Optimizing distortion riskmetrics with distributional uncertainty

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Abstract

Optimization of distortion riskmetrics with distributional uncertainty has wide applications in finance and operations research. Distortion riskmetrics include many commonly applied risk measures and deviation measures, which are not necessarily monotone or convex. One of our central findings is a unifying result that allows to convert an optimization of a non-convex distortion riskmetric with distributional uncertainty to a convex one induced from the concave envelope of the distortion function, leading to practical tractability. A sufficient condition to the unifying equivalence result is the novel notion of closedness under concentration, a variation of which is also shown to be necessary for the equivalence. Our results include many special cases that are well studied in the optimization literature, including but not limited to optimizing probabilities, Value-at-Risk, Expected Shortfall, Yaari's dual utility, and differences between distortion risk measures, under various forms of distributional uncertainty. We illustrate our theoretical results via applications to portfolio optimization, optimization under moment constraints, and preference robust optimization.

Keywords: risk measures; deviation measures; distributionally robust optimization; convexification; conditional expectation

1 Introduction

Riskmetrics, such as measures of risk and variability, are common tools to represent preferences, model decisions under risks, and quantify different types of risks. To fix terms, we refer to *riskmetrics* as any mapping from a set of random variables to the real line, and *risk measures* as riskmetrics that are monotone in the sense of Artzner et al. (1999).

In this paper, we focus on *distortion riskmetrics* which is a large class of commonly used measures of risk and variability; see Wang et al. (2020a) for the terminology "distortion riskmetrics". Distortion riskmetrics include L-functionals (Huber and Ronchetti, 2009) in statistics, Yaari's dual

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utilities (Yaari, 1987) in decision theory, distorted premium principles (Wang et al., 1997) in insurance, and spectral risk measures (Acerbi, 2002) in finance; see Wang et al. (2020a) for further examples. After a normalization, increasing distortion riskmetrics are *distortion risk measures*, which include, in particular, the two most important risk measures used in current banking and insurance regulation, the Value-at-Risk (VaR) and the Expected Shortfall (ES). Moreover, convex distortion riskmetrics are the building blocks (via taking a supremum) for all convex risk functionals (Liu et al., 2020), including classic risk measures (Artzner et al., 1999; Föllmer and Schied, 2002) and deviation measures (Rockafellar et al., 2006).

When riskmetrics are evaluated on distributions that are subject to uncertainty, decisions should be taken with respect to the worst (or best) possible values a riskmetric attains over a set of alternative distributions; giving rise to the active subfield of distributionally robust optimization. The set of alternative distributions, the *uncertainty set*, may be characterized by moment constraints (e.g., Popescu (2007)), parameter uncertainty (e.g., Delage and Ye (2010)), probability constraints (e.g., Wiesemann et al. (2014)), Wasserstein distances (e.g., Pflug and Wozabal (2007), Esfahani and Kuhn (2018), Blanchet and Murthy (2019), and Gao and Kleywegt (2023)), and ϕ-divergence (e.g., Jiang and Guan (2016)), amongst others. Distributionally robust optimization problems have been studied under the framework of expected utility (e.g., Popescu (2007) and Chen et al. (2011)) and further under shortfall risk measures (e.g., Delage et al. (2022)). As an important class of risk measures, distortion risk measures have also been considered as a natural choice of objectives for distributionally robust optimization. Popular distortion risk measures such as VaR and ES are studied extensively in this context; see e.g., Natarajan et al. (2008) and Zhu and Fukushima (2009).

Optimization of convex distortion risk measures, i.e., distortion riskmetrics with an increasing and concave distortion function, is relatively well understood under distributional uncertainty; see Cornilly et al. (2018), Li (2018), and Liu et al. (2020) for some recent work. Nevertheless, many distortion riskmetrics are not convex or monotone. For example, in the Cumulative Prospect Theory of Tversky and Kahneman (1992), the distortion function is typically assumed to be inverse-S-shaped; in financial risk management, the popular risk measure VaR has a non-concave distortion function, and the inter-quantile difference (Wang et al., 2020b) has a distortion function that is neither concave nor monotone. Another example is the difference between two distortion risk measures, which is clearly not increasing or convex in general. Optimizing non-convex distortion riskmetrics under distributional uncertainty is difficult and results are available only for special cases; see Li et al. (2018), Cai et al. (2018), Zhu and Shao (2018), Wang et al. (2019), and Bernard et al. (2020), all with an increasing distortion function.

There is, however, a notable common feature in the above mentioned literature when a non-convex distortion risk metric is involved. For numerous special cases, one often obtains an equivalence between the optimization problem with non-convex distortion riskmetric and that with a convex one. Inspired by this observation, the aim of this paper is to address:

What conditions provide equivalence between a non-convex riskmetric and a convex one, that is induced by the concave envelope of the distortion function, in the setting of distributional uncertainty?

An answer to this question is still missing in the literature. In this sense, we offer a novel perspective on distributionally robust optimization problems by converting optimization problems with non-convex objectives to their convex counterpart. Transforming an optimization problem with a non-convex objective to a convex one through approximation and via a direct equivalence has been studied by Zymler et al. (2013) and Cai et al. (2020). Both contributions, however, consider uncertainty sets described by some special forms of constraints. A unifying framework applicable to numerous uncertainty sets and the entire class of distortion riskmetrics is however missing and at the core of this paper.

The main novelty of our results is three-fold: first, we obtain a unifying result (Theorem 1) that allows, under distributional uncertainty, to convert an optimization problem of a non-convex distortion riskmetric to an optimization problem with a convex one, where the convex one is induced via the concave envelope of the distortion function. The result covers, to the authors' best knowledge, all known equivalences between optimization problems of non-convex and convex riskmetrics, where the convex one is induced by the concave envelope of the distortion function with distributional uncertainty. The proof requires techniques beyond the ones used in the existing literature, as we do not make assumptions such as monotonicity, positiveness, and continuity. Our framework can also be applied to settings with atomic probability space or with uncertainty sets of multi-dimensional distributions. Second, we introduce the concept of *closedness under concentration* as a sufficient condition to establish the equivalence, and it is also a necessary condition on the set of optimizers given that the equivalence holds (Theorem 2). We show how the properties of closedness under concentration within a collection of intervals $\mathcal{I}$ and closedness under concentration for all intervals can be verified through direct analysis and provide numerous examples. Third, the classes of distortion riskmetrics and uncertainty formulations considered in this paper include all special cases studied in the literature; examples are presented in Sections 3-4. In particular, our class of riskmetrics include all practically used risk measures and variability measures (some via taking a sup), dual utilities with inverse-S-shaped distortion functions of Tversky and Kahneman (1992), and differences between two dual utilities or distortion risk measures. Our uncertainty formulations include both supremum and infimum problems,[1] moment constraints, convex order/risk measure constraints, marginal constraints in risk aggregation with dependence uncertainty (e.g., Embrechts et al. (2015)), preference robust optimization (e.g., Armbruster and Delage (2015) and Guo and Xu (2021)), and some one-dimensional and multi-dimensional uncertainty sets induced by Wasserstein metrics.

The generality of our work distinguishes it from the large literature on distributional robust optimization cited above. Our work is of analytical and probabilistic nature, and we focus on theoretical equivalence results which will be also illustrated via numerical implementations. The target problems are formulated in Section 2. Sections 3 is devoted to our main contribution of the equivalence of an optimization problem with a non-convex riskmetric and the convex one induced from the concave envelope of the distortion function under distributional uncertainty. We illus-

[1] Thus we provide a universal treatment of worst-case and best-case risk values. Calculating best-case risk values allows us to solve economic decision making problems where optimal distributions are chosen to minimize the risk.

trate by many examples the concepts of closedness under conditional expectation and closedness under concentration, and distinguish them in several practical settings. Section 4 demonstrates the equivalence results in multi-dimensional settings. In addition to a general multi-dimensional model with a concave loss function, we solve a robust risk aggregation problem with ambiguity on both the marginal distributions and the dependence structure. In Section 5, our results are used to solve optimization problems with uncertainty sets defined via moment constraints. In particular, we generalize a few well-known results in the literature on optimization and worst-case values of risk measures. Sections 6 and 7 contain numerical illustrations of optimizing differences between two distortion riskmetrics, portfolio optimization, and preference robust optimization. Some concluding remarks are put in Section 8. Complete proofs of all results are relegated to Appendix B.

2 Distortion riskmetrics with distributional uncertainty

2.1 Problem formulation

Throughout, we work with an atomless probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Denote by $\mathcal{L}^p$, $p \in [1, \infty)$, the space of random variables with finite p-th moment. Let $\mathcal{L}^\infty$ represent the set of bounded random variables and let $\mathcal{L}^0$ represent the space of all random variables. For $n \in \mathbb{N}$, A represents a set of actions, ρ is an objective functional, $f : A \times \mathbb{R}^n \to \mathbb{R}$ is a loss function, and $\mathbf{X}$ is an n-dimensional random vector with distributional uncertainty. Many problems in distributionally robust optimization have the form

$$\min_{\mathbf{a} \in A} \sup_{F_{\mathbf{X}} \in \widetilde{\mathcal{M}}} \rho(f(\mathbf{a}, \mathbf{X})), \tag{1}$$

where $F_{\mathbf{X}}$ denotes the distribution of $\mathbf{X}$ and $\widetilde{\mathcal{M}}$ is a set of plausible distributions for $\mathbf{X}$. We will first focus on the inner problem

$$\sup_{F_{\mathbf{X}} \in \widetilde{\mathcal{M}}} \rho(f(\mathbf{a}, \mathbf{X})), \tag{2}$$

which we may rewrite as

$$\sup_{F_Y \in \mathcal{M}} \rho(Y), \tag{3}$$

where F_Y denotes the distribution of Y and $\mathcal{M}$ is a set of distributions on $\mathbb{R}$. We suppress the reliance on $\mathbf{a}$ as it remains constant in the inner problem (2). The supremum in (3) is typically referred to as the *worst-case risk measure* in the literature if ρ is monotone.[2] The problem (3) can also represent an optimal decision problem, where ρ is an objective to maximize, and a decision maker chooses an optimal distribution from the set $\mathcal{M}$ which is interpreted as an action set instead of an uncertainty set (i.e., no uncertainty in this problem). Since the two problems share the same mathematical formulation (3), we will navigate through our results mainly with the first interpretation of worst-case risk under uncertainty.

[2] A risk measure $\rho : \mathcal{L}^p \to \mathbb{R}$ is *monotone* if $\rho(X) \leqslant \rho(Y)$ for all $X, Y \in \mathcal{L}^p$ with $X \leqslant Y$.

Denote by $\mathcal{H}$ the set of functions $h : [0, 1] \mapsto \mathbb{R}$ with bounded variation satisfying $h(0) = 0$. For $p \in [1, \infty]$ and $h \in \mathcal{H}$, a *distortion riskmetric* $\rho_h : \mathcal{L}^p \to \mathbb{R}$ is defined as

$$\rho_h(Y) = \int_0^\infty h(\mathbb{P}(Y > x))\,\mathrm{d}x + \int_{-\infty}^0 (h(\mathbb{P}(Y > x)) - h(1))\,\mathrm{d}x, \quad Y \in \mathcal{L}^p, \tag{4}$$

whenever the above integrals are finite; see Proposition 6 below for a sufficient condition. The function $h \in \mathcal{H}$ is called a *distortion function*. Note that we allow h to be non-monotone; if h is increasing and $h(1) = 1$, then ρ_h is a distortion risk measure. The distortion riskmetric ρ_h is convex if and only if h is concave; see Wang et al. (2020b) for this and other properties of ρ_h.

In this paper we consider the objective functional ρ in (1) to be a distortion riskmetric ρ_h for some $h \in \mathcal{H}$, as the class of distortion riskmetrics includes a large class of objective functionals of interest. Note that a general analysis of (3) also covers the infimum problem $\inf_{F_Y \in \mathcal{M}} \rho_h(Y)$, since $-\rho_h = \rho_{-h}$ is again a distortion riskmetric. This illustrates an advantage of studying distortion riskmetrics over monotone ones, as our analysis unifies best- and worst-case risk evaluations. Best-case risk measures are also of practical importance. In particular, they may represent risk minimization problems through the second interpretation of (3), where $\mathcal{M}$ represents a set of possible actions (see Section 3.4 for some examples).

If ρ_h is not convex, or equivalently, h is not concave, problems such as (1) and (3) are often highly nontrivial. However, the optimization problem of maximizing $\rho_{h^*}(Y)$ over $F_Y \in \mathcal{M}$, where h^* is the smallest concave distortion function dominating h, can often be solved either analytically or through numerical methods. Note that ρ_h is mixture concave (i.e., $F_X \mapsto \rho(X)$ is concave) if and only if h is concave by Theorem 3 of Wang et al. (2020b). As a consequence, if $f(\mathbf{a}, \mathbf{x})$ is convex in $\mathbf{a}$ (for instance, in portfolio selection, a common choice is $f(\mathbf{a}, \mathbf{x}) = \mathbf{a}^\top \mathbf{x}$), then the optimization (1) for ρ_{h^*},

$$\min_{\mathbf{a} \in A} \sup_{F_{\mathbf{X}} \in \widetilde{\mathcal{M}}} \rho_{h^*}(f(\mathbf{a}, \mathbf{X})),$$

has an objective $\rho_{h^*}(f(\mathbf{a}, \mathbf{X}))$ which is convex in $\mathbf{a}$ and concave in $F_{\mathbf{X}}$. This is a standard convex-concave minimax problem in the optimization literature and various computational methods exist (e.g., Korpelevich (1976), Nemirovski (2004), and Ouyang and Xu (2021)). To utilize this observation for optimizing ρ_h, the crucial condition is

$$\sup_{F_{\mathbf{X}} \in \widetilde{\mathcal{M}}} \rho_h(f(\mathbf{a}, \mathbf{X})) = \sup_{F_{\mathbf{X}} \in \widetilde{\mathcal{M}}} \rho_{h^*}(f(\mathbf{a}, \mathbf{X})),$$

that is, with $Y = f(\mathbf{a}, \mathbf{X})$,

$$\sup_{F_Y \in \mathcal{M}} \rho_h(Y) = \sup_{F_Y \in \mathcal{M}} \rho_{h^*}(Y). \tag{5}$$

Also note that $\rho_h \leqslant \rho_{h^*}$ always holds, and hence for (5), it suffices to study the "$\geqslant$" inequality.

The main contribution of this paper is a sufficient condition on the uncertainty set $\mathcal{M}$ that guarantees the equivalence (5). We will also obtain a necessary condition for (5). The equivalence (5) makes the optimization problem (1) for ρ_h much more tractable in various settings, which will be illustrated through the examples in the following sections.

2.2 Notation and preliminaries

For $p \geqslant 1$ and $n \in \mathbb{N}$, we denote by $\mathcal{M}_p^n$ the set of all distributions on $\mathbb{R}^n$ with finite p-th moment. Let $\mathcal{M}_\infty^n$ be the set of n-dimensional distributions of bounded random variables. For $p \in [1, \infty]$, write $\mathcal{M}_p^1 = \mathcal{M}_p$ for simplicity. The set inclusion $\subset$ and terms like "increasing" and "decreasing" are in the non-strict sense. For $X, Y \in \mathcal{L}^p$, we write $X \stackrel{\mathrm{d}}{=} Y$ to represent that X and Y have the same distribution. For a distribution $F \in \mathcal{M}_1$, let its left- and right-quantile functions be given respectively by

$$F^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F(x) \geqslant \alpha\} \text{ and } F^{-1+}(\alpha) = \inf\{x \in \mathbb{R} : F(x) > \alpha\}, \quad \alpha \in [0,1],$$

with the convention $\inf(\varnothing) = \infty$. For $x, y \in \mathbb{R}$, we write $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. For $x \in \mathbb{R}$, we write $x_+ = x \vee 0$ and $x_- = -(-x)_+$. Since $h \in \mathcal{H}$ is of bounded variation, its discontinuity points are at most countable and the left- and right-limits exist at each of these points. We write

$$h(t^+) = \begin{cases} \lim_{x \downarrow t} h(x), & t \in [0,1), \\ h(1), & t = 1, \end{cases} \quad \text{and} \quad h(t^-) = \begin{cases} \lim_{x \uparrow t} h(x), & t \in (0,1], \\ h(0), & t = 0, \end{cases}$$

and the upper semicontinuous modification of h is denoted by

$$\hat{h}(t) = h(t^+) \vee h(t^-) \vee h(t), \quad t \in (0,1), \quad \text{with } \hat{h}(0) = 0 \text{ and } \hat{h}(1) = h(1).$$

Note that $\hat{h}(t) = h(t)$ at all continuous points of h, and we do not make any modification at the points 0 and 1 even if h has a jump at these points. For $h \in \mathcal{H}$ and $t \in [0,1]$, define its concave and convex envelopes h^* and h_* respectively by

$$h^*(t) = \inf\{g(t) : \ g \in \mathcal{H}, \ g \geqslant h, \ g \text{ is concave on } [0,1]\},$$

$$h_*(t) = \sup\{g(t) : \ g \in \mathcal{H}, \ g \leqslant h, \ g \text{ is convex on } [0,1]\}.$$

Both h^* and h_* are continuous functions on $(0,1)$ for all $h \in \mathcal{H}$, and if h is continuous at 0 and 1, then so are h^* and h_* (see Figure 4 below for an illustration of h and h^*). Denote by $\mathcal{H}^*$ (resp. $\mathcal{H}_*$) the set of concave (resp. convex) functions in $\mathcal{H}$. Note that for all $h \in \mathcal{H}$, we have $h^* \in \mathcal{H}^*$ and $h_* \in \mathcal{H}_*$. As a well-known property of the convex and concave envelopes of a continuous h (e.g., Brighi and Chipot (1994)), h^* (resp. h_*) differs from h on a union of disjoint open intervals, and h^* (resp. h_*) is linear on these intervals. The functions h, $\hat{h}$, h^* and $(\hat{h})^*$ are illustrated in Figure 1.

