \mathbb{E}\left[(1+\theta)U_j^\theta\right] = \mathbb{E}\left[f_i(X_i)(1+\theta)U_i^\theta\right] \\
\mathbb{E}^\eta\left[\prod_{i=1}^d f_i(X_i)\right] &= \mathbb{E}\left[\prod_{i=1}^d f_i(X_i)(1+\theta)U_i^\theta\right] = \prod_{i=1}^d \mathbb{E}^\eta\left[f_i(X_i)\right],
\end{aligned}$$

which proves the independence preserving property. The other cases follow similarly.

ii) Follows directly from Proposition 2(i).

iii) Again, we only show this only for $\eta(\mathbf{X}) = S(\mathbf{X})^\theta / \mathbb{E}\left[S(\mathbf{X})^\theta\right]$. Let $\mathbf{X} \in \mathcal{X}^d$ be independent. It easily follows that $F^\eta_{X_i}(x) = \mathbb{E}\left[\mathbb{1}_{\{X_i \leqslant x\}}(1+\theta)U_i^\theta\right] = F_{X_i}(x)^{\theta+1}$. The claim then follows from (Liu et al., 2021, Prop. 2(ii)), given the convexity of $u \mapsto u^{\theta+1}$.

Proof of Proposition 5

For part ii) of the proposition, we will need to use the following Lemma.

Lemma 1. *Consider a uniform random variable U, an increasing function $f : [0,1] \to \mathbb{R}$, and two increasing and non-negative functions $\ell_1, \ell_2 : [0,1] \to \mathbb{R}_+$, such that for $u \in (0,1]$, $\ell_1(u) > 0$ and $\ell_2(u) > 0$. If $\ell_2(u)/\ell_1(u)$ is increasing on $(0,1]$, then*

$$\mathbb{E}\left[f(U)\frac{\ell_1(U)}{\mathbb{E}[\ell_1(U)]}\right] \leqslant \mathbb{E}\left[f(U)\frac{\ell_2(U)}{\mathbb{E}[\ell_2(U)]}\right].$$

Proof. For $i = 1, 2$, $\ell_i^*(u) = \frac{\ell_i(u)}{\mathbb{E}[\ell_i(U)]}$ is a density on $[0, 1]$. Denote the associated distributions by $L_i(p) = \int_0^p \ell_i^*(u)du$, $i = 1, 2$. By Theorem 3.A.26 in Shaked and Shanthikumar (2007), $\ell_1(U)$ is smaller than $\ell_2(U)$ in the Lorenz order, equivalently,

$$\frac{\ell_1(U)}{\mathbb{E}[\ell_1(U)]} \preceq_{\text{cx}} \frac{\ell_2(U)}{\mathbb{E}[\ell_2(U)]}.$$

The convex ordering implies, by Theorem 3.A.5 in Shaked and Shanthikumar (2007), that for all $0 < p < 1$,

$$\begin{aligned}
&\frac{1}{\mathbb{E}[\ell_1(U)]}\int_0^p F^{-1}_{\ell_1(U)}(u)du \geqslant \frac{1}{\mathbb{E}[\ell_2(U)]}\int_0^p F^{-1}_{\ell_2(U)}(u)du \\
\overset{\ell_1,\ell_2 \text{ incr.}}{\Longleftrightarrow}\ &\frac{1}{\mathbb{E}[\ell_1(U)]}\int_0^p \ell_1\left(F_U^{-1}(u)\right)du \geqslant \frac{1}{\mathbb{E}[\ell_2(U)]}\int_0^p \ell_2\left(F_U^{-1}(u)\right)du \\
\overset{U\sim Unif}{\Longleftrightarrow}\ &\frac{1}{\mathbb{E}[\ell_1(U)]}\int_0^p \ell_1(u)du \geqslant \frac{1}{\mathbb{E}[\ell_2(U)]}\int_0^p \ell_2(u)du \\
\Longleftrightarrow\ & L_1(p) \geqslant L_2(p).
\end{aligned}$$

The claim then follows from Lemma A.1 in Wang et al. (2015), after noting that $\mathbb{E}\left[f(U)\frac{\ell_i(U)}{\mathbb{E}[\ell_i(U)]}\right]$ is a distortion risk measure with distortion function L_i. □