While in general ρ_h and $\rho_{\hat{h}}$ are different functionals, one has $\rho_h(Y) = \rho_{\hat{h}}(Y)$ for any random variable Y with continuous quantile function; see Lemma 1 of Wang et al. (2020a). Moreover, $h^* = (\hat{h})^* \geqslant \hat{h} \geqslant h$ and the four functions are all equal if h is concave. Below, we provide a new result on convex envelopes of distortion functions h that are not necessarily monotone or continuous,

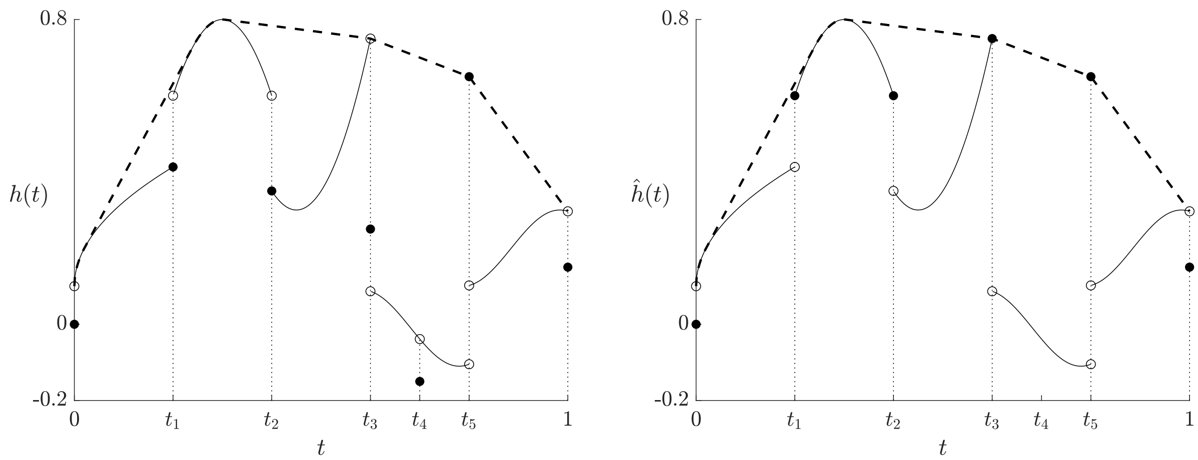

Figure 1: An example of h (left) and $\hat{h}$ (right) with the set of discontinuity points $\{t_1, t_2, t_3, t_4, t_5\}$ excluding 0 and 1; the dashed lines represent h^* and $(\hat{h})^*$, which are identical by Proposition 1

which may be of independent interest.

Proposition 1. *For any $h \in \mathcal{H}$, we have $h^* = (\hat{h})^*$ and the set $\{t \in [0,1] : \hat{h}(t) \neq h^*(t)\}$ is the union of some disjoint open intervals. Moreover, h^* is linear on each of the above intervals.*

In the sequel, we mainly focus on h^*, which will be useful when optimizing ρ_h in (3). A similar result to Proposition 1 holds for h_*, useful in the corresponding infimum problem, where the upper semicontinuous modification of h is replaced by the lower semicontinuous one. This follows directly from Proposition 1 by setting $g = -h$ which gives $\rho_g = -\rho_h$ and $h_* = -g^*$.

For all distortion functions $h \in \mathcal{H}$, from Proposition 1, there exist (countably many) disjoint open intervals on which $\hat{h} \neq h^*$. Using a similar notation to Wang et al. (2019), we define the set

$$\mathcal{I}_h = \left\{(1-b, 1-a) : \hat{h} \neq h^* \text{ on } (a,b),\ \hat{h}(a) = h^*(a),\ \hat{h}(b) = h^*(b)\right\}.$$

The set $\mathcal{I}_h$ is straightforward to identify in practice; see Section 3.2 for examples of commonly used distortion riskmetrics and their corresponding sets $\mathcal{I}_h$.

3 Equivalence between non-convex and convex riskmetrics

3.1 Concentration and the main equivalence result

In this section, we introduce the concept of concentration, and use this concept to explain our main equivalence results, Theorems 1 and 2. For a distribution $F \in \mathcal{M}_1$ and an interval $C \subset [0,1]$ (when speaking of an interval in $[0,1]$, we exclude singletons or empty sets), we define the *C-concentration of F*, denote by F^C, as the distribution of the random variable

$$F^{-1}(U)\mathbb{1}_{\{U \notin C\}} + \mathbb{E}[F^{-1}(U) | U \in C]\mathbb{1}_{\{U \in C\}}, \tag{6}$$

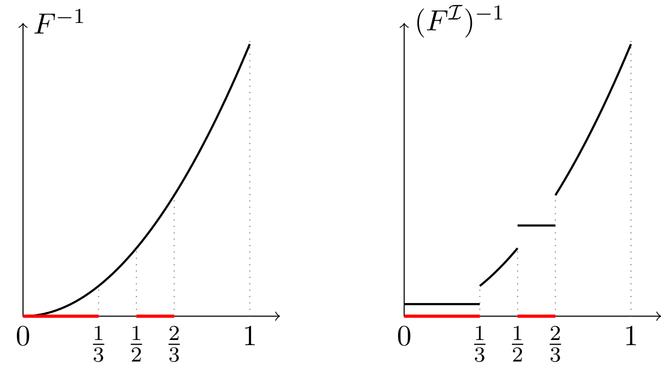

Figure 2: Left panel: quantile function of F; right panel: quantile function of $F^{\mathcal{I}}$ where $\mathcal{I} = \{(0, 1/3), (1/2, 2/3)\}$

where $U \sim \mathrm{U}[0,1]$ is a standard uniform random variable. In other words, F^C is obtained by concentrating the probability mass of $F^{-1}(U)$ on $\{U \in C\}$ at its conditional expectation, whereas the rest of the distribution remains unchanged. For $F \in \mathcal{M}_1$ and $0 \leqslant a < b \leqslant 1$, it is clear that the left-quantile function of $F^{(a,b)}$ is given by

$$F^{-1}(t)\mathbb{1}_{\{t \notin (a,b]\}} + \frac{\int_a^b F^{-1}(u)\,\mathrm{d}u}{b-a}\mathbb{1}_{\{t \in (a,b]\}}, \quad t \in [0,1]. \tag{7}$$

For a collection $\mathcal{I}$ of (possibly infinitely many) non-overlapping intervals in $[0,1]$, let $F^{\mathcal{I}}$ be the distribution corresponding to the left-quantile function given by the left-continuous version of

$$F^{-1}(t)\mathbb{1}_{\{t \notin \bigcup_{C \in \mathcal{I}} C\}} + \sum_{C \in \mathcal{I}} \frac{\int_C F^{-1}(u)\,\mathrm{d}u}{\lambda(C)}\mathbb{1}_{\{t \in C\}}, \qquad t \in [0,1], \tag{8}$$

where λ is the Lebesgue measure; see Figure 2 for an illustration.

Definition 1. Let $\mathcal{M}$ be a set of distributions in $\mathcal{M}_1$ and $\mathcal{I}$ be a collection of intervals in $[0,1]$. We say that (a) $\mathcal{M}$ is *closed under concentration within* $\mathcal{I}$ if $F^{\mathcal{I}} \in \mathcal{M}$ for all $F \in \mathcal{M}$; (b) $\mathcal{M}$ is *closed under concentration for all intervals* if for all $F \in \mathcal{M}$, we have $F^C \in \mathcal{M}$ for all intervals $C \subset [0,1]$; (c) $\mathcal{M}$ is *closed under conditional expectation* if for all $F_X \in \mathcal{M}$ and σ-algebra $\mathcal{G} \subseteq \mathscr{F}$, we have $F_{\mathbb{E}[X|\mathcal{G}]} \in \mathcal{M}$.

The relationship between the three properties of closedness in Definition 1 is discussed in Propositions 2 and 3 below. Generally, (c)$\Rightarrow$(b)$\Rightarrow$(a) if $\mathcal{I}$ is finite. Our main equivalence result is summarized in the following theorem.

Theorem 1. *For $\mathcal{M} \subset \mathcal{M}_1$ and $h \in \mathcal{H}$, the following hold.*

(i) If $h = \hat{h}$, i.e., h is upper semicontinuous on $(0,1)$, and $\mathcal{M}$ is closed under concentration within $\mathcal{I}_h$, then

$$\sup_{F_Y \in \mathcal{M}} \rho_h(Y) = \sup_{F_Y \in \mathcal{M}} \rho_{h^*}(Y). \tag{9}$$

(ii) If $\mathcal{M}$ is closed under concentration for all intervals, then (9) holds.

(iii) If $h = \hat{h}$, $\mathcal{M}$ is closed under concentration within $\mathcal{I}_h$, and the second supremum in (9) is attained by some $F \in \mathcal{M}$, then $F^{\mathcal{I}_h}$ attains both suprema.

Sketch of the proof. Here, we provide a sketch of the proof ideas of (i) and (ii), a complete proof is delegated to Appendix B. For (i), let $h = \hat{h}$ and $\mathcal{M}$ be closed under concentration within $\mathcal{I}_h$. Since $\sup_{F_X\in\mathcal{M}} \rho_{\hat{h}}(X) \leqslant \sup_{F_X\in\mathcal{M}} \rho_{h^*}(X)$, it suffices to show that for all $F_Y \in \mathcal{M}$, there exists a $F_Z \in \mathcal{M}$, such that $\rho_{\hat{h}}(Z) \geqslant \rho_{h^*}(Y)$. Take $Z_{\mathcal{I}_h} \sim G = F_Y^{\mathcal{I}_h} \in \mathcal{M}$ and write $g(t) = 1 - \hat{h}(1-t)$ and $g_*(t) = 1 - h^*(1-t)$ for $t \in [0,1]$. Next, we show that on each interval $(a,b) \in \mathcal{I}_h$ where g_* is linear it holds that

$$\int_{(a,b)} F_Y^{-1}(t)\,\mathrm{d}g_*(t) = (g_*(b) - g_*(a))\frac{\int_a^b F_Y^{-1}(t)\,\mathrm{d}t}{b-a} = (g(b)-g(a))\frac{\int_a^b F_Y^{-1}(t)\,\mathrm{d}t}{b-a}$$
$$= \int_{(a,b]} G^{-1}(t)\,\mathrm{d}g(t) + G^{-1+}(a)(g(a^+) - g(a)).$$

Thus, we have $\rho_{\hat{h}}(Z_{\mathcal{I}_h}) = \rho_{h^*}(Y)$ and (i) follows.

For (ii), we first prove the case where $\mathcal{I}_h$ is finite and h has finitely many discontinuity points. In this case, $\sup_{F_X\in\mathcal{M}} \rho_{\hat{h}}(X) = \sup_{F_X\in\mathcal{M}} \rho_{h^*}(X)$ holds and it remains to prove that

$$\sup_{F_X\in\mathcal{M}} \rho_h(X) = \sup_{F_X\in\mathcal{M}} \rho_{\hat{h}}(X). \tag{10}$$

We define the following sets

$$\hat{J} = \{t \in J_h : \hat{h}(t) \neq h(t)\}, \quad \hat{J}_+ = \{t \in \hat{J} : \hat{h}(t) = \hat{h}(t^+)\}, \quad \text{and} \quad \hat{J}_- = \hat{J} \setminus \hat{J}_+.$$

For $n > 0$, write intervals

$$A_s^n = \begin{cases} (1-s-1/\sqrt{n}, 1-s+1/n), & s \in \hat{J}_-, \\ (1-s-1/n, 1-s+1/\sqrt{n}), & s \in \hat{J}_+. \end{cases}$$

Let $\mathcal{I}^n = \{A_s^n : s \in \hat{J}\}$. By taking into account all discontinuity points (finitely many) and by dominated convergence theorem, we show that,

$$\sup_{F_X\in\mathcal{M}} \rho_h(X) \geqslant \lim_{n\to\infty} \rho_h(Z_{\mathcal{I}^n}) = \rho_{\hat{h}}(Y) \quad \text{for all} \quad F_Y \in \mathcal{M}\,,$$

where

$$Z_{\mathcal{I}^n} = F_Y^{-1}(U)\mathbb{1}_{\{U\notin \bigcup_{s\in\hat{J}} A_s^n\}} + \sum_{s\in\hat{J}} \mathbb{E}[F_Y^{-1}(U)|U\in A_s^n]\mathbb{1}_{\{U\in A_s^n\}}.$$

Thus Equation (10) holds.

The cases where $\mathcal{I}_h$ is countable or h has countably many discontinuity points follows by approximating the original distortion function by its finite version and taking a limit. $\square$

Both suprema in (9) may be infinite, and this is discussed in Remark 6 in Appendix A.2. The proof of Theorem 1 is more technical than similar results in the literature because of the challenges arising from non-monotonicity, non-positivity, and discontinuity of h; see Figure 1 for a sample of possible complications. In (ii), h does not need to be upper semicontinuous on $(0,1)$ for (9) to hold because closedness under concentration for all intervals in (ii) is stronger than the condition in (i).

Remark 1. For $\mathcal{M} \subset \mathcal{M}_1$ and $h \in \mathcal{H}$, if $h = \hat{h}$ and $F^C \in \mathcal{M}$ for all $F \in \mathcal{M}$ and $C \in \mathcal{I}_h$, then the equivalence relation (9) also holds. If $\mathcal{I}_h$ is finite, then this condition is generally stronger than closedness under concentration within $\mathcal{I}_h$ in (i).

With Theorem 1, we can convert an optimization problem of a non-convex distortion riskmetric to a convex optimisation, whose objective is a distortion riskmetric where the distortion function is the concave envelope of the distortion function of the non-convex riskmetric. As a result, the new problem has a convex objective. In many practical examples, the worst-case distortion riskmetric depends on the parameters of the ambiguity sets, and thus changes in the parameters will affect the quality of the decisions derived; see Sections 5 and 7 for further discussion.

Although closedness under concentration is generally weaker than closedness under conditional expectation, verifying closedness under concentration is usually as difficult as closedness under conditional expectation. Therefore, for the sufficiency of our equivalence (9), checking closedness under conditional expectation is more convenient for most practical situations. However, the necessity of the equivalence result relies on the property of closedness under concentration for the set of optimizers; see Theorem 2 below. Moreover, compared with closedness under conditional expectation, closedness under concentration is useful in some special but realistic problems; see Section 3.4 for examples.

A natural question from Theorem 1 is whether our key condition of closedness under concentration is necessary in some sense for the equivalence (9) to hold.[3] It is immediate to notice that adding any distributions F_Z satisfying $\rho_{h^*}(Z) < \sup_{F_Y \in \mathcal{M}} \rho_{h^*}(Y)$ to the set $\mathcal{M}$ does not affect the equivalence, and therefore we turn our attention to the set of maximizers instead of the whole set $\mathcal{M}$. In the next result, we show that closedness under concentration within $\mathcal{I}_h$ of the set of maximizers of (3) is necessary for the equivalence (9) to hold.