We now proceed with the proof of Proposition 5.

i) For any increasing function f we have

$$\begin{aligned}
\mathbb{E}^{\eta(\mathbf{X})}[f(X_i)] &= \frac{1}{d}\sum_{j=1}^d \mathbb{E}[f(X_i)g(U_j)] \\
&\leqslant \frac{1}{d}\sum_{j=1}^d \mathbb{E}[f(X_i)g(U_i)] \\
&= \mathbb{E}[f(X_i)g(U_i)] = \mathbb{E}^{\eta_i(X_i)}[f(X_i)],
\end{aligned}$$

where the inequality is implied by the pairs $(f(X_i), g(U_j))$ and $(f(X_i), g(U_i))$ having the same marginal distributions, with the latter pair being comonotonic.

In the special case of comonotonicity we have $U_i = U_j,\ i \neq j$, such that the inequality above becomes equality.

ii) The case of independence is immediate. For the more general case of stochastic increasingness, let $A = S$ and consider an increasing function f. Without loss of generality let $i = 1$. Then,

$$\begin{aligned}
\mathbb{E}^{\eta_1(X_1)}[f(X_1)] &= \mathbb{E}\left[f(F^{-1}_{X_1}(U_1))\frac{U_1^\theta}{\mathbb{E}[U_1^\theta]}\right] = \mathbb{E}\left[f(F^{-1}_{X_i}(U_1))\frac{\ell_1(U_1)}{\mathbb{E}[\ell_1(U_1)]}\right] \\
\mathbb{E}^{\eta(\mathbf{X})}[f(X_1)] &= \mathbb{E}\left[f(F^{-1}_{X_1}(U_1))\frac{U_1^\theta\mathbb{E}[U_2^\theta\cdot\ldots\cdot U_d^\theta|U_1]}{\mathbb{E}[U_1^\theta\cdot\ldots\cdot U_d^\theta]}\right] = \mathbb{E}\left[f(F^{-1}_{X_1}(U_1))\frac{\ell_2(U_1)}{\mathbb{E}[\ell_2(U_1)]}\right],
\end{aligned}$$

where $\ell_1(u) = u^\theta$ and $\ell_2(u) = u^\theta \mathbb{E}\big[U_2^\theta \cdot \ldots \cdot U_d^\theta | U_1 = u\big]$. Then the ratio

$$\frac{\ell_2(u)}{\ell_1(u)} = \mathbb{E}\big[U_2^\theta \cdot \ldots \cdot U_d^\theta | U_1 = u\big]$$

is increasing by the assumption of stochastic increasingness. By applying Lemma 1, it follows that

$$\mathbb{E}^{\eta_1(X_1)}[f(X_1)] \leqslant \mathbb{E}^{\eta(\mathbf{X})}[f(X_1)].$$

Proof of Propositions 6 and 7

For Proposition 6 we note that

$$F_{X_i}^{\eta_i(X_i)}(x) = \int_0^{F_{X_i}(x)} \frac{u^{-\theta}(1-u)^{-\theta}}{B(1-\theta, 1-\theta)} \mathrm{d}u = G(F_{X_i}(x)),$$

where G is a Beta$(1-\theta, 1-\theta)$ distribution. For the given parameters, the distribution G satisfies $G(1/2) = 1/2$, $G(u) < u$ for $u \in (1/2, 1)$, and $G(u) > u$ for $u \in (0, 1/2)$. The stated result follows directly from these observations.

For Proposition 7i), the equality $F_{X_i}^{\eta_S(\mathbf{X})} = F_{X_i}^{\eta_i(X_i)}$ is an immediate consequence of independence. To show $F_{X_i}^{\eta_M(\mathbf{X})} \preceq_{\mathrm{QS}} F_{X_i}^{\eta_i(X_i)}$ it is enough to notice that, by independence,

$$\begin{aligned} F_{X_i}^{\eta_M(\mathbf{X})}(x) &= \lambda_i G(F_{X_i}(x)) + \sum_{j \neq i} \lambda_j F_{X_i}(x) \\ &= \lambda_i F_{X_i}^{\eta_i(X_i)}(x) + (1 - \lambda_i) F_{X_i}(x). \end{aligned}$$

For Proposition 7ii), comonotonicity implies $\eta_M(\mathbf{X}) = \eta_i(X_i),\ i = 1, \ldots, d$, which leads to $F_{X_i}^{\eta_i(X_i)} = F_{X_i}^{\eta_M(\mathbf{X})}$. On the other hand, since $U_i = U_j$ for all i, j, we have

$$\eta_S(\mathbf{X}) = \frac{U_i^{-d\theta} \bar{U}_i^{-d\theta}}{B(1 - d\theta, 1 - d\theta)}, \quad 0 < d\theta < 1,\ i = 1, \ldots, d,$$

and it holds that $F_{X_i}^{\eta_S(\mathbf{X})}(x) = \tilde{G}(F_{X_i}(x))$, where $\tilde{G}$ is a Beta$(1 - d\theta, 1 - d\theta)$ distribution. As $1 - d\theta < 1 - \theta$, we have that $\tilde{G}(u) < G(u)$ for $u \in (1/2, 1)$, and $\tilde{G}(u) > G(u)$ for $u \in (0, 1/2)$. From this, the relation $F_{X_i}^{\eta_i(X_i)} \preceq_{\mathrm{QS}} F_{X_i}^{\eta_S(\mathbf{X})}$ follows.

C Technical background for Examples 3, 5, and 6

In Example 3, we discussed the case where $\mathbf{X}$ is independent. Here we characterize the marginal post-stress distributions for specific dependence structures, in the case of stressing mechanisms of the form (4).

For the needs of this section we introduce some additional notation:

$$\eta^{A,\theta}(\mathbf{X}) = \frac{A(\mathbf{X})^{\theta}}{\mathbb{E}\left[A(\mathbf{X})^{\theta}\right]}, \quad A \in \{S, K\}, \quad \theta > 0,$$
$$\bar{\eta}^{A,\theta}(\mathbf{X}) = \frac{A(\mathbf{X})^{-\theta}}{\mathbb{E}\left[A(\mathbf{X})^{-\theta}\right]}, \quad A \in \{\bar{S}, \bar{K}\}, \quad \theta \in (0,1),$$

Proposition 8.

i) Let $\mathbf{X}$ *be independent. Then, for* $i = 1, \ldots, d$*,*

$$F_{X_i}^{\eta^{A,\theta}}(x) = F_{X_i}(x)^{1+\theta}, \qquad A \in \{K, S\}$$
$$\bar{F}_{X_i}^{\bar{\eta}^{A,\theta}}(x) = \bar{F}_{X_i}(x)^{1-\theta}, \qquad A \in \{\bar{K}, \bar{S}\}.$$

ii) Let $\mathbf{X}$ *be comonotonic. Then, for* $i = 1, \ldots, d$*,*

$$F_{X_i}^{\eta^{K,\theta}}(x) = F_{X_i}(x)^{1+\theta}$$
$$F_{X_i}^{\eta^{S,\theta}}(x) = F_{X_i}(x)^{1+d\theta}$$
$$\bar{F}_{X_i}^{\bar{\eta}^{\bar{K},\theta}}(x) = \bar{F}_{X_i}(x)^{1-\theta}$$
$$\bar{F}_{X_i}^{\bar{\eta}^{\bar{S},\theta}}(x) = \bar{F}_{X_i}(x)^{1-d\theta},$$

where for the last case we assume $\theta < 1/d$*.*

iii) Assume that $U_1 = V, \ldots, U_k = V,\ U_{k+1} = 1 - V, \ldots, U_d = 1 - V$*, for some* $V \in \mathcal{U}$*. Then,*

$$F_{X_i}^{\eta^{S,\theta}}(x) = B\big(F_{X_i}(x); k\theta + 1, (d-k)\theta + 1\big), \quad i = 1, \ldots, k$$
$$F_{X_i}^{\eta^{S,\theta}}(x) = B\big(F_{X_i}(x); (d-k)\theta + 1, k\theta + 1\big), \quad i = k+1, \ldots, d,$$
$$F_{X_i}^{\bar{\eta}^{\bar{S},\theta}}(x) = B\big(F_{X_i}(x); 1 - (d-k)\theta, 1 - k\theta\big), \quad i = 1, \ldots, k$$
$$F_{X_i}^{\bar{\eta}^{\bar{S},\theta}}(x) = B\big(F_{X_i}(x); 1 - k\theta, 1 - (d-k)\theta\big), \quad i = k+1, \ldots, d,$$

where $B(\cdot; a, b)$ *is the Beta*(a,b) *cumulative distribution function and in the last two cases we assume that* $\theta < \min\{1/k, 1/(d-k)\}$*.*