Theorem 2. *For $\mathcal{M} \subset \mathcal{M}_1$ and $h \in \mathcal{H}$ such that $h \neq h^*$, suppose that the set of maximizers $\mathcal{M}_{\text{opt}} = \arg\max_{F_Y \in \mathcal{M}} \rho_h(Y)$ is non-empty. If the equivalence (9) holds, i.e., $\sup_{F_Y \in \mathcal{M}} \rho_h(Y) = \sup_{F_Y \in \mathcal{M}} \rho_{h^*}(Y)$, then $\mathcal{M}_{\text{opt}}$ is closed under concentration within $\mathcal{I}_h$.*

If the equivalence (9) holds, then each $F \in \mathcal{M}_{\text{opt}}$ also maximizes the problem $\sup_{F_Y \in \mathcal{M}} \rho_{h^*}(Y)$. Conversely, if $h = \hat{h}$, then this condition and closedness of $\mathcal{M}_{\text{opt}}$ under concentration within $\mathcal{I}_h$ together are necessary (by Theorem 2) and sufficient (by Theorem 1) for the equivalence (9) to hold. If the maximizer F of the original problem (3) is unique, then by Theorem 2, F must be equal to $F^{\mathcal{I}_h}$. The equivalence (9) does not imply closedness under concentration within $\mathcal{I}_h$ of the uncertainty set $\mathcal{M}$ itself; an example showing this is discussed in Remark 2.

[3]We thank an anonymous referee for raising this question.

3.2 Some examples of distortion riskmetrics

We provide a few examples of distortion riskmetrics ρ_h commonly used in decision theory and finance, and obtain their corresponding set $\mathcal{I}_h$. The Value-at-Risk (VaR) and the Expected Shortfall (ES) are the most popular risk measures in practice. We introduce them first, followed by an inverse-S-shaped distortion function of Tversky and Kahneman (1992).

Example 1 (VaR and ES)**.** For $Y \in \mathcal{L}^0$, using the sign convention of McNeil et al. (2015), VaR is defined as the left-quantile, and upper VaR (VaR^+) is defined as the right-quantile; that is,

$$\mathrm{VaR}_\alpha(Y) = F_Y^{-1}(\alpha), \quad \alpha \in (0,1] \quad \text{and} \quad \mathrm{VaR}_\alpha^+(Y) = F_Y^{-1+}(\alpha), \quad \alpha \in [0,1).$$

ES at level α is defined as

$$\mathrm{ES}_\alpha(Y) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_t(Y)\,\mathrm{d}t, \quad \alpha \in (0,1),\ Y \in \mathcal{L}^1.$$

Both VaR_α and ES_α belong to the class of distortion riskmetrics. Take $\alpha \in (0,1)$. Let $h(t) = \mathbb{1}_{(1-\alpha,1]}(t)$, $t \in [0,1]$. It follows that $h \in \mathcal{H}$ and $\hat{h}(t) = \mathbb{1}_{[1-\alpha,1]}(t)$, $t \in [0,1]$. In this case, $\rho_h = \mathrm{VaR}_\alpha$. Moreover, $h^*(t) = \frac{t}{1-\alpha} \wedge 1$, $t \in [0,1]$ and $\rho_{h^*} = \mathrm{ES}_\alpha$. Since h^* and $\hat{h}$ differ on $(0, 1-\alpha)$, we have $\mathcal{I}_h = \{(\alpha, 1)\}$.

Example 2 (TK distortion riskmetrics)**.** The following function h is an inverse-S-shaped distortion function (see also Figure 4):

$$h(t) = \frac{t^\gamma}{(t^\gamma + (1-t)^\gamma)^{1/\gamma}}, \quad t \in [0,1],\ \gamma \in (0,1). \tag{11}$$

Distortion riskmetrics with distortion function (11) are commonly used in behavioural economics and finance; see e.g., Tversky and Kahneman (1992). For simplicity, we call such distortion riskmetrics *TK distortion riskmetrics*. Typical values of γ are in $[0.5, 0.9]$; see Wu and Gonzalez (1996). For h in (11), it is clear that $h = \hat{h}$ on $[0,1]$ by continuity of h. We have $h^* \neq h$ on $(t_0, 1)$, for some $t_0 \in (0,1)$, and h^* is linear on $[t_0, 1]$. Thus, $\mathcal{I}_h = \{(0, 1-t_0)\}$. An example of h in (11) and its concave envelope h^* are plotted in Figure 3 (left).

For $h_1, h_2 \in \mathcal{H}$, we write $h = h_1 - h_2 \in \mathcal{H}$ and consider the difference between two distortion riskmetrics, that is

$$\rho_h = \rho_{h_1} - \rho_{h_2}. \tag{12}$$

Such type of distortion riskmetrics measure the difference or disagreement between two utilities, risk attitudes, or capital requirements. Determining the upper and lower bounds, or the largest absolute values of such measures of disagreement, is of interest in practice but rarely studied in the literature; see e.g., David (1998) for a history and the use of the inter-quantile range in statistics, dating back to the work of C. F. Gauss. Note that $h_1 - h_2$ is in general not monotone or concave even when h_1 and h_2 themselves have the specified properties. Below we show some examples of distortion riskmetrics taking the form of (12).

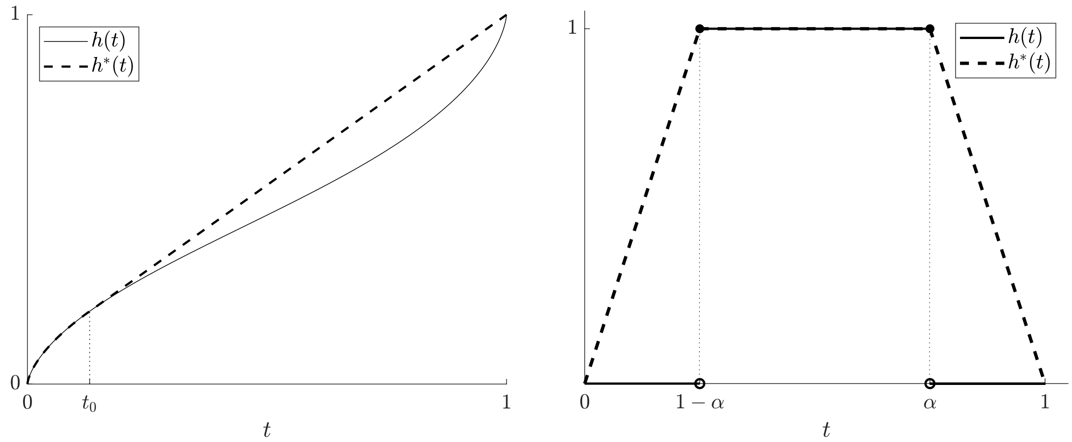

Figure 3: Left panel: h and h^* for the TK distortion riskmetric with $\gamma = 0.7$ in Example 2; right panel: h and h^* for the inter-quantile range in Example 3

Example 3 (Inter-quantile range and inter-ES range)**.** For $\alpha \in [1/2, 1)$, we take $h_1(t) = \mathbb{1}_{[1-\alpha,1]}(t)$ and $h_2(t) = \mathbb{1}_{(\alpha,1]}(t)$, $t \in [0,1]$. It follows that $h(t) = h_1(t) - h_2(t) = \mathbb{1}_{\{1-\alpha \leqslant t \leqslant \alpha\}}$, $t \in [0,1]$, $\hat{h} = h$, and

$$\rho_h(X) = F_X^{-1+}(\alpha) - F_X^{-1}(1-\alpha), \quad X \in \mathcal{L}^0.$$

Correspondingly, we have $h^*(t) = t/(1-\alpha) \wedge 1 + (\alpha - t)/(1-\alpha) \wedge 0$, $t \in [0,1]$, and

$$\rho_{h^*}(X) = \mathrm{ES}_\alpha(X) + \mathrm{ES}_\alpha(-X), \quad X \in \mathcal{L}^1.$$

This distortion riskmetric ρ_h is called an inter-quantile range and ρ_{h^*} is called an inter-ES range. As the distortion functions h^* and $\hat{h}$ differ on the open intervals $(0, 1-\alpha)$ and $(\alpha, 1)$, we have $\mathcal{I}_h = \{(\alpha, 1), (0, 1-\alpha)\}$. The distortion functions h and h^* are displayed in Figure 3 (right).

Example 4 (Difference of two inverse-S-shaped distortion functions)**.** We take h_1 and h_2 to be the inverse-S-shaped distortion functions in (11), with parameters $\gamma_1 = 0.8$ and $\gamma_2 = 0.7$, respectively. By calculation, the function $h = h_1 - h_2$ is convex on $[0, 0.3770]$, concave on $[0.3770, 1]$, and as seen in Figure 4 not monotone. The concave envelope h^* is linear on $[0, 0.7578]$ and $h^* = h$ on $[0.7578, 1]$. Thus, we have $\mathcal{I}_h = \{(0.2422, 1)\}$. The graphs of the distortion functions h_1, h_2, h, and h^* are displayed in Figure 4.

The functions in $\mathcal{H}$ are a.e. differentiable, and for an absolutely continuous function $h \in \mathcal{H}$, let h' be a (representative) function on $[0,1]$ that is a.e. equal to the derivative of h. If $h \in \mathcal{H}$ is left-continuous or $\mathrm{VaR}_t(Y)$ is continuous with respect to $t \in (0,1)$, the risk measure ρ_h in (4) has representation

$$\rho_h(Y) = \int_0^1 \mathrm{VaR}_{1-t}(Y)\,\mathrm{d}h(t), \quad Y \in \mathcal{L}^p; \tag{13}$$

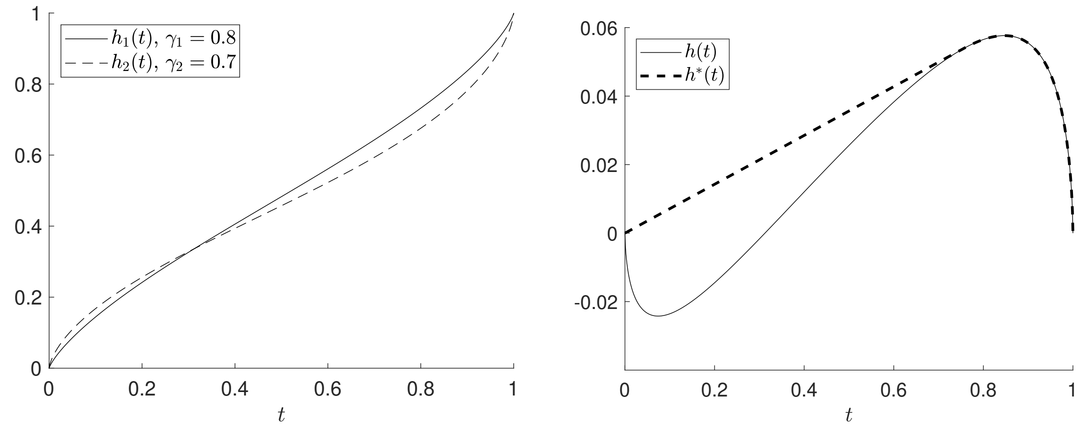

Figure 4: Left panel: inverse-S-shaped distortion functions h_1 and h_2 in Example 4; right panel: $h = h_1 - h_2$ and h^* of the same example

see Lemma 1 of Wang et al. (2020a). If $h \in \mathcal{H}$ is absolutely continuous it holds

$$\rho_h(Y) = \int_0^1 \mathrm{VaR}_{1-t}(Y) h'(t)\,\mathrm{d}t, \quad Y \in \mathcal{L}^p. \tag{14}$$

Another example of a recently introduced distortion riskmetric with concave distortion function may be of independent interest in risk management.

Example 5 (Second-order superquantile)**.** As introduced by Rockafellar and Royset (2018), a second-order superquantile is defined as

$$\mathrm{SSQ}_\alpha(Y) = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{ES}_t(Y)\,\mathrm{d}t, \quad \alpha \in (0,1),\ Y \in \mathcal{L}^2.$$

By Theorem 2.4 of Rockafellar and Royset (2018), SSQ_α is a distortion riskmetric with a concave distortion function h given by

$$h(t) = \begin{cases} \frac{t}{1-\alpha}\left(1 + \log\frac{1-\alpha}{t}\right), & 0 \leqslant t < 1-\alpha, \\ 1, & 1-\alpha \leqslant t \leqslant 1. \end{cases}$$

Clearly, $\mathrm{SSQ}_\alpha \geqslant \mathrm{ES}_\alpha$. The difference $\mathrm{SSQ}_\alpha - \mathrm{ES}_\alpha$ between second-order superquantile and ES, which has a similar interpretation as $\mathrm{ES}_\alpha - \mathrm{VaR}_\alpha$, is a distortion riskmetric with a non-concave and non-monotone distortion function g, and the set $\mathcal{I}_g$ contains a single interval of the form $(0, \beta)$ for some $\beta \in [\alpha, 1)$.

3.3 Closedness under concentration for all intervals

In this section, we present some technical results and specific examples about closedness under concentration for all intervals and under conditional expectation. The proposition below clarifies the relationship between closedness under concentration for all intervals and closedness under conditional expectation.

Proposition 2. *Closedness under conditional expectation implies closedness under concentration for all intervals, but the converse is not true.*

Example 6. To provide insights into the difference between the two properties, we show that the following set is closed under concentration for all intervals but that it is not closed under conditional expectation:

$$\mathcal{M} = \{F \in \mathcal{M}_1 : F \text{ is finitely supported in } [0,1]\}.$$

To see that $\mathcal{M}$ is not closed under conditional expectation, let $X = \mathbb{1}_{\{U>V\}}$ where $U, V \sim \mathrm{U}[0,1]$ are independent. Then $\mathbb{E}[X|U] = \mathbb{E}[\mathbb{1}_{\{U>V\}}|U] = U$. As the distribution of X is in $\mathcal{M}$ but the distribution of U is not in $\mathcal{M}$, the set is not closed under conditional expectation.

Example 6 suggests that the difference between closedness under concentration for all intervals and closedness under conditional expectation is subtle but explicit. Generally speaking, we can construct a set closed under concentration for all intervals by taking a dense but discrete subset of a set that is closed under conditional expectation. It also indicates that the concept of closedness under concentration naturally arises in discrete setups.

Example 7. We present 6 classes of sets $\mathcal{M}$ that are closed under conditional expectation, and hence also under concentration for all intervals.

1. (Moment conditions) For $p > 1$, $m \in \mathbb{R}$, and $v > 0$, the set

   $$\mathcal{M}(p, m, v) = \{F_Y \in \mathcal{M}_p : \mathbb{E}[Y] = m,\ \mathbb{E}[|Y - m|^p] \leqslant v^p\}$$

   is closed under conditional expectation by Jensen's inequality. The set $\mathcal{M}(p, m, v)$ corresponds to distributional uncertainty with moment information, and the setting $p = 2$ (mean and variance constraints) is the most commonly studied.

2. (Mean-covariance conditions) For $n \in \mathbb{N}$, $\mathbf{a} \in \mathbb{R}^n$, $\boldsymbol{\mu} \in \mathbb{R}^n$, and $\Sigma \in \mathbb{R}^{n\times n}$ positive semidefinite, let

   $$\mathcal{M}^{\mathrm{mv}}(\mathbf{a}, \boldsymbol{\mu}, \Sigma) = \{F_{\mathbf{a}^\top \mathbf{X}} \in \mathcal{M}_2 : F_{\mathbf{X}} \in \mathcal{M}_2^n,\ \mathbb{E}[\mathbf{X}] = \boldsymbol{\mu},\ \mathrm{var}(\mathbf{X}) \preceq \Sigma\},$$

   where $\mathbf{X} = (X_1, \dots, X_n)$, $\mathbb{E}[\mathbf{X}] = (\mathbb{E}[X_1], \dots, \mathbb{E}[X_n])$, $\mathrm{var}(\mathbf{X})$ is the covariance matrix of $\mathbf{X}$, and $B' \preceq B$ means that the matrix $B - B'$ is positive semidefinite for two positive semidefinite symmetric matrices B and B'. With a simple verification in Appendix A.1, $\mathcal{M}^{\mathrm{mv}}(\mathbf{a}, \boldsymbol{\mu}, \Sigma) = \mathcal{M}(2, \mathbf{a}^\top \boldsymbol{\mu}, (\mathbf{a}^\top \Sigma \mathbf{a})^{1/2})$.

3. (Convex function conditions) For $n \in \mathbb{N}$, $\mathbf{a} \in \mathbb{R}^n$, $K \subset \mathbb{N}$, a collection $\mathbf{f} = (f_k)_{k\in K}$ of convex functions on $\mathbb{R}^n$, and a vector $\mathbf{x} = (x_k)_{k\in K} \in \mathbb{R}^{|K|}$, let

   $$\mathcal{M}^{\mathbf{f}}(\mathbf{a}, \mathbf{x}) = \{F_{\mathbf{a}^\top \mathbf{X}} \in \mathcal{M}_1 : \mathbb{E}[f_k(\mathbf{X})] \leqslant x_k \text{ for all } k \in K\}.$$

   The set $\mathcal{M}^{\mathbf{f}}$ corresponds to distributional uncertainty with constraints on expected losses or test functions. The set $\mathcal{M}^{\mathbf{f}}$ includes $\mathcal{M}(p, m, v)$ as a special case. We can verify that $\mathcal{M}^{\mathbf{f}}$ is closed under conditional expectation by Jensen's inequality.