Proof. i) For $A \in \{S, K\}$

$$\begin{aligned} F_{X_i}^{\psi_\theta, A}(x) &= \mathbb{E}\left[\mathbb{1}_{\{X_i \leqslant x\}}(1+\theta)U_i^{\theta}\right] \\ &= \int_{-\infty}^{x} (1+\theta) F_{X_i}(x)^{\theta} dF_{X_i}(x) \\ &= F_{X_i}(x)^{1+\theta}. \end{aligned}$$

For $A \in \{\bar{K}, \bar{S}\}$, the argument is analogous.

ii) Note that $\mathbb{E}[U^a] = 1 + a$, for $a > -1$. Let $U = U_i,\ i = 1, \dots, d$, $\bar{U} = 1 - U$. Then

$$\begin{aligned}
\eta^{K,\theta}(\mathbf{X}) &= (1+\theta)U^{\theta} \\
\eta^{S,\theta}(\mathbf{X}) &= (1+d\theta)U^{d\theta} \\
\bar{\eta}^{\bar{K},\theta}(\mathbf{X}) &= (1-\theta)\bar{U}^{-\theta} \\
\bar{\eta}^{\bar{S},\theta}(\mathbf{X}) &= (1-d\theta)\bar{U}^{-d\theta}.
\end{aligned}$$

The marginal distributions follow from the same argument as in part i).

iii) We have $\eta^{S,\theta}(\mathbf{X}) = c \cdot V^{k\theta}(1-V)^{(d-k)\theta}$, for a constant c. Then,

$$\mathbb{Q}^{\eta^{S,\theta}}(V \leqslant v) = \int_0^v c v^{k\theta}(1-v)^{(d-k)\theta} dv = B\big(v; k\theta + 1, (d-k)\theta + 1\big),$$

from which the result follows. The other cases are similar.

□

In Example 5 we considered the case of multivariate Pareto distribution and Clayton (survival) copulas. Here we state this result formally, in the slightly more general setting where we only specify the copula of $\mathbf{X}$ rather than the full multivariate distribution. (Note that, by invariance of η, the post-stress copula $C_{\mathbf{X}}^{\eta}$ only depends on the baseline copula $C_{\mathbf{X}}$ and not the marginal distributions of $\mathbf{X}$.) The *Clayton copula*,

$$C_{\lambda}^{Cl}(\mathbf{u}) = \left(\sum_{i=1}^{d} u_i^{-\lambda} - d + 1\right)^{-1/\lambda}, \quad \lambda > 0,$$

is a special case of an Archimedean copula with generator $\phi(t) = (1+t)^{-1/\lambda}$. This copula has pairwise Kendall's rank correlations of $r_K(X_i, X_j) = \frac{\lambda}{\lambda+2}$.

Proposition 9.

i) Let $\mathbf{X}$ have a Clayton copula, $C_{\mathbf{X}} = C_{\lambda}^{Cl}$. Then, $\mathbf{X}$ also has a Clayton copula under $\mathbb{Q}^{\eta^{K,\theta}}$, $C_{\mathbf{X}}^{\eta^{K,\theta}} = C_{\lambda/(1+\theta)}^{Cl}$.

ii) Let $\mathbf{X}$ have a Clayton survival copula, $\bar{C}_{\mathbf{X}} = C_{\lambda}^{Cl}$. Then, $\mathbf{X}$ also has a Clayton survival copula under $\mathbb{Q}^{\bar{\eta}^{\bar{K},\theta}}$, $\bar{C}_{\mathbf{X}}^{\bar{\eta}^{\bar{K},\theta}} = C_{\lambda/(1-\theta)}^{Cl}$.

Proof. i) Consider the random vector $\mathbf{X}$, with multivariate distribution and density

$$F_{\mathbf{X}}(\mathbf{x}) = H(\mathbf{x}; \alpha) := \left(\sum_{i=1}^{d} x_i^{-1} + 1\right)^{-\alpha}, \quad \mathbf{x} > \mathbf{0}$$

$$f_{\mathbf{X}}(\mathbf{x}) = h(\mathbf{x}; \alpha) := \alpha(\alpha+1)\dots(\alpha+d-1)\left(\sum_{i=1}^{d} x_i^{-1} + 1\right)^{-\alpha-d} (x_1 \cdot \dots \cdot x_d)^{-2}$$

with parameter $\alpha > 0$. It is easily checked that $C_{\mathbf{X}} = C^{Cl}_{1/\alpha}$ and $F_{X_i}(x) = \left(x_i^{-1} + 1\right)^{-\alpha}$. By direct calculation we obtain, for $\mathbf{x} > \mathbf{0}$,

$$
\begin{aligned}
F_{\mathbf{X}}^{\eta^{K,\theta}}(\mathbf{x}) &= \frac{\mathbb{E}\left[F_{\mathbf{X}}(\mathbf{X})^{\theta}\mathbb{1}_{\{\mathbf{X}\leqslant\mathbf{x}\}}\right]}{\mathbb{E}\left[F_{\mathbf{X}}(\mathbf{X})^{\theta}\right]} \\
&= \left(\int_0^{\infty}\cdots\int_0^{\infty} F_{\mathbf{X}}(\mathbf{t})^{\theta} f_{\mathbf{X}}(\mathbf{t}) dt_1 \dots dt_d\right)^{-1}\left(\int_0^{x_d}\cdots\int_0^{x_1} F_{\mathbf{X}}(\mathbf{t})^{\theta} f_{\mathbf{X}}(\mathbf{t}) dt_1 \dots dt_d\right) \\
&= \left(\int_0^{\infty}\cdots\int_0^{\infty}\left(\sum_{i=1}^{d} t_i^{-1}+1\right)^{-\alpha(\theta+1)-d}(t_1\cdot\ldots\cdot t_d)^{-2} dt_1\dots dt_d\right)^{-1} \\
&\quad\left(\int_0^{x_d}\cdots\int_0^{x_1}\left(\sum_{i=1}^{d} t_i^{-1}+1\right)^{-\alpha(\theta+1)-d}(t_1\cdot\ldots\cdot t_d)^{-2} dt_1\dots dt_d\right) \\
&= \left(\int_0^{\infty}\cdots\int_0^{\infty} h(\mathbf{t};\alpha(\theta+1)) dt_1\dots dt_d\right)^{-1}\left(\int_0^{x_d}\cdots\int_0^{x_1} h(\mathbf{t};\alpha(\theta+1)) dt_1\dots dt_d\right) \\
&= H(\mathbf{x};\alpha(\theta+1)).
\end{aligned}
$$

Since under $\mathbb{Q}^{\eta^{K,\theta}}$ the distribution of $\mathbf{X}$ remains within the same family, but with the different parameter $\alpha(\theta+1)$, it follows that its copula is $C^{Cl}_{1/(\alpha(\theta+1))}$.

ii) The proof proceeds similarly, starting at a different choice of distribution for $\mathbf{X}$, with multivariate survival function

$$
\bar{F}_{\mathbf{X}}(\mathbf{x}) = \left(\sum_{i=1}^{d} x_i + 1\right)^{-\alpha}, \quad \mathbf{x} > \mathbf{0},\ \alpha > 0.
$$

□

In Example 6 we stated that, when starting from a baseline independent $\mathbf{X}$, we can generate a post-stress Archimedean copula, when stressing by a mixture, over the exponent, of mechanisms of the form (4). This is proved below.

Proposition 10. *Let $\mathbf{X}$ be independent and G be a distribution on $\mathbb{R}_+$ with $G(0) = 0$. Define the stressing mechanism*

$$
\eta(\mathbf{X}) = \int_0^{\infty} t^d A(\mathbf{X})^{t-1} dG(t).
$$

Then, under $\mathbb{Q}^{\eta}$, for $A = S$ (resp. $A = \bar{S}$) $\mathbf{X}$ has an Archimedean copula (resp. survival copula), with generator given by

$$
\phi(u) = \int_0^{\infty} e^{-tu} dG(t).
$$

Proof. For $A = S$, first note that

$$
\eta(\mathbf{X}) = \int_0^{\infty} t^d U_1^{t-1}\dots U_d^{t-1} dG(t) = \int_0^{\infty} \eta^{S,t-1}(\mathbf{X}) dG(t).
$$