4. (Convex order conditions) For $K \subset \mathbb{N}$ and a collection of random variables $\mathbf{Z} = (Z_k)_{k \in K} \in (\mathcal{L}^1)^{|K|}$, let
$$\mathcal{M}^{\mathrm{cx}}(\mathbf{Z}) = \{F_Y \in \mathcal{M}_1 : Y \leqslant_{\mathrm{cx}} Z_k \text{ for all } k \in K\},$$
where $\leqslant_{\mathrm{cx}}$ is the inequality in convex order.[4] Similar to the above examples, $\mathcal{M}^{\mathrm{cx}}(\mathbf{Z})$ is closed under conditional expectation (cf. Remark 7 in Appendix A.2).

5. (Distortion conditions) For $K \subset \mathbb{N}$, a collection $\mathbf{h} = (h_k)_{k \in K} \in (\mathcal{H}^*)^{|K|}$ and a vector $\mathbf{x} = (x_k)_{k \in K} \in \mathbb{R}^{|K|}$, let
$$\mathcal{M}^{\mathbf{h}}(\mathbf{x}) = \{F_Y \in \mathcal{M}_1 : \rho_{h_k}(Y) \leqslant x_k \text{ for all } k \in K\}.$$
The set $\mathcal{M}^{\mathbf{h}}$ corresponds to distributional uncertainty with constraints on preferences modeled by convex dual utilities. It is closed under conditional expectation by noting that ρ_{h_k} preserves convex order (see e.g., Theorem 2 of Wang et al. (2020b)).

6. (Marginal conditions) For given univariate distributions $F_1, \dots, F_n \in \mathcal{M}_1$, let
$$\mathcal{M}^S(F_1, \dots, F_n) = \{F_{X_1 + \cdots + X_n} \in \mathcal{M}_1 : X_i \sim F_i,\ i = 1, \dots, n\}.$$
In other words, $\mathcal{M}^S$ is the set of all possible aggregate risks $X_1 + \cdots + X_n$ with given marginal distributions of $X_1, \dots, X_n$; see Embrechts et al. (2015) for some results on $\mathcal{M}^S$. Generally, $\mathcal{M}^S$ is not closed under concentration for all intervals or conditional expectation, since closedness under concentration for all intervals is stronger than joint mixability (Wang and Wang, 2016). In the special case where $F_1 = \cdots = F_n = \mathrm{U}[0,1]$, Proposition 1 and Theorem 5 of Mao et al. (2019) imply that $\mathcal{M}^S$ is closed under conditional expectation if and only if $n \geqslant 3$.

Remark 2. The uncertainty set $\mathcal{M}(p, m, v)$ of the moment condition in Example 7 can be restricted to the set
$$\overline{\mathcal{M}}(p, m, v) = \{F_Y \in \mathcal{M}_p : \mathbb{E}[Y] = m,\ \mathbb{E}[|Y - m|^p] = v^p\},$$
which is the "boundary" of $\mathcal{M}(p, m, v)$. For $\mathcal{M} = \mathcal{M}(p, m, v)$, the suprema on both sides of (9) are obtained by some distributions in $\overline{\mathcal{M}}(p, m, v)$; see Theorem 5. As a direct consequence, we get
$$\sup_{F_Y \in \overline{\mathcal{M}}(p,m,v)} \rho_{h^*}(Y) = \sup_{F_Y \in \mathcal{M}(p,m,v)} \rho_{h^*}(Y) = \sup_{F_Y \in \mathcal{M}(p,m,v)} \rho_h(Y) = \sup_{F_Y \in \overline{\mathcal{M}}(p,m,v)} \rho_h(Y).$$
Hence, equivalence holds even though $\overline{\mathcal{M}}(p, m, v)$ is not closed under concentration for any interval. By Theorem 2, the set of optimizers is closed under concentration within $\mathcal{I}_h$ for each $h \in \mathcal{H}$.

For a distribution $F \in \mathcal{M}_1$ and a collection $\mathcal{I}$ of disjoint intervals in $[0,1]$, we have the following result regarding to the distribution $F^{\mathcal{I}}$.

[4] Precisely, we write $G \leqslant_{\mathrm{cx}} (\leqslant_{\mathrm{icx}}) F$ if $\int \phi \, \mathrm{d}G \leqslant \int \phi \, \mathrm{d}F$ for all (increasing) convex functions ϕ such that the above two integrals are well defined.

Proposition 3. *Let $\mathcal{I}$ be a collection of disjoint intervals in $[0,1]$ and $\mathcal{M}$ be a set of distributions. If $\mathcal{M}$ is closed under concentration for all intervals and $\mathcal{I}$ is finite, or $\mathcal{M}$ is closed under conditional expectation, then $\mathcal{M}$ is closed under concentration within $\mathcal{I}$.*

If $\mathcal{I}$ is infinite, closedness under concentration for all intervals may not be sufficient for closedness under concentration within $\mathcal{I}$; see Remark 8 in Appendix A.2 for a technical explanation. An infinite $\mathcal{I}_h$ does not appear for any distortion riskmetrics in practice.

3.4 Examples of closedness under concentration within $\mathcal{I}$ but not for all intervals

In practice, it is more tractable to check closedness under concentration within a specific collection of intervals $\mathcal{I}$ than closedness under concentration for all intervals or under conditional expectation. In this section, we show several examples for closedness under concentration within some $\mathcal{I}$.

For distortion functions h such that $\mathcal{I}_h = \{(p,1)\}$ (resp. $\mathcal{I}_h = \{(0,p)\}$) for some $p \in (0,1)$, the result in Theorem 1 (i) only requires $\mathcal{M}$ to be closed under concentration within $\{(p,1)\}$ (resp. $\{(0,p)\}$). Such distortion functions include the inverse-S-shaped distortion functions in (11), those of VaR_p, and VaR_p^+, and that of the difference between the second-order superquantile and ES in Example 5. Below we present some more concrete examples.

Example 8 ($\mathcal{M}$ has two elements)**.** Let $p \in (0,1)$ and $\mathcal{M} = \{\mathrm{U}[0,1],\, p\delta_{p/2} + (1-p)\mathrm{U}[p,1]\}$ where $\delta_{p/2}$ is the point-mass at $p/2$. We can check that $\mathcal{M}$ is closed under concentration within $\{(0,p)\}$ but $\mathcal{M}$ is not closed under concentration for all intervals. Indeed, any set closed under concentration for all intervals and containing $\mathrm{U}[0,1]$ has infinitely many elements. In general, a finite set which contains any non-degenerate distribution is not closed under conditional expectation in an atomless probability space, since there are infinitely many possible distributions for the conditional expectation of a given non-constant random variable. Another similar example that is closed under concentration within $\{(0,p)\}$ is the set of all possible distributions of the sum of several Pareto risks; see Example 5.1 of Wang et al. (2019).

Example 9 (VaR and ES)**.** As we see from Example 1, if $\rho_h = \mathrm{VaR}_\alpha^+$ for some $\alpha \in (0,1)$, then ρ_{h^*} is ES_α and $\mathcal{I}_h = \{(\alpha,1)\}$. Theorem 1 (i) implies that if $\mathcal{M}$ is closed under concentration within $\{(\alpha,1)\}$, then

$$\sup_{F_Y \in \mathcal{M}} \mathrm{VaR}_\alpha^+(Y) = \sup_{F_Y \in \mathcal{M}} \mathrm{ES}_\alpha(Y).$$

This observation leads to (with some modifications) the main results in Wang et al. (2015) and Li et al. (2018) on the equivalence between VaR and ES.

Example 10 (TK distortion riskmetric)**.** If we take h to be an inverse-S-shaped distortion function in (11), then $\mathcal{I}_h = \{(0, 1-t_0)\}$ for some $t_0 \in (0,1)$, and ρ_h is the TK distortion riskmetric. As a direct consequence of Theorem 1 (i), if $\mathcal{M}$ is closed under concentration within $\{(0,1-t_0)\}$, then

$$\sup_{F_Y \in \mathcal{M}} \rho_h(Y) = \sup_{F_Y \in \mathcal{M}} \rho_{h^*}(Y).$$

This result implies Theorem 4.11 of Wang et al. (2019) on the robust risk aggregation problem based on dual utilities with inverse-S-shaped distortion functions.

Example 11 (Wasserstein ball, 1-dimensional)**.** Optimization problems under the uncertainty set of a Wasserstein ball are common in literature when quantifying the discrepancy between a benchmark distribution and alternative scenarios; see e.g., Blanchet and Murthy (2019) and Gao and Kleywegt (2023). We discuss the application of the concept of concentration to optimization with Wasserstein distances. For $p \geqslant 1$ and $F, G \in \mathcal{M}_p$, the p-*Wasserstein distance* between F and G is defined as

$$W_p(F,G) = \left(\int_0^1 \left|F^{-1}(u) - G^{-1}(u)\right|^p \,\mathrm{d}u\right)^{1/p}.$$

For $\varepsilon \geqslant 0$, the uncertainty set of an ε-*Wasserstein ball* around a benchmark distribution $\widetilde{G} \in \mathcal{M}_p$ is given by

$$\mathcal{M}(\widetilde{G},\varepsilon) = \{F \in \mathcal{M}_p : W_p(F,\widetilde{G}) \leqslant \varepsilon\}.$$

Suppose that the benchmark distribution $\widetilde{G}$ has a quantile function that is constant on each element in some collection of disjoint intervals $\widetilde{\mathcal{I}} \subset [0,1]$. As shown in Appendix A.1, $\mathcal{M}(\widetilde{G},\varepsilon)$ is closed under concentration within $\mathcal{I}$ for all $\mathcal{I} \subset \widetilde{\mathcal{I}}$. Using this closedness property and Theorem 1 (i), the equivalence

$$\sup_{F_Y \in \mathcal{M}(\widetilde{G},\varepsilon)} \rho_h(Y) = \sup_{F_Y \in \mathcal{M}(\widetilde{G},\varepsilon)} \rho_{h^*}(Y) \tag{15}$$

holds for all $h \in \mathcal{H}$ such that $\mathcal{I}_h \subset \widetilde{\mathcal{I}}$.

Remark 3. In general, if the quantile function $\widetilde{G}$ in Example 11 is not constant on some interval in $\widetilde{\mathcal{I}}$, then $\mathcal{M}(\widetilde{G},\varepsilon)$ is not closed under concentration within $\widetilde{\mathcal{I}}$ (see Appendix A.1 for a proof). Thus, although it is stringent, the conditions imposed on $\widetilde{G}$ is also necessary for $\mathcal{M}(\widetilde{G},\varepsilon)$ to be closed under concentration within $\widetilde{\mathcal{I}}$. For instance, the worst-case VaR_α over $\mathcal{M}(\widetilde{G},\varepsilon)$ is generally different from the worst-case ES_α over $\mathcal{M}(\widetilde{G},\varepsilon)$ as obtained in Proposition 4 of Liu et al. (2022). We also refer to Bernard et al. (2020) who consider a Wasserstein ball together with moment constraints.

Example 12 (Wasserstein ball, n-dimensional)**.** For $n \in \mathbb{N}$, $p \geqslant 1$, $a \geqslant 1$ and $F, G \in \mathcal{M}_p^n$, the p-Wasserstein distance on $\mathbb{R}^n$ between F and G is defined as

$$W_{a,p}^n(F,G) = \inf_{\mathbf{X}\sim F,\ \mathbf{Y}\sim G} (\mathbb{E}[\|\mathbf{X}-\mathbf{Y}\|_a^p])^{1/p},$$

where $\|\cdot\|_a$ is the $\mathcal{L}^a$-norm on $\mathbb{R}^n$. Similarly to the 1-dimensional case, for $\varepsilon \geqslant 0$, an ε-Wasserstein ball on $\mathbb{R}^n$ around a benchmark distribution $\widetilde{G} \in \mathcal{M}_p^n$ is defined as

$$\mathcal{M}^n(\widetilde{G},\varepsilon) = \{F \in \mathcal{M}_p^n : W_{a,p}^n(F,\widetilde{G}) \leqslant \varepsilon\}.$$

In a portfolio selection problem, we consider the worst-case riskmetric of a linear combination of random losses. Suppose that there exists a random vector $\mathbf{Z} \sim \widetilde{G}$ such that $\mathbf{Z} \geqslant \mathbf{0}$ and $\mathbb{P}(\mathbf{Z} = \mathbf{0}) = p_0$

for some $p_0 \in (0,1]$. For $\varepsilon \geqslant 0$, $\mathbf{w} \in [0,\infty)^n$, $p > 1$, $a > 1$ and $\mathbf{Z} \in (\mathcal{L}^p)^n$, as shown in Appendix A.1, the uncertainty set

$$\{F_{\mathbf{w}^\top \mathbf{X}} \in \mathcal{M}_p : F_{\mathbf{X}} \in \mathcal{M}^n(F_{\mathbf{Z}}, \varepsilon)\}$$

is closed under concentration within $\{(0,t)\}$ for all $t \leqslant p_0$. For a practical example, assume that an investor holds a portfolio of bonds (for simplicity, assume that they have the same maturity). The loss vector $\mathbf{X} \geqslant \mathbf{0}$ from this portfolio at maturity has an estimated benchmark loss distribution $\widetilde{G}$, and the probability of no default from these bonds (i.e., $\mathbf{X} = \mathbf{0}$) is estimated as $p_0 > 0$ (usually quite large). Suppose that the investor uses a distortion riskmetric with an inverse-S-shaped distortion function h given in (11) of Example 2, and considers a Wasserstein ball around $\widetilde{G}$ with radius ε. Note that $\mathcal{I}_h = \{(0,t)\}$ for some $t \in (0,1)$ from Example 10. By Theorem 1 (i), we obtain an equivalence result on the worst-case riskmetrics for the portfolio with weight vector $\mathbf{w}$,

$$\sup_{F_{\mathbf{X}} \in \mathcal{M}^n(\widetilde{G},\varepsilon)} \rho_h(\mathbf{w}^\top \mathbf{X}) = \sup_{F_{\mathbf{X}} \in \mathcal{M}^n(\widetilde{G},\varepsilon)} \rho_{h^*}(\mathbf{w}^\top \mathbf{X}),$$

whenever $t \in (0, p_0]$.

Remark 4. By definition, the operation of concentration usually changes the support of the original distribution. For example, for a distribution F_X, standard uniform random variable U, and an interval $C = (a,b) \subseteq (0,1)$, suppose that $F_X \neq F_X^C$. The density function of F_X^C becomes infinite at $\mathbb{E}[X|U \in C]$ and 0 everywhere else on $[F_X^{-1}(a), F_X^{-1}(b)]$. Thus the supports of F_X and F_X^C are different. This feature makes closedness under concentration unsuitable for ambiguity sets induced by ϕ-divergence (Ben-Tal et al. (2013)), which requires the support of the distributions in comparison to be the same.

Example 13 (Optimal hedging strategy)**.** Suppose that an investor is willing to hedge her random loss X only when it exceeds some certain level $l \in \mathbb{R}$. Mathematically, for a fixed $X \in \mathcal{L}^1$ continuously distributed on $(F_X^{-1}(p_0), F_X^{-1}(1))$ such that $\mathbb{P}(X \leqslant l) = p_0$ for some $p_0 \in (0,1)$ and $l \in \mathbb{R}$, define the set of measurable functions

$$\mathcal{V} = \{V : \mathbb{R} \to \mathbb{R} \mid x \mapsto x - V(x) \text{ is increasing, } V(x) = 0 \text{ for all } x \leqslant l\}$$

representing possible hedging strategies. Let $g : \mathbb{R} \to \mathbb{R}$ be an increasing and convex function. The final payoff obtained by a hedging strategy $V \in \mathcal{V}$ is given by $X - V(X) + g(\mathbb{E}[V(X)])$, where $g(\mathbb{E}[V(X)])$ is a fixed cost of the hedging strategy that depends on the expected value of $V(X)$ calculated by a risk-neutral seller in the market using the same probability measure $\mathbb{P}$. As shown in Appendix A.1, the action set in this optimization problem,

$$\mathcal{M} = \{F_{X - V(X) + g(\mathbb{E}[V(X)])} \in \mathcal{M}_1 : V \in \mathcal{V}\},$$

is closed under concentration within $\{(p,1)\}$ for all $p \in [p_0, 1)$. On the other hand, it is obvious that $\mathcal{M}$ is not closed under concentration for all intervals or closed under conditional expectation since the quantiles of the distributions in $\mathcal{M}$ are fixed beyond the interval $(p_0, 1)$. The above closedness under concentration property allows us to use Theorem 1 to convert the optimal hedging problem for ρ_h with an inverse-S-shaped distortion function h as in (11) to a convex version ρ_{h^*}.

Example 14 (Risk choice)**.** Suppose that an investor is faced with a random loss $X \in \mathcal{L}^1$. The distortion function h of her riskmetric is inverse-S-shaped with $\mathcal{I}_{-h} = \{(p,1)\}$ for some $p \in (0,1)$. Suppose that p is known to the seller. Since the investor is averse to risk for large losses, the seller may provide her with the option to stick to the initial investment or to convert the upper part of the random loss into a fixed payment to avoid large loss. Specifically, we consider the set $\mathcal{M} = \{F_X, F_X^{(p,1)}\}$ containing two elements, where $\mathbb{P}(X \leqslant u) = p$ for some $u \in \mathbb{R}$. It is clear that $\mathcal{M}$ is closed under concentration within $\{(p,1)\}$ but not closed under conditional expectation. We assume that the costs of the two investment strategies are calculated by expectation and thus are the same. By (i) of Theorem 1, it follows that the risk minimization problem satisfies

$$\min_{F_Y \in \mathcal{M}} \rho_h(Y) = \min_{F_Y \in \mathcal{M}} \rho_{h_*}(Y) = \rho_{h_*}(X),$$

where the last equality follows from Theorem 3 of Wang et al. (2020a). By (iii) of Theorem 1, we further have the minimum of the original problem $\min_{F_Y \in \mathcal{M}} \rho_h(Y)$ is obtained by $F_X^{(p,1)}$; intuitively, the investor will choose to convert the upper part of her loss into a fixed payment.

3.5 Atomic probability space

The definition of closedness under concentration in Definition 1 requires the assumption of an atomless probability space since a uniform random variable is used in the setup. It may be of practical interest in some economic and optimization settings to assume a finite probability space. In this section, we let the sample space be $\Omega_n = \{\omega_1, \dots, \omega_n\}$ for $n \in \mathbb{N}$ and the probability measure $\mathbb{P}_n$ be such that $\mathbb{P}_n(\omega_i) = 1/n$ for all $i = 1, \dots, n$ (such a space is called *adequate* in economics). The possible distributions in such a probability space are supported by at most n points each with probability a multiple of $1/n$, and we denote by $\mathcal{M}_{[n]}$ the set of these distributions.

Define the collection of intervals $\mathcal{I}_n = \{(j/n, k/n] : j,k \in \mathbb{N} \cup \{0\},\ j < k \leqslant n\}$. We say a set of distributions $\mathcal{M} \subset \mathcal{M}_{[n]}$ is *closed under grid concentration* within $\mathcal{I} \subset \mathcal{I}_n$ if for all $F \in \mathcal{M}$, the distribution of the random variable

$$F^{-1}(U_n)\mathbb{1}_{\{U_n \notin \bigcup_{C \in \mathcal{I}} C\}} + \sum_{C \in \mathcal{I}} \mathbb{E}[F^{-1}(U_n) | U_n \in C] \mathbb{1}_{\{U_n \in C\}}$$

is also in $\mathcal{M}$, where U_n is a random variable such that $U_n(\omega_i) = i/n$ for all $i = 1, \dots, n$. For a distribution F with finite mean and $(a,b] \in \mathcal{I}_n$, it is straightforward that the left-quantile function of $F^{(a,b]}$ is given by (7). The following equivalence result holds with additional assumption $\mathcal{I}_h \subset \mathcal{I}_n$. The proof can be obtained directly from that of Theorem 1.

Proposition 4. *Let $\mathcal{M} \subset \mathcal{M}_{[n]}$ and $h \in \mathcal{H}$. If $h = \hat{h}$, $\mathcal{I}_h \subset \mathcal{I}_n$ and $\mathcal{M}$ is closed under grid concentration within $\mathcal{I}_h$, then*

$$\sup_{F_Y \in \mathcal{M}} \rho_h(Y) = \sup_{F_Y \in \mathcal{M}} \rho_{h^*}(Y).$$

We note that the condition $\mathcal{I}_h \subset \mathcal{I}_n$ in Proposition 4 is satisfied by all distortion functions h which are linear (or constant) on each of $((j-1)/n, j/n]$, $j = 1, \dots, n$. It is common to assume such a distortion function h in an adequate probability space of n states, since any distribution function can only take values in $\{j/n : j = 0, \dots, n\}$.

In a similar spirit to Theorem 2, we give a necessary condition of the equivalence result in the context of atomic probability spaces. Its proof follows directly from that of Theorem 2.

Proposition 5. *For $\mathcal{M} \subset \mathcal{M}_{[n]}$ and $h \in \mathcal{H}$ such that $h \neq h^*$ and $\mathcal{I}_h \subset \mathcal{I}_n$, suppose that the set of maximizers, $\mathcal{M}_{\text{opt}} = \arg\max_{F_Y \in \mathcal{M}} \rho_h(Y)$, is non-empty. If $\sup_{F_Y \in \mathcal{M}} \rho_h(Y) = \sup_{F_Y \in \mathcal{M}} \rho_{h^*}(Y)$, then $\mathcal{M}_{\text{opt}}$ is closed under grid concentration within $\mathcal{I}_h$.*

4 Multi-dimensional setting

Our main equivalence results in Theorems 1 and 2 are stated under the context of one-dimensional random variables. In this section, we discuss their generalization to multi-dimensional framework with a few additional steps.

In the multi-dimensional setting, closedness under concentration is not easy to define, as quantile functions are not naturally defined for multivariate distributions. Nevertheless, closedness under conditional expectation can be analogously formulated. For $n \in \mathbb{N}$, we say that $\mathcal{M} \subset \mathcal{M}^n$ is *closed under conditional expectation*, if for all $F_{\mathbf{X}} \in \mathcal{M}$, the distribution of any conditional expectation of $\mathbf{X}$ is in $\mathcal{M}$. The following theorem states the multi-dimensional version of our main equivalence result using closedness under conditional expectation.

Theorem 3. *For $\widetilde{\mathcal{M}} \subset \mathcal{M}_1^n$, increasing function $h \in \mathcal{H}$ and $f : A \times \mathbb{R}^n \to \mathbb{R}$ concave in the second argument, if $\widetilde{\mathcal{M}}$ is closed under conditional expectation, then for all $\mathbf{a} \in A$,*

$$\sup_{F_{\mathbf{X}} \in \widetilde{\mathcal{M}}} \rho_h(f(\mathbf{a}, \mathbf{X})) = \sup_{F_{\mathbf{X}} \in \widetilde{\mathcal{M}}} \rho_{h^*}(f(\mathbf{a}, \mathbf{X})). \tag{16}$$

If $h = \hat{h}$ and the second supremum in (16) is attained by some $F_{\mathbf{X}} \in \widetilde{\mathcal{M}}$, then $F^{\mathcal{I}_h}_{f(\mathbf{a},\mathbf{X})}$ attains both suprema. Moreover, if f is linear in the second component, then (16) holds for all $h \in \mathcal{H}$ (not necessarily monotone).

Remark 5. If we assume that f is convex (instead of concave) in the second argument in Theorem 3 and keep the other assumptions, then for an increasing h,

$$\inf_{F_{\mathbf{X}} \in \widetilde{\mathcal{M}}} \rho_h(f(\mathbf{a}, \mathbf{X})) = \inf_{F_{\mathbf{X}} \in \widetilde{\mathcal{M}}} \rho_{h_*}(f(\mathbf{a}, \mathbf{X})).$$

This statement follows by noting $\rho_{-h} = -\rho_h$. The case of a decreasing h is similar.

Theorem 3 is similar to Theorem 3.4 of Cai et al. (2020) which states the equivalence (16) for increasing h and a specific set $\widetilde{\mathcal{M}}$ which is a special case in Example 15 below. In contrast, our result applies to a more general set $\widetilde{\mathcal{M}}$ and the infimum problem. Moreover, our result applies to non-monotone h when $f(\mathbf{a},\mathbf{x})$ is linear in $\mathbf{x}$. The setting of a function f linear in the second argument often appears in portfolio selection problems where $f(\mathbf{a},\mathbf{X})=\mathbf{a}^\top\mathbf{X}$; see Example 12 and Section 6.

Example 15. Similarly to Example 7, we give examples of sets of multi-dimensional distributions closed under conditional expectation.

1. (Convex function conditions) For $n\in\mathbb{N}$, a convex set $B\subset\mathbb{R}^n$, set Ψ of convex functions on $\mathbb{R}^n$, and a mapping $\pi:\Psi\to\mathbb{R}$, let

$$\widetilde{\mathcal{M}}(B,\Psi,f)=\{F_{\mathbf{X}}\in\mathcal{M}_1^n:\mathbb{P}(\mathbf{X}\in B)=1,\ \mathbb{E}[\psi(\mathbf{X})]\leqslant\pi(\psi)\text{ for all }\psi\in\Psi\}.$$

It is clear that $\widetilde{\mathcal{M}}(B,\Psi,f)$ is closed under conditional expectation due to Jensen's inequality. The uncertainty set proposed by Delage et al. (2014) and used in Theorem 3.4 of Cai et al. (2020) can be obtained as a special case of this setting by taking $\Psi=\{f_1,\dots,f_n\}\cup\{g_1,\dots,g_n\}\cup\Phi$, where $f_i:(x_1,\dots,x_n)\mapsto x_i$, $g_i:(x_1,\dots,x_n)\mapsto -x_i$ for all $i=1,\dots,n$, and Φ is a set of convex functions. The specification for π is that $\pi(f_i)=m_i\in\mathbb{R}$, $\pi(g_i)=-m_i$, $\pi(\phi)=0$ for all $i=1,\dots,n$, $\phi\in\Phi$.

2. (Mean-covariance conditions) For $n\in\mathbb{N}$, a convex set $B\subset\mathbb{R}^n$, $\boldsymbol{\mu}=(\mu_1,\dots,\mu_n)\in\mathbb{R}^n$, and $\Sigma\in\mathbb{R}^{n\times n}$ positive semidefinite, let

$$\widetilde{\mathcal{M}}^{\mathrm{mv}}(B,\boldsymbol{\mu},\Sigma)=\{F_{\mathbf{X}}\in\mathcal{M}_2^n:\mathbb{P}(\mathbf{X}\in B)=1,\ \mathbb{E}[\mathbf{X}]=\boldsymbol{\mu},\ \mathrm{var}(\mathbf{X})\preceq\Sigma\}.$$

The set $\widetilde{\mathcal{M}}^{\mathrm{mv}}(B,\boldsymbol{\mu},\Sigma)$ was proposed by Delage and Ye (2010) and is a special case of $\widetilde{\mathcal{M}}(B,\Psi,f)$. Similarly to the construction in point 1, one can obtain $\widetilde{\mathcal{M}}^{\mathrm{mv}}(B,\boldsymbol{\mu},\Sigma)$ from $\widetilde{\mathcal{M}}(B,\Psi,f)$ by setting $\Psi=\{f_1,\dots,f_n\}\cup\{g_1,\dots,g_n\}\cup\Phi$, where $f_i:(x_1,\dots,x_n)\mapsto x_i$, $g_i:(x_1,\dots,x_n)\mapsto -x_i$ for all $i=1,\dots,n$, and

$$\Phi=\left\{\phi:\mathbb{R}^n\to\mathbb{R}:\phi(\mathbf{x})=\left(\mathbf{a}^\top(\mathbf{x}-\boldsymbol{\mu})\right)^2/2-\mathbf{a}^\top\Sigma\mathbf{a}\text{ for some }\mathbf{a}\in\mathbb{R}^n\right\},$$

with the same specification of π that $\pi(f_i)=\mu_i\in\mathbb{R}$, $\pi(g_i)=-\mu_i$, $\pi(\phi)=0$ for all $i=1,\dots,n$, $\phi\in\Phi$.

3. (Distortion conditions) For $n\in\mathbb{N}$, $K\subset\mathbb{N}$, $\mathbf{a}=(\mathbf{a}_k)_{k\in K}\in\mathbb{R}^{n\times|K|}$, $\mathbf{h}=(h_k)_{k\in K}\in(\mathcal{H}^*)^{|K|}$ and $\mathbf{x}=(x_k)_{k\in K}\in\mathbb{R}^{|K|}$, the set

$$\widetilde{\mathcal{M}}^{\mathbf{h}}(\mathbf{a},\mathbf{x})=\{F_{\mathbf{X}}\in\mathcal{M}_1^n:\rho_{h_k}(\mathbf{a}_k^\top\mathbf{X})\leqslant x_k\text{ for all }k\in K\}$$

is closed under conditional expectation. In portfolio optimization problems, this setting incorporates distributional uncertainty with constraints on convex distortion risk measures of the total loss. In particular, optimization with the riskmetrics chosen as ES is common in the literature; see e.g., Rockafellar and Uryasev (2002), where ES is called CVaR.

4. (Convex order conditions) For $n \in \mathbb{N}$ and random vectors $\mathbf{Z}_k \in (\mathcal{L}^1)^n$, $k \in K \subset \mathbb{N}$, we naturally extend from part 5 of Example 7 and obtain that the set

$$\widetilde{\mathcal{M}}^{\mathrm{cx}}(\mathbf{Z}) = \{F_{\mathbf{X}} \in \mathcal{M}_1^n : \mathbf{X} \leqslant_{\mathrm{cx}} \mathbf{Z}_k \text{ for all } k \in K\}$$

is closed under conditional expectation.

Next, we discuss a multi-dimensional problem setting involving concentrations of marginal distributions. For $n \in \mathbb{N}$, we assume that marginal distributions of an n-dimensional distribution in $\mathcal{M}_1^n$ are uncertain and are in some sets $\mathcal{F}_1, \dots, \mathcal{F}_n \subset \mathcal{M}_1$. For $F_1, \dots, F_n \in \mathcal{M}_1$, define the set

$$\mathcal{D}(F_1, \dots, F_n) = \{\text{cdf of } (X_1, \dots, X_n) : X_i \sim F_i,\ i = 1, \dots, n\},$$

which is the set of all possible joint distributions with specified marginals; see Embrechts et al. (2015). For $\mathbf{a} \in A$, $h \in \mathcal{H}$ and $\mathcal{F}_1, \dots, \mathcal{F}_n \subset \mathcal{M}_1$, the worst-case distortion riskmetric can be represented as

$$\sup_{F_1 \in \mathcal{F}_1, \dots, F_n \in \mathcal{F}_n} \sup_{F_{\mathbf{X}} \in \mathcal{D}(F_1, \dots, F_n)} \rho_h(f(\mathbf{a}, \mathbf{X})). \tag{17}$$

The outer problem of (17) is a robust risk aggregation problem (see Embrechts et al. (2013, 2015) and item 6 of Example 7), which is typically nontrivial in general when h is not concave. With additional uncertainty of the marginal distributions, (17) can be converted to a problem with a convex objective given that $\mathcal{F}_1, \dots, \mathcal{F}_n$ are closed under concentration.

Theorem 4. *For $\mathcal{F}_1, \dots, \mathcal{F}_n \subset \mathcal{M}_1$, increasing $h \in \mathcal{H}$ with $h = \hat{h}$, and $f : A \times \mathbb{R}^n \to \mathbb{R}$ increasing, supermodular and positively homogeneous in the second argument, if $\mathcal{F}_1, \dots, \mathcal{F}_n$ are closed under concentration within $\mathcal{I}_h$, then the following hold.*[5]

(i) For all $\mathbf{a} \in A$,

$$\sup_{F_1 \in \mathcal{F}_1, \dots, F_n \in \mathcal{F}_n} \sup_{F_{\mathbf{X}} \in \mathcal{D}(F_1, \dots, F_n)} \rho_h(f(\mathbf{a}, \mathbf{X})) = \sup_{F_1 \in \mathcal{F}_1, \dots, F_n \in \mathcal{F}_n} \sup_{F_{\mathbf{X}} \in \mathcal{D}(F_1, \dots, F_n)} \rho_{h^*}(f(\mathbf{a}, \mathbf{X})). \tag{18}$$

(ii) If the supremum of the right-hand side of (18) is attained by some $F_1 \in \mathcal{F}_1, \dots, F_n \in \mathcal{F}_n$ and $F \in \mathcal{D}(F_1, \dots, F_n)$, then for all $\mathbf{a} \in A$, $F_1^{\mathcal{I}_h}, \dots, F_n^{\mathcal{I}_h}$ and a comonotonic random vector $(X_1^{\mathcal{I}_h}, \dots, X_n^{\mathcal{I}_h})$ with $X_i^{\mathcal{I}_h} \sim F_i^{\mathcal{I}_h}$, $i = 1, \dots, n$ attain the suprema on both sides of (18).[6]

As one of the most important potential applications of Theorem 4, we can take the ambiguity sets $\mathcal{M}_1, \dots, \mathcal{M}_n$ as $\mathcal{M}(p_1, m_1, v_1), \dots, \mathcal{M}(p_n, m_n, v_n)$ with moment conditions in Example 7, and thus solve the robust risk aggregation problem with moment constraints of the marginal distributions. Some examples of functions on $\mathbb{R}^n$ that are supermodular and positively homogeneous are given below. These functions are concave due to Theorem 3 of Marinacci and Montrucchio (2008).