Hence, as η is a mixture of stressing mechanisms it is indeed itself a stressing mechanism. (Note that we here extend the definition of $\eta^{S,\theta}$ to $\theta > -1$.) Now

$$\begin{aligned}
F_{\mathbf{X}}^{\eta}(\mathbf{x}) &= \mathbb{E}\left[\mathbb{1}_{\{\mathbf{X}\leqslant\mathbf{x}\}}\eta(\mathbf{X})\right] \\
&= \int_0^{\infty} \mathbb{E}\left[\mathbb{1}_{\{\mathbf{X}\leqslant\mathbf{x}\}}\eta^{S,t-1}(\mathbf{X})\right] dG(t) \\
&= \int_0^{\infty} \prod_{i=1}^{d} F_{X_i}^{\eta^{S,t-1}}(x_i) dG(t) \quad \text{(by independence preserving)} \\
&= \int_0^{\infty} \prod_{i=1}^{d} F_{X_i}(x_i)^t dG(t) \quad \text{(by Proposition 8i)).}
\end{aligned}$$

Hence the joint distribution of $\mathbf{X}$ under $\mathbb{Q}^{\eta}$ can be understood as a mixture of power-transformed distributions with respect to $G(t)$, see Denuit et al. (2006, Def. 7.2.12). The link to Archimedean copulas follows from the frailty construction of Marshall and Olkin (1988); see Denuit et al. (2006, Sec. 4.7.5.2) for a succinct discussion. The case $A = \bar{S}$ follows similarly. □

D Applying stressing mechanisms to raw data

The formulation of the stressing mechanisms in Section 4 assumes availability of the joint distribution of $\mathbf{X}$ to the end-user, see (1). Here, we briefly describe how to compute the stressing mechanisms based only on simulated or real data, without an explicit expression for the – potentially unknown – copula and marginal distribution functions. The idea follows from generating an empirical version of the quantities needed for computing the stressing mechanism η and expectations under the measure $\mathbb{Q}^{\eta}$.

Suppose that there are n data points that represent iid realizations of $(X_1, \ldots, X_d)$. For each observation $\mathbf{x}^j = (x_1^j, \ldots, x_d^j)$, $j = 1, \ldots, n$, and each $i = 1, \ldots, d$, we can define an empirical version of U_i as $u_i^j = \hat{F}_i(x_i^j)$, the normalized rank of X_i, defined via

$$\hat{F}_i(x) = \frac{1}{n+1} \sum_{j=1}^{n} \mathbb{1}_{\{x_i^j \leqslant x\}}, \quad x \in \mathbb{R}.$$

The function $\hat{F}_i(x)$ is a version of the empirical distribution; normalization by $1/(n+1)$ is used to prevent u_i^j from taking values 0 and 1 (which could potentially lead to infinite values of η), and this adjustment ensures $\mathbb{E}[\hat{F}_i(X_i)] = 1/2$.

Let A be a function on $[0,1]^d$ which generates a mixture stressing mechanism as in (3) or the case of Spearman mechanisms in (4). More precisely, we consider $A(u_1, \ldots, u_d) = \sum_{i=1}^{d} \lambda_i g_i(u_i)$ for (3) and $A(u_1, \ldots, u_d) = \prod_{i=1}^{d} u_i^{\theta}$ for (4), such that the corresponding stressing mechanism η is given by

$$\eta(\mathbf{X}) = \frac{A(\mathbf{U})}{\mathbb{E}[A(\mathbf{U})]}, \quad \mathbf{X} \in \mathcal{X}^d.$$

We can analogously define A for dual Spearman stressing mechanisms and for the Kendall case, involving the empirical Kendall's core; these are omitted here. Then, we can define an empirical version of the stressing mechanism $\hat{\eta}$ via

$$\hat{\eta}(\mathbf{x}^j) = \frac{A(u_1^j, \ldots, u_d^j)}{\sum_{l=1}^n A(u_1^l, \ldots, u_d^l)}, \quad \text{for } j = 1, \ldots, n.$$