[5] For a function $f : \mathbb{R}^n \to \mathbb{R}$, we say f is *supermodular* if $f(\mathbf{x}) + f(\mathbf{y}) \leqslant f(\mathbf{x} \wedge \mathbf{y}) + f(\mathbf{x} \vee \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$; f is *positively homogeneous* if $f(\lambda \mathbf{x}) = \lambda f(\mathbf{x})$ for all $\lambda \geqslant 0$ and $\mathbf{x} \in \mathbb{R}^n$.

[6] A random vector $(X_1, \dots, X_n) \in (\mathcal{L}^1)^n$ is called *comonotonic* if there exists a random variable $Z \in \mathcal{X}$ and increasing functions $f_1, \dots, f_n$ on $\mathbb{R}$ such that $X_i = f_i(Z)$ almost surely for all $i = 1, \dots, n$.

Example 16 (Supermodular and positively homogeneous functions)**.** For $n \in \mathbb{N}$, the following functions $f : \mathbb{R}^n \to \mathbb{R}$ are supermodular and positively homogeneous. Write $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. The last two examples are from economic literature, showing potential economic applications of Theorem 4. For (v) and (vi), negative real numbers K and L stand for negated capital investment and negated labor investment, respectively.

(i) (Linear function) $f : \mathbf{x} \mapsto \mathbf{a}^\top \mathbf{x}$ for $\mathbf{a} \in \mathbb{R}^n$. The function is increasing for $\mathbf{a} \in \mathbb{R}^n_+$.

(ii) (Geometric mean) $f : \mathbf{x} \mapsto -(\prod_{i=1}^n |x_i|)^{1/n}$ on $\mathbb{R}^n_-$ for odd n. The function is also increasing on $\mathbb{R}^n_-$.

(iii) (Negated p-norm) $f : \mathbf{x} \mapsto -\|\mathbf{x}\|_p$ for $p \geqslant 1$. The function is increasing on $\mathbb{R}^n_-$.

(iv) (Sum of functions) $f : \mathbf{x} \mapsto \sum_{i=1}^n f_i(x_i)$ for positively homogeneous functions $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R}$. The function is increasing if $f_1, \dots, f_n$ are increasing.

(v) (Cobb-Douglas production function) The Cobb-Douglas production function (Cobb and Douglas (1928), negated to represent loss) is defined as $f : (K, L) \mapsto -A|K|^{1-\alpha}|L|^\alpha$ for $A > 0$ and $\alpha \in [0, 1]$. The function is increasing in $(K, L) \in \mathbb{R}^2_-$.

(vi) (CES production function) The negated constant elasticity of substitution (CES) production function (Arrow et al. (1961)) is defined as $f : (K, L) \mapsto -A\left((1-\alpha)|K|^r + \alpha|L|^r\right)^{\frac{1}{r}}$ for $A > 0$, $\alpha \in [0, 1]$, and $r \leqslant 1$. The function is increasing in $(K, L) \in \mathbb{R}^2_-$.

In practice, we can choose a loss function as a mixture of those shown in Example 16. For example, we consider the loss function $f : \mathbf{x} \mapsto \mathbf{a}^\top \mathbf{x} - (\prod_{i=1}^n |x_i|)^{1/n}$ on $\mathbb{R}^n_-$ for odd n, which represents the aggregate loss of a portfolio with a geometric penalty. Such a loss function satisfies the assumption in Theorem 4. The geometric penalty term $-(\prod_{i=1}^n |x_i|)^{1/n}$ can also be replaced by the negated p-norm $-||\mathbf{x}||_p$ for $p \geqslant 1$. Another possible choice of the loss function is that we take $f_i : x \mapsto x_+$ in Example 16 (iv). The resulting loss function can be used to represent aggregate insurance losses.

5 One-dimensional uncertainty set with moment constraints

A popular example of an uncertainty set closed under concentration for all intervals is that of distributions with specified moment constraints as in Example 7. We investigate this uncertainty set in detail and offer in this section some general results, which generalize several existing results in the literature; none of the results in the literature include non-monotone and non-convex distortion functions. Non-monotone distortion functions create difficulties because of possible complications at their discontinuity points.

For $p > 1$, $m \in \mathbb{R}$ and $v > 0$, we recall the set of interest in Example 7:

$$\mathcal{M}(p, m, v) = \{F_Y \in \mathcal{M}_p : \mathbb{E}[Y] = m,\ \mathbb{E}[|Y - m|^p] \leqslant v^p\}.$$

Let $q \in [1,\infty]$ be the Hölder conjugate of p, namely $q = (1-1/p)^{-1}$, or equivalently, $1/p + 1/q = 1$. For all $h \in \mathcal{H}^*$ or $h \in \mathcal{H}_*$, we denote by

$$\|h' - x\|_q = \left(\int_0^1 |h'(t) - x|^q \,\mathrm{d}t\right)^{1/q}, \ q < \infty \ \text{ and } \ \|h' - x\|_\infty = \max_{t\in[0,1]} |h'(t) - x|, \quad x \in \mathbb{R}. \tag{19}$$

We introduce the following quantities:

$$c_{h,q} = \underset{x\in\mathbb{R}}{\arg\min} \|h' - x\|_q \quad \text{and} \quad [h]_q = \min_{x\in\mathbb{R}} \|h' - x\|_q = \|h' - c_{h,q}\|_q.$$

We set $[h]_q = \infty$ if h is not continuous. It is clear that $c_{h,q}$ is unique for $q > 1$. The quantity $[h]_q$ may be interpreted as a q-central norm of the function h and $c_{h,q}$ as its q-center. Note that for $q = 2$ and h continuous, $[h]_2 = \|h' - h(1)\|_2$ and $c_{h,2} = h(1)$. We also note that the optimization problem is trivial if $[h]_q = 0$, which corresponds to the case that $h' = h(1)\mathbb{1}_{[0,1]}$ and ρ_h is a linear functional, thus a multiple of the expectation. In this case, the supremum and infimum are attained by all random variables whose distributions are in $\mathcal{M}(p,m,v)$, and they are equal to $mh(1)$. Furthermore, for $h \in \mathcal{H}^*$ or $h \in \mathcal{H}_*$, and $q > 1$, we define a function on $[0,1]$ by

$$\phi_h^q(t) = \frac{|h'(1-t) - c_{h,q}|^q}{h'(1-t) - c_{h,q}} [h]_q^{1-q} \quad \text{if } h'(1-t) - c_{h,q} \neq 0, \quad \text{and } \phi_h^q(t) = 0 \text{ otherwise.}$$

In case $q = 2$, for $t \in [0,1]$, $\phi_h^2(t) = (h'(1-t) - h(1))\|h' - h(1)\|_2^{-1}$ if $\|h' - h(1)\|_2 > 0$ and 0 otherwise. We summarize our findings in the following theorem.

Theorem 5. *For any $h \in \mathcal{H}$, $m \in \mathbb{R}$, $v > 0$ and $p > 1$, we have*

$$\sup_{F_Y\in\mathcal{M}(p,m,v)} \rho_h(Y) = mh(1) + v[h^*]_q \quad \text{and} \quad \inf_{F_Y\in\mathcal{M}(p,m,v)} \rho_h(Y) = mh(1) - v[h_*]_q. \tag{20}$$

Moreover, if $h = \hat{h}$, $0 < [h_*]_q < \infty$ and $0 < [h^*]_q < \infty$, then the supremum and infimum in (20) are attained by a random variable X such that $F_X \in \mathcal{M}(p,m,v)$ with its quantile function uniquely specified as a.e. equal to $m + v\phi_{h^*}^q$ and $m - v\phi_{h_*}^q$, respectively.

The proof of Theorem 5 follows from a combination of Lemmas A.1 and A.2 in Appendix B.4 and Theorem 1. Note that for $h \in \mathcal{H}^*$ (resp. $h \in \mathcal{H}_*$) and $q > 1$, ϕ_h^q is increasing (resp. decreasing) on $[0,1]$. Hence, ϕ_h^q (resp. $-\phi_h^q$) in Theorem 5 indeed determines a quantile function.

The following proposition concerns the finiteness of ρ_h on $\mathcal{L}^p$.

Proposition 6. *For any $h \in \mathcal{H}$ and $p \in [1,\infty]$, ρ_h is finite on $\mathcal{L}^p$ if $[h^*]_q < \infty$ and $[h_*]_q < \infty$.*

As a special case of Proposition 6, ρ_h is always finite on $\mathcal{L}^1$ if h is convex or concave with bounded h' because $[h^*]_\infty < \infty$ and $[h_*]_\infty < \infty$.

As a common example of the general result in Theorem 5, below we collect our findings for the case of VaR.

Corollary 1. *For $\alpha \in (0,1)$, $p > 1$, $m \in \mathbb{R}$ and $v > 0$, we have*

$$\sup_{F_Y \in \mathcal{M}(p,m,v)} \mathrm{VaR}_\alpha(Y) = \max_{F_Y \in \mathcal{M}(p,m,v)} \mathrm{ES}_\alpha(Y) = m + v\alpha \left(\alpha^p(1-\alpha) + (1-\alpha)^p\alpha\right)^{-1/p},$$

and

$$\inf_{F_Y \in \mathcal{M}(p,m,v)} \mathrm{VaR}_\alpha(Y) = \min_{F_Y \in \mathcal{M}(p,m,v)} \mathrm{ES}^L_\alpha(Y) = m - v(1-\alpha) \left(\alpha^p(1-\alpha) + (1-\alpha)^p\alpha\right)^{-1/p},$$

where

$$\mathrm{ES}^L_\alpha(Y) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_t(Y)\,\mathrm{d}t, \quad Y \in \mathcal{L}^1.$$

We see from Theorem 5 that if $h = \hat{h}$, then the supremum and the infimum of $\rho_h(Y)$ over $F_Y \in \mathcal{M}(p,m,v)$ are always attainable. However, in case $h \neq \hat{h}$, the supremum or infimum may no longer be attainable as a maximum or minimum. We illustrate this in Example 17 below.

Example 17 (VaR and ES, $p = 2$)**.** Take $\alpha \in (0,1)$, $p = 2$ and $\rho_h = \mathrm{VaR}_\alpha$, which implies $\rho_{h^*} = \mathrm{ES}_\alpha$. Corollary 1 gives $\sup_{F_Y \in \mathcal{M}(2,m,v)} \mathrm{VaR}_\alpha(Y) = \sup_{F_Y \in \mathcal{M}(2,m,v)} \mathrm{ES}_\alpha(Y) = m + v\sqrt{\alpha/(1-\alpha)}$. This is the well-known Cantalli-type formula for ES. By Lemma A.1, the unique left-quantile function of the random variable Z that attains the supremum of ES_α is given by $F_Z^{-1}(t) = m + v(\mathbb{1}_{(\alpha,1]}(t)/(1-\alpha) - 1)\sqrt{(1-\alpha)/\alpha}$, $t \in [0,1]$ a.e. We thus have $\mathrm{VaR}_\alpha(Z) = m - v\sqrt{(1-\alpha)/(\alpha)}$, and hence Z does not attain $\sup_{F_Y \in \mathcal{M}(2,m,v)} \mathrm{VaR}_\alpha(Y)$. It follows by the uniqueness of F_Z that the supremum of $\mathrm{VaR}_\alpha(Y)$ over $F_Y \in \mathcal{M}(2,m,v)$ cannot be attained. However, the supremum of VaR^+_α is attained by Z since $\mathrm{VaR}^+_\alpha(Z) = m + v\sqrt{(1-\alpha)/(\alpha)}$.

Example 18 (Difference of two TK distortion riskmetrics)**.** Take $p = 2$ and $h = h_1 - h_2$ to be the difference between two inverse-S-shaped functions in (11) with parameters the same as those in Example 4 ($\gamma_1 = 0.8$, $\gamma_2 = 0.7$). By Theorem 5, the worst-case distortion riskmetrics under the uncertainty set $\mathcal{M}(2,m,v)$ are given by $\sup_{F_Y \in \mathcal{M}(2,m,v)} \rho_h(Y) = \sup_{F_Y \in \mathcal{M}(2,m,v)} \rho_{h^*}(Y) = 0.3345v$, and the unique left-quantile function of the random variable Z attaining both suprema above is given by $F_Z^{-1}(t) = m + 2.9892 \cdot h^{*\prime}(1-t)v$, $t \in [0,1]$ a.e. The worst-case distortion riskmetrics obtained above are independent of the mean m as $h(1) = h_1(1) - h_2(1) = 0$, which is sensible since ρ_h and ρ_{h^*} only incorporate the disagreement between two distortion riskmetrics. Similarly, we can calculate the infimum of $\rho_h(Y)$ over $Y \in \mathcal{M}(2,m,v)$, and thus obtain the largest absolute difference between the two preferences numerically represented by ρ_{h_1} and ρ_{h_2}.

6 Related optimization problems

In this section, we discuss the applications of our main results to some related optimization problems commonly investigated in the literature by including the outer problem of (1).

6.1 Portfolio optimization

Our equivalence results can be applied to robust portfolio optimization problems. For an uncertainty set $\widetilde{\mathcal{M}} \subset \mathcal{M}_p^n$ with $p \in [1, \infty]$, let the random vector $\mathbf{X} = (X_1, \ldots, X_n) \sim F_{\mathbf{X}} \in \widetilde{\mathcal{M}}$, representing the random losses from n risky assets. For $A \subset \mathbb{R}^n$, denote by a vector $\mathbf{a} \in A$ the amounts invested in each of the n risky assets. For a distortion function $h \in \mathcal{H}$ and distortion riskmetric $\rho_h : \mathcal{L}^p \to \mathbb{R}$, we aim to solve the robust portfolio optimization problem given by

$$\min_{\mathbf{a} \in A} \left(\sup_{F_{\mathbf{X}} \in \widetilde{\mathcal{M}}} \rho_h(\mathbf{a}^\top \mathbf{X}) + \beta(\mathbf{a}) \right), \tag{21}$$

where $\beta : \mathbb{R}^n \to \mathbb{R}$ is a penalty function of risk concentration. Note that β is irrelevant for the inner problem of (21). For a general non-concave h, there is no known algorithm to solve the inner problem of (21). The outer optimization problem is also nontrivial in general. Therefore, we usually cannot obtain closed-form solutions of (21) using classical results of optimization problems for non-convex risk measures. However, as a direct consequence of Theorems 1 and 3, the following proposition converts (21) to an equivalent optimization problem with the objective functional ρ_{h^*} being convex and mixture concave, which is usually technically tractable to solve. The proof of Proposition 7 follows directly from Theorems 1 and 3.

Proposition 7. *Let $h \in \mathcal{H}$, $n \in \mathbb{N}$, $A \subset \mathbb{R}^n$, and $\widetilde{\mathcal{M}} \subset \mathcal{M}_1^n$.*

(i) if $h = \hat{h}$ and the set $\{F_{\mathbf{a}^\top \mathbf{X}} \in \mathcal{M}_1 : F_{\mathbf{X}} \in \widetilde{\mathcal{M}}\}$ is closed under concentration within $\mathcal{I}_h$ for all $\mathbf{a} \in A$, then

$$\min_{\mathbf{a} \in A} \left(\sup_{F_{\mathbf{X}} \in \widetilde{\mathcal{M}}} \rho_h(\mathbf{a}^\top \mathbf{X}) + \beta(\mathbf{a}) \right) = \min_{\mathbf{a} \in A} \left(\sup_{F_{\mathbf{X}} \in \widetilde{\mathcal{M}}} \rho_{h^*}(\mathbf{a}^\top \mathbf{X}) + \beta(\mathbf{a}) \right). \tag{22}$$

(ii) if the set $\{F_{\mathbf{a}^\top \mathbf{X}} \in \mathcal{M}_1 : F_{\mathbf{X}} \in \widetilde{\mathcal{M}}\}$ is closed under concentration for all intervals for all $\mathbf{a} \in A$, then (22) holds.

(iii) If $\widetilde{\mathcal{M}}$ is closed under conditional expectation, then (22) holds.

As commented for Theorem 1, closedness under concentration is generally as difficult to verify as closedness under conditional expectation, and thus does not add practical convenience in the context of Proposition 7. Therefore, the property of closedness under concentration for the aggregation set $\{F_{\mathbf{a}^\top \mathbf{X}} \in \mathcal{M}_1 : F_{\mathbf{X}} \in \widetilde{\mathcal{M}}\}$ is usually checked through closedness under conditional expectation for sufficiency of the equivalence result.

6.2 Preference robust optimization

We are also able to solve the preference robust optimization problem with distributional uncertainty. For $n \in \mathbb{N}$, an n-dimensional action set A, a set of plausible distributions $\widetilde{\mathcal{M}} \subset \mathcal{M}_1^n$, and a set of possible probability perceptions $\mathcal{G} \subset \mathcal{H}$, the problem is formulated as follows:

$$\min_{\mathbf{a} \in A} \sup_{F_{\mathbf{X}} \in \widetilde{\mathcal{M}}} \sup_{h \in \mathcal{G}} \rho_h(f(\mathbf{a}, \mathbf{X})). \tag{23}$$

Preference robust optimization refers to the situation when the objective is not completely known, e.g., h is in the set $\mathcal{G}$ but not identified. Therefore, optimization is performed under the worst-case preference in the set $\mathcal{G}$. Also note that the form $\sup_{h \in \mathcal{G}} \rho_h$ includes (but is not limited to) all coherent risk measures via the representation of Kusuoka (2001). See Delage and Li (2018) for the problem of (23) without distributional uncertainty (thus, only the minimum and the second supremum), which was further studied by Wang and Xu (2020) for optimization problems of robust spectral risk measures. We have the following result whose proof follows from Theorems 1 and 3.

Proposition 8. *Let $\widetilde{\mathcal{M}} \subset \mathcal{M}_1^n$ and $A \subset \mathbb{R}^n$ with $n \in \mathbb{N}$.*

(i) If $h = \hat{h}$ and the set $\{F_{f(\mathbf{a},\mathbf{X})} \in \mathcal{M}_1 : F_{\mathbf{X}} \in \widetilde{\mathcal{M}}\}$ is closed under concentration within $\mathcal{I}_h$ for all $\mathbf{a} \in A$, then for all $\mathcal{G} \subset \mathcal{H}$,

$$\min_{\mathbf{a} \in A} \sup_{F_{\mathbf{X}} \in \widetilde{\mathcal{M}}} \sup_{h \in \mathcal{G}} \rho_h(f(\mathbf{a}, \mathbf{X})) = \min_{\mathbf{a} \in A} \sup_{F_{\mathbf{X}} \in \widetilde{\mathcal{M}}} \sup_{h \in \mathcal{G}} \rho_{h^*}(f(\mathbf{a}, \mathbf{X})). \tag{24}$$

(ii) If the set $\{F_{f(\mathbf{a},\mathbf{X})} \in \mathcal{M}_1 : F_{\mathbf{X}} \in \widetilde{\mathcal{M}}\}$ is closed under concentration for all intervals for all $\mathbf{a} \in A$, then (24) holds for all $\mathcal{G} \subset \mathcal{H}$.

(iii) If $\mathcal{G}$ is a set of increasing functions in $\mathcal{H}$, $f : A \times \mathbb{R}^n \to \mathbb{R}$ is concave in the second component, and $\widetilde{\mathcal{M}}$ is closed under conditional expectation, then (24) holds.

The preference robust optimization problem without distributional uncertainty (i.e., problem (23) with only the minimum and the second supremum) is generally difficult to solve when the distortion function h is not concave. However, when the distribution of the random variable is not completely known, we can transfer the original non-convex problem of a distortion riskmetric to its convex counterpart induced from the concave envelope of the distortion function using (24), provided that the set of plausible distributions is well structured.

7 Applications and numerical illustrations

Following the discussion in Section 6, we provide several applications of our theoretical results to portfolio management for specific sets of plausible distributions. None of the considered optimization problems in this section are convex, and we provide numerical calculations or approximation for the solutions to these optimization problems.[7]

[7] The processors we use are Intel(R) Xeon(R) CPU E5-2690 v3 @ 2.60GHz 2.59GHz (2 processors). The numerical results are calculated by MATLAB.

7.1 Difference of risk measures under moment constraints

We demonstrate a price competition problem as an application of optimizing the difference between two risk measures shown in Example 18. Similar to the portfolio management problem discussed in Section 6.1, we consider n risky assets with random losses $X_1, \dots, X_n \in \mathcal{L}^2$ that are only known to have a fixed mean and a constrained covariance. That is, we choose the set

$$\widetilde{\mathcal{M}} = \{F_{\mathbf{X}} \in \mathcal{M}_2^n : \mathbb{E}[\mathbf{X}] = \boldsymbol{\mu},\ \mathrm{var}(\mathbf{X}) \preceq \Sigma\},$$

for $\boldsymbol{\mu} \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n\times n}$ positive semidefinite. For an n-dimensional $\mathbf{a} \in A$, the set of all possible distributions of aggregate portfolio losses

$$\{F_{\mathbf{a}^\top \mathbf{X}} \in \mathcal{M}_2 : F_{\mathbf{X}} \in \widetilde{\mathcal{M}}\} = \mathcal{M}^{\mathrm{mv}}(\mathbf{a}, \boldsymbol{\mu}, \Sigma) = \mathcal{M}\left(2, \mathbf{a}^\top \boldsymbol{\mu}, \left(\mathbf{a}^\top \Sigma \mathbf{a}\right)^{1/2}\right) \tag{25}$$

is closed under concentration for all intervals as is shown in Example 7. Let $\rho_{h_1} : \mathcal{L}^2 \to \mathbb{R}$ be an investor's own price of the portfolio, while $\rho_{h_2} : \mathcal{L}^2 \to \mathbb{R}$ is her opponent's price of the same portfolio. We choose h_1 and h_2 to be the inverse-S-shaped distortion functions in (11), with parameters the same as those in Example 18 ($\gamma_1 = 0.8$ and $\gamma_2 = 0.7$). Write $h = h_1 - h_2$. For an action set $A = \{(a_1, \dots, a_n) \in [0,1]^n : \sum_{i=1}^n a_i = 1\}$, the investor chooses the optimal $\mathbf{a}^* \in A$, such that the worst-case overpricing from her opponent is minimized.

From the calculation in Example 18, we get

$$\begin{aligned} D(\Sigma) &:= \min_{\mathbf{a}\in A} \sup_{F_{\mathbf{X}} \in \widetilde{\mathcal{M}}} \left(\rho_{h_1}(\mathbf{a}^\top \mathbf{X}) - \rho_{h_2}(\mathbf{a}^\top \mathbf{X})\right) \\ &= \min_{\mathbf{a}\in A} \sup_{F_Y \in \mathcal{M}^{\mathrm{mv}}(\mathbf{a}, \boldsymbol{\mu}, \Sigma)} \rho_{h^*}(Y) = 0.3345 \times \min_{\mathbf{a}\in A} \left(\mathbf{a}^\top \Sigma \mathbf{a}\right)^{1/2}. \end{aligned} \tag{26}$$

We note that optimizing $\rho_{h_1} - \rho_{h_2}$ is generally nontrivial since the difference between two distortion functions $h_1 - h_2$ is not necessarily monotone, concave, or continuous, even though h_1 and h_2 themselves may have these properties. The generality of our equivalence result allows us to convert the original problem to the much simpler form (26), which can be solved efficiently.[8] Table 1 demonstrates the optimal values of $\mathbf{a}^*$ and D for different choices of Σ.

7.2 Preference robust portfolio optimization with moment constraints

Next, we discuss an example of preference robust optimization with distributional uncertainty using the results in Sections 5. Similarly to Section 7.1, we consider the set of plausible aggregate portfolio loss distributions

$$\mathcal{M}^{\mathrm{mv}}(\mathbf{a}, \boldsymbol{\mu}, \Sigma) = \{F_{\mathbf{a}^\top \mathbf{X}} \in \mathcal{M}_2 : F_{\mathbf{X}} \in \mathcal{M}_2^n,\ \mathbb{E}[\mathbf{X}] = \boldsymbol{\mu},\ \mathrm{var}(\mathbf{X}) \preceq \Sigma\}$$

[8] The convex problem (26) is solved numerically by the constrained nonlinear multivariable function "fmincon" with the interior-point method.

Table 1: Optimal results in (26) for difference between two TK distortion riskmetrics

n	Σ	$\mathbf{a}^*$	D
3	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$(0.333,\ 0.333,\ 0.333)$	0.193
3	$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$	$(0.300,\ 0.400,\ 0.300)$	0.150
3	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$	$(0.997,\ 0.002,\ 0.001)$	0.335
5	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$	$(0.438,\ 0.219,\ 0.146,\ 0.110,\ 0.088)$	0.221

and the action set $A = \{(a_1, \dots, a_n) \in [0,1]^n : \sum_{i=1}^n a_i = 1\}$ representing the weights the investor assigns to each random loss. The investor considers TK distortion riskmetrics, however, she is not certain about the parameter γ of the distortion function h. Thus, the investor consider the set of TK distortion riskmetrics with distortion functions in

$$\mathcal{G} = \{h \in \mathcal{H} : h = h^{\gamma},\ \gamma \in [0.5, 0.9]\},$$

which is approximately the two-sigma confidence interval of γ in Wu and Gonzalez (1996).[9] Therefore, the investor aims to find a optimal portfolio given the uncertainty in the riskmetrics. To penalize deviations from the benchmark parameter $\gamma = 0.71$ (Wu and Gonzalez, 1996), the investor use the term $\mathrm{e}^{c(\gamma-0.71)^2}$ for some $c \geqslant 0$. Here we choose the exponential penalty only to ensure that it is nonnegative. We could also choose other forms of the penalty functions such as the quadratic penalty or absolute difference penalty, which will not change the results qualitatively. Since the set $\mathcal{M}^{\mathrm{mv}}(\mathbf{a}, \boldsymbol{\mu}, \Sigma)$ is closed under concentration for all intervals for all $\mathbf{a} \in A$, Proposition 8, (25), and Theorem 5 lead to

$$\begin{aligned} V(\boldsymbol{\mu}, \Sigma) &:= \min_{\mathbf{a}\in A} \sup_{F_Y \in \mathcal{M}^{\mathrm{mv}}(\mathbf{a},\boldsymbol{\mu},\Sigma)} \sup_{\gamma\in[0.5,0.9]} \left(\rho_{h^{\gamma}}(Y) - \mathrm{e}^{c(\gamma-0.71)^2}\right) \\ &= \min_{\mathbf{a}\in A} \sup_{F_Y \in \mathcal{M}\left(2, \mathbf{a}^\top\boldsymbol{\mu}, \left(\mathbf{a}^\top \Sigma \mathbf{a}\right)^{1/2}\right)} \sup_{\gamma\in[0.5,0.9]} \left(\rho_{(h^{\gamma})^*}(Y) - \mathrm{e}^{c(\gamma-0.71)^2}\right) \\ &= \min_{\mathbf{a}\in A} \sup_{\gamma\in[0.5,0.9]} \left(\mathbf{a}^\top \boldsymbol{\mu} + \left(\mathbf{a}^\top \Sigma \mathbf{a}\right)^{1/2} [(h^{\gamma})^*]_2 - \mathrm{e}^{c(\gamma-0.71)^2}\right). \end{aligned} \tag{27}$$

We calculate the optimal values V for different choices of parameters (n, c, $\boldsymbol{\mu}$ and Σ) and

[9] The aggregate least-square estimate of γ in Section 5 of Wu and Gonzalez (1996) is 0.71 with standard deviation 0.1.

report them in Table 2, where $\mathbf{a}^*$ and $\hat{\gamma}$ represent the optimal weights and the parameters of the inverse-S-shaped distortion function, respectively. Note that the last optimization problem in (27) can be calculated numerically.[10]

Table 2: Optimal values in (27) for TK distortion riskmetrics

n	c	$\boldsymbol{\mu}$	Σ	$\mathbf{a}^*$	$\hat{\gamma}$	V
3	0	$(1,1,1)$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	(0.333, 0.333, 0.333)	0.500	2.38
3	30	$(2,1,1)$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	(0.000, 0.500, 0.500)	0.678	0.284
3	30	$(1,1,1)$	$\begin{pmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{pmatrix}$	(0.300, 0.400, 0.300)	0.690	0.173
3	50	$(1.2,1,1)$	$\begin{pmatrix}1&1&1\\1&2&1\\1&1&3\end{pmatrix}$	(0.195, 0.537, 0.269)	0.678	0.520
5	50	$(1,1,1,1,1)$	$\begin{pmatrix}1&0&0&0&0\\0&2&0&0&0\\0&0&3&0&0\\0&0&0&4&0\\0&0&0&0&5\end{pmatrix}$	(0.438, 0.219, 0.146, 0.110, 0.088)	0.694	0.254

7.3 Portfolio optimization with marginal constraints

A special case of the portfolio optimization problem introduced in Section 6.1, which is of interest in robust risk aggregation (see e.g., Blanchet et al. (2020)), is to take $\widetilde{\mathcal{M}}$ to be the Fréchet class defined as

$$\widetilde{\mathcal{M}}(F_1,\dots,F_n) = \{F_{\mathbf{X}} \in \mathcal{M}_1^n : X_i \sim F_i,\ i=1,\dots,n\}, \tag{28}$$

for some known marginal distributions $F_1,\dots,F_n \in \mathcal{M}_1$. In this case, although the left-hand side of (22) is generally difficult to solve, for $A \subset \mathbb{R}_+^n$, the right-hand side of (22) can be rewritten using convexity and comonotonicity as

$$\min_{\mathbf{a}\in A}\left(\mathbf{a}^\top(\rho_{h^*}(X_1),\dots,\rho_{h^*}(X_n)) + \beta(\mathbf{a})\right), \tag{29}$$

where $X_i \sim F_i$, $i=1,\dots,n$. We see that (29) is a linear optimization problem with a penalty β, which often admits closed-form solutions when β is properly chosen. For any given $\mathbf{a} \in A$, we define

$$\mathcal{M}(\mathbf{a},F_1,\dots,F_n) = \{F_{\mathbf{a}^\top\mathbf{X}} \in \mathcal{M}_1 : X_i \sim F_i,\ i=1,\dots,n\}. \tag{30}$$

[10] We solved the inner problem of (27) by grid screening different γ from 0.5 to 0.9 with a grid size 0.002. The outer problem (27) is convex and is solved numerically by the constrained nonlinear multivariable function "fmincon" with the interior-point method.

The set $\mathcal{M}(\mathbf{a}, F_1, \ldots, F_n)$ is the weighted version of $\mathcal{M}^S(F_1, \ldots, F_n)$ in Example 7. Note that $\mathcal{M}(\mathbf{a}, F_1, \ldots, F_n)$ is generally neither closed under concentration for all intervals nor closed under conditional expectation. However, for a special case where $\mathbf{a} = (1/n, \ldots, 1/n)$ and $F_1 = \cdots = F_n$, the set $\mathcal{M}(\mathbf{a}, F_1, \ldots, F_n)$ is asymptotically closed under concentration for all intervals by Theorem 3.5 of Mao and Wang (2015). Therefore, even though $\mathcal{M}(\mathbf{a}, F_1, \ldots, F_n)$ is not closed under concentration for all intervals for some $\mathbf{a} \in A$, our result of the problem (29) is a good approximation of the original problem for large n. Such asymptotic equivalence between worst-case riskmetrics of aggregate risks with equal weights and unequal marginals has already been well studied in the literature; see e.g., Theorem 3.3 of Embrechts et al. (2015) for the VaR/ES pair and Theorem 3.5 of Cai et al. (2018) for distortion risk measures.

We conduct numerical calculations to illustrate the equivalence between both sides in (22). We choose the action set $A_{a,b} = \{(x_1, \ldots, x_n) \in [a, b]^n : \sum_{i=1}^n x_i = 1\}$, for $0 \leqslant a < 1/n < b \leqslant 1$ and the penalty function β to be the $\mathcal{L}^2$-norm multiplied by a scaler $c \geqslant 0$, namely $c\|\cdot\|_2$, where the scaler c is a tuning parameter of the $\mathcal{L}^2$ penalty. We first solve the optimization problems separately for the well-known VaR/ES pair at the level of 0.95. Specifically, the two problems are given by

$$V_{\mathrm{VaR}}(a, b, F_1, \ldots, F_n) = \min_{\mathbf{a} \in A_{a,b}} \left(\sup_{F_{\mathbf{X}} \in \mathcal{M}(F_1, \ldots, F_n)} \mathrm{VaR}_{0.95}(\mathbf{a}^\top \mathbf{X}) + c\|\mathbf{a}\|_2 \right), \tag{31}$$

$$\begin{aligned} V_{\mathrm{ES}}(a, b, F_1, \ldots, F_n) &= \min_{\mathbf{a} \in A_{a,b}} \left(\sup_{F_{\mathbf{X}} \in \mathcal{M}(F_1, \ldots, F_n)} \mathrm{ES}_{0.95}(\mathbf{a}^\top \mathbf{X}) + c\|\mathbf{a}\|_2 \right) \\ &= \min_{\mathbf{a} \in A_{a,b}} \left(\mathbf{a}^\top (\mathrm{ES}_{0.95}(F_1), \ldots, \mathrm{ES}_{0.95}(F_n)) + c\|\mathbf{a}\|_2 \right). \end{aligned} \tag{32}$$

The inner VaR problem is calculated using the rearrangement algorithm (RA) of Puccetti and Rüschendorf (2012) and Embrechts et al. (2013), which is a well-adopted approach to approximate the sharp VaR bound of aggregate losses with given marginal distributions. The optimal value of the outer ES problem is obtained by minimizing the sum of a linear combination of ES and the 2-norm of the vector $\mathbf{a}$, which can be done efficiently.[11] In particular, if the marginals of the random losses are identical (i.e., $F_1 = \cdots = F_n = F$), the optimal solution is $\mathbf{a}^* = (1/n, \ldots, 1/n)$ and $V_{\mathrm{ES}}(a, b, F_1, \ldots, F_n) = \mathrm{ES}_{0.95}(F) + c/\sqrt{n}$. We consider the following marginal distributions

(i) F_i follows a Pareto distribution with scale parameter 1 and shape parameter $3 + (i-1)/(n-1)$ for $i = 1, \ldots, n$;

(ii) F_i is normally distributed with parameters $\mathrm{N}(1, 1 + (i-1)/(n-1))$, for $i = 1, \ldots, n$;

(iii) F_i follows an exponential distribution with parameter $1 + (i-1)/(n-1)$, for $i = 1, \ldots, n$.