Finally, to estimate the post-stress distribution of $\mathbf{X}$, we adjust the probability at each point $\mathbf{x}^j$ from $1/n$ (as in the empirical distribution) to $\hat{\eta}(\mathbf{x}^j)$. From there, we obtain a stressed empirical (joint) distribution defined as

$$F_n^{\hat{\eta}}(\mathbf{x}) = \sum_{j=1}^n \hat{\eta}(\mathbf{x}^j) \mathbb{1}_{\{\mathbf{x}^j \leqslant \mathbf{x}\}}, \quad \mathbf{x} \in \mathbb{R}^d,$$

where the subscript "n" emphasizes that the post-stress distribution depends on the number of data points n. The next result of the Glivenko-Cantelli type justifies the above empirical version of the stressing mechanism. It is shown that $\hat{\eta}$ produces an empirical post-stress distribution that serves as a good approximation to the post-stress distribution computed with $\eta(\mathbf{X})$, that is, under the assumption that $F_{\mathbf{X}}$ is fully available to the end-user.

Proposition 11. *Suppose that the data $\mathbf{x}^j$, $j = 1, 2, \ldots,$ are iid realizations of $\mathbf{X}$ and A is continuous with $\mathbb{E}[A(\mathbf{U})] < \infty$. Then, $F_n^{\hat{\eta}} \to F_{\mathbf{X}}^{\eta}$ at each point as $n \to \infty$ almost surely.*

We finally note that the computation of other quantities of interest, such as stressed expectations of functions of $\mathbf{X}$, follow similarly, e.g., for some $f \in \mathbb{F}^d$ it is

$$\mathbb{E}^{\hat{\eta}}[f(\mathbf{X})] = \sum_{j=1}^n \hat{\eta}(\mathbf{x}^j) f(\mathbf{x}^j).$$

This illustrates that stressing the model via a change of measure does not require re-evaluations of the function f, which, in realistic applications may be computationally expensive. Hence our suggested stress-testing framework is computationally efficient, consistently with the arguments of Pesenti et al. (2019).

Proof of Proposition 11

Fix $\mathbf{t} \in \mathbb{R}^d$ and let $f : \mathbf{t} \mapsto \mathbb{1}_{\{\mathbf{x} \leqslant \mathbf{t}\}}$. We need to verify

$$\int_{\mathbb{R}^d} f \mathrm{d}F_n^{\hat{\eta}} \to \int_{\mathbb{R}^d} f \mathrm{d}F_{\mathbf{X}}^{\eta} \quad \text{almost surely.} \tag{D.4}$$

Note that

$$\int_{\mathbb{R}^d} f \mathrm{d}F_n^{\hat{\eta}} = \sum_{j=1}^n \hat{\eta}(\mathbf{x}^j) f(\mathbf{x}^j) = \frac{\sum_{j=1}^n A(u_1^j, \ldots, u_d^j) f(\mathbf{x}^j)}{\sum_{j=1}^n A(u_1^j, \ldots, u_d^j)} = \frac{\frac{1}{n}\sum_{j=1}^n A(u_1^j, \ldots, u_d^j) f(\mathbf{x}^j)}{\frac{1}{n}\sum_{j=1}^n A(u_1^j, \ldots, u_d^j)},$$

and

$$\int_{\mathbb{R}^d} f \mathrm{d}F_{\mathbf{X}}^{\eta} = \frac{\mathbb{E}[A(\mathbf{U}) f(\mathbf{X})]}{\mathbb{E}[A(\mathbf{U})]}.$$

The desired convergence (D.4) follows from the continuity of A, the continuity of the marginals of $\mathbf{X}$, and the well-known fact that the pseudo-sample $\{(u_1^j, \ldots, u_d^j) : j = 1, \ldots, n\}$ behaves similarly to an iid copy of $\mathbf{U}$ as $n \to \infty$ in the sense of e.g. Ruschendorf (1976). To be more specific on the last point, we can safely treat $\{(A(u_1^j, \ldots, u_d^j), \mathbf{x}^j) : j = 1, \ldots, n\}$ as an iid copy of $(A(\mathbf{U}), \mathbf{X})$ in asymptotic analyses; see also Genest and Rivest (1993) and Section 7.5 of McNeil et al. (2015).