We choose n to be 3, 10, and 20. For comparison, we calculate the value $n\|\Delta\mathbf{a}^*\|_2$, where $\Delta\mathbf{a}^*$ is the difference between the optimal weights of the non-convex problem and the convex problem.

[11]The outer problems of (31) and (32) is solved numerically by the "GlobalSearch" function with the constrained nonlinear multivariable function "fmincon" and the sequential quadratic programming (SQP) algorithm. This guarantees that the optimal solution we find obtains a global minimum of the problem. The same method is also applied when solving outer problems of (33) and (34).

In addition, we calculate the absolute differences between the optimal values obtained by the two problems, $\Delta V = V_{\mathrm{ES}} - V_{\mathrm{VaR}} \geqslant 0$, and the percentage differences $\Delta V / V_{\mathrm{VaR}}$. Tables 3 and 4 show the numerical results that compare both optimization problems with two choices of the action sets $A_{a,b}$. The computation time is reported (in seconds). We observe that the optimal values obtained in the two problems get closer and become approximately the same as n gets larger. As explained before, this is because the set of plausible distributions $\mathcal{M}(F_1, \ldots, F_n)$ is asymptotically equal to a set closed under concentration for all intervals.

Next, we consider a TK distortion riskmetric with parameter $\gamma = 0.7$. Due to the non-concavity of h, there are no known ways of directly solving the non-convex optimization problem

$$\min_{\mathbf{a} \in A_{a,b}} \left(\sup_{F_{\mathbf{X}} \in \mathcal{M}(F_1,\ldots,F_n)} \rho_h(\mathbf{a}^\top \mathbf{X}) + c\|\mathbf{a}\|_2 \right). \tag{33}$$

We may get an approximation of (33) using a lower bound of ρ_h in (33) produced with the dependence structure created by the rearrangement algorithm (RA);[12] for simplicity, we denote by V_h this lower bound. On the other hand, by (22), the convex counterpart of (33) can be written (using Theorem 1) as

$$\begin{aligned} V_{h^*}(a, b, F_1, \ldots, F_n) &= \min_{\mathbf{a} \in A_{a,b}} \left(\sup_{F_{\mathbf{X}} \in \mathcal{M}(F_1,\ldots,F_n)} \rho_{h^*}(\mathbf{a}^\top \mathbf{X}) + c\|\mathbf{a}\|_2 \right) \\ &= \min_{\mathbf{a} \in A_{a,b}} \left(\mathbf{a}^\top (\rho_{h^*}(X_1), \ldots, \rho_{h^*}(X_n)) + c\|\mathbf{a}\|_2 \right), \end{aligned} \tag{34}$$

where $X_i \sim F_i$ for $i = 1, \ldots, n$. We calculate the absolute differences between the optimal values of the convex and non-convex problems $\Delta V = V_{h^*} - V_h \geqslant 0$, and the percentage differences $\Delta V / V_h$. Tables 5 and 6 compare the numerical results of the two optimization problems with different choices of $A_{a,b}$. We observe that the percentage differences between the RA lower bound V_h for the non-convex problem (33) and the minimum value V_{h^*} of the convex problem (34) are roughly between 10% to 20%. According to our previous discussion in this section, the theoretical worst-case distortion riskmetric in (33) is close to that in (34) when n goes to infinity. However, the RA lower bound for (33) is not expected to be very close to the true minimum in (33), and hence the differences between the solution of (33) and the optimal value in (34) are smaller than the observed numbers.

Note that, by transforming an optimization problem with a non-convex riskmetric to a convex one induced from the concave envelope of the distortion function, we significantly reduce the computational time of calculating bounds with negligible errors, as shown in Tables 3-6.

[12] Such a dependence structure obviously provides a lower bound for the worst-case value in (33). In theory, the result from RA is thus not an optimal dependence structure for (33). In our numerical results, this lower bound is very close to an upper bound only for the case of VaR and ES but not for the case of TK distortion riskmetrics.

Table 3: Comparison of the numerical results of the two optimization problems (31) and (32) for $\mathrm{VaR}_{0.95}$ and $\mathrm{ES}_{0.95}$ with $a = 0$ and $b = 1$

		c	V_{VaR}	time	V_{ES}	time	$n\|\Delta\mathbf{a}^*\|_2$	ΔV	$\Delta V/V_{\mathrm{VaR}}$ (%)
(i) Pareto	$n = 3$	2.5	3.547	31.53	3.741	0.72	8.88×10^{-5}	0.194	5.48
	$n = 10$	3.0	3.197	153.83	3.215	1.39	9.18×10^{-4}	0.0178	0.558
	$n = 20$	4.0	3.156	424.17	3.159	9.37	3.53×10^{-5}	2.68×10^{-3}	0.0850
(ii) Normal	$n = 3$	4.0	5.766	31.19	5.785	0.18	1.39×10^{-3}	0.0186	0.323
	$n = 10$	2.0	4.082	97.30	4.083	0.77	1.18×10^{-3}	3.24×10^{-5}	7.93×10^{-4}
	$n = 20$	3.0	4.132	431.79	4.132	4.66	2.69×10^{-3}	1.88×10^{-5}	4.55×10^{-4}
(iii) Exp	$n = 3$	3.0	4.251	26.78	4.405	0.07	0.331	0.155	3.64
	$n = 10$	4.0	3.892	118.23	3.893	0.50	9.74×10^{-4}	2.92×10^{-4}	7.52×10^{-3}
	$n = 20$	7.0	4.230	543.03	4.230	3.47	3.08×10^{-4}	4.47×10^{-5}	1.06×10^{-3}

Table 4: Comparison of the numerical results of the two optimization problems (31) and (32) for $\mathrm{VaR}_{0.95}$ and $\mathrm{ES}_{0.95}$ with $a = 1/(2n)$ and $b = 2/n$

		c	V_{VaR}	time	V_{ES}	time	$n\|\Delta\mathbf{a}^*\|_2$	ΔV	$\Delta V/V_{\mathrm{VaR}}$ (%)
(i) Pareto	$n = 3$	2.5	3.546	54.59	3.741	0.19	6.58×10^{-4}	0.194	5.48
	$n = 10$	3.0	3.204	146.63	3.220	1.60	1.99×10^{-4}	0.0160	0.498
	$n = 20$	4.0	3.162	847.13	3.163	10.08	1.69×10^{-3}	2.23×10^{-3}	0.0706
(ii) Normal	$n = 3$	4.0	5.766	57.31	5.785	0.19	1.32×10^{-3}	0.0187	0.324
	$n = 10$	2.0	4.084	166.25	4.084	0.79	0	2.94×10^{-5}	7.20×10^{-4}
	$n = 20$	3.0	4.133	691.91	4.133	5.91	0	1.99×10^{-5}	4.82×10^{-4}
(iii) Exp	$n = 3$	3.0	4.369	48.58	4.422	0.09	1.04×10^{-3}	0.0533	1.22
	$n = 10$	4.0	3.916	115.18	3.916	0.50	2.54×10^{-5}	1.38×10^{-4}	3.52×10^{-3}
	$n = 20$	7.0	4.236	665.05	4.236	3.48	2.73×10^{-4}	4.04×10^{-5}	9.54×10^{-4}

8 Concluding remarks

We introduced the new concept of closedness under concentration, which is, in the context of distributional uncertainty, a sufficient condition to transform an optimization problem with a non-convex distortion riskmetric to its convex counterpart induced from the concave envelope of the distortion function. This concept is genuinely weaker than closedness under conditional expectation, and our main result unifies and improves many existing results in the literature. Many sets of plausible distributions commonly used in the literature of finance, optimization, and risk management are closed under concentration within some $\mathcal{I}$. Moreover, by focusing on distortion riskmetrics whose distortion functions are not necessarily monotone, concave, or continuous, we are able to solve optimization problems for the class of functionals larger than classical risk measures or deviation measures. In particular, we are able to obtain bounds on differences between two distortion riskmetrics, which represent measures of disagreement between two utilities/risk attitudes. Our result can also be applied to solve the popular problem of optimizing risk measures under moment

Table 5: Comparison of the numerical results of the two optimization problems (33) and (34) for TK distortion riskmetrics with $a = 0$ and $b = 1$

		c	V_h	time	V_{h^*}	time	$n\|\Delta \mathbf{a}^*\|_2$	ΔV	$\Delta V/V_h$ (%)
(i) Pareto	$n = 3$	1.0	1.076	144.75	1.185	0.23	0.488	0.109	10.2
	$n = 10$	2.0	1.047	220.03	1.237	1.42	0	0.190	18.1
	$n = 20$	4.0	1.301	826.64	1.501	8.24	0	0.200	15.4
(ii) Normal	$n = 3$	0.5	1.240	60.76	1.493	0.16	0.0784	0.253	20.4
	$n = 10$	0.5	1.141	246.31	1.363	0.72	1.28	0.222	19.4
	$n = 20$	0.5	1.103	1503.35	1.316	2.80	1.78	0.213	19.3
(iii) Exp	$n = 3$	1.0	1.305	49.79	1.427	0.23	0.360	0.122	9.32
	$n = 10$	2.0	1.313	198.43	1.484	1.62	0.184	0.171	13.0
	$n = 20$	2.0	1.120	850.12	1.286	10.91	0.158	0.166	14.8

Table 6: Comparison of the numerical results of the two optimization problems (33) and (34) for TK distortion riskmetrics with $a = 1/(2n)$ and $b = 2/n$

		c	V_h	time	V_{h^*}	time	$n\|\Delta \mathbf{a}^*\|_2$	ΔV	$\Delta V/V_h$ (%)
(i) Pareto	$n = 3$	1.0	1.077	73.21	1.185	0.25	0.469	0.109	10.11
	$n = 10$	2.0	1.047	248.38	1.237	2.29	0.378	0.191	18.2
	$n = 20$	4.0	1.301	638.24	1.501	12.21	0	0.200	15.4
(ii) Normal	$n = 3$	0.5	1.240	179.68	1.493	0.19	0.0784	0.253	20.4
	$n = 10$	0.5	1.146	389.97	1.363	0.76	0.660	0.217	19.0
	$n = 20$	0.5	1.103	1563.84	1.316	3.39	1.63	0.213	19.3
(iii) Exp	$n = 3$	1.0	1.304	52.66	1.430	0.25	0.107	0.126	9.65
	$n = 10$	2.0	1.312	236.15	1.485	2.27	0.214	0.172	13.1
	$n = 20$	2.0	1.119	879.73	1.289	10.10	0.141	0.170	15.2

constraints. In particular, we obtain the worst- and best-case distortion riskmetrics when the underlying random variable has a fixed mean and bounded p-th moment.

We demonstrate the applicability of our result by numerically calculating the solution to optimizing the difference between risk measures, preference robust optimization and portfolio optimization under marginal constraints. In all numerical examples, the original non-convex problem is converted or well approximated by a convex one which can be solved efficiently.

Our condition of closedness under concentration within $\mathcal{I}$ in Theorem 1 is sufficient but not necessary for the equivalence of optimization problems with non-convex distortion riskmetrics and convex ones induced from the concave envelopes of the distortion functions under distributional uncertainty. A necessary condition of the equivalence is closedness under concentration of the set of maximizers in Theorem 2. An open question is to find a *necessary and sufficient* condition on the uncertainty set $\mathcal{M}$ itself such that the desired equivalence holds. Pinning down such a condition may facilitate many more applications in decision theory, finance, game theory, and operations research.

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Statements and declarations

All authors declare that they have no conflicts of interest.

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Technical appendices

A Omitted technical details from the paper

In this appendix, we present technical details for some examples and as well as some technical remarks omitted from the paper.

A.1 Proofs of claims in some Examples

Proof of the claim in Example 7. We show that $\mathcal{M}^{\mathrm{mv}}(\mathbf{a},\boldsymbol{\mu},\Sigma)$ is equivalent to

$$\{F_S \in \mathcal{M}_2 : \mathbb{E}[S] = \mathbf{a}^\top\boldsymbol{\mu},\ \mathrm{var}(S) \leqslant \mathbf{a}^\top\Sigma\mathbf{a}\} = \mathcal{M}\left(2, \mathbf{a}^\top\boldsymbol{\mu}, \left(\mathbf{a}^\top\Sigma\mathbf{a}\right)^{1/2}\right).$$

For a proof of the equivalence between the sets with fixed mean and covariance matrix, see Popescu (2007). Indeed, it is clear that $\mathcal{M}^{\mathrm{mv}}(\mathbf{a},\boldsymbol{\mu},\Sigma) \subset \mathcal{M}(2, \mathbf{a}^\top\boldsymbol{\mu}, (\mathbf{a}^\top\Sigma\mathbf{a})^{1/2})$. On the other hand, for all $F_S \in \mathcal{M}(2, \mathbf{a}^\top\boldsymbol{\mu}, (\mathbf{a}^\top\Sigma\mathbf{a})^{1/2})$, we write $\mathbf{a} = (a_1,\dots,a_n)$, $\boldsymbol{\mu} = (\mu_1,\dots,\mu_n)$, and take $\mathbf{X} = (X_1,\dots,X_n)$ such that $X_i = (S - \mathbf{a}^\top\boldsymbol{\mu})/(na_i) + \mu_i$, for $i = 1,\dots,n$. It follows that $F_S = F_{\mathbf{a}^\top\mathbf{X}} \in \mathcal{M}^{\mathrm{mv}}(\mathbf{a},\boldsymbol{\mu},\Sigma)$. Therefore, we have $\mathcal{M}^{\mathrm{mv}}(\mathbf{a},\boldsymbol{\mu},\Sigma) = \mathcal{M}(2, \mathbf{a}^\top\boldsymbol{\mu}, (\mathbf{a}^\top\Sigma\mathbf{a})^{1/2})$. $\square$

Proof of the claim in Example 11 and Remark 3. We will show that $\mathcal{M}(\widetilde{G},\varepsilon)$ is closed under concentration within $\mathcal{I}$ for all $\mathcal{I} \subset \widetilde{\mathcal{I}}$. Write $\mathcal{I} = \{C_i : i \in K\}$ for some $K \subset \mathbb{N}$. For all $i \in K$ and $F \in \mathcal{M}(\widetilde{G},\varepsilon)$, we have $\widetilde{G}^{-1}(u) = c_i$ for $u \in C_i$ for some $c_i \in \mathbb{R}$. For all $i \in K$, by Jensen's inequality,

$$\frac{1}{\lambda(C_i)}\int_{C_i}\left|F^{-1}(u) - \widetilde{G}^{-1}(u)\right|^p \mathrm{d}u \geqslant \left|\frac{\int_{C_i} F^{-1}(u)\,\mathrm{d}u}{\lambda(C_i)} - c_i\right|^p = \frac{1}{\lambda(C_i)}\int_{C_i}\left|(F^{C_i})^{-1}(u) - \widetilde{G}^{-1}(u)\right|^p \mathrm{d}u.$$

It follows that

$$\begin{aligned}(W_p(F,\widetilde{G}))^p - (W_p(F^{C_i},\widetilde{G}))^p &= \int_0^1 \left|F^{-1}(u) - \widetilde{G}^{-1}(u)\right|^p \mathrm{d}u - \int_0^1 \left|(F^{C_i})^{-1}(u) - \widetilde{G}^{-1}(u)\right|^p \mathrm{d}u \\ &= \int_{C_i} \left|F^{-1}(u) - \widetilde{G}^{-1}(u)\right|^p \mathrm{d}u - \int_{C_i} \left|(F^{C_i})^{-1}(u) - \widetilde{G}^{-1}(u)\right|^p \mathrm{d}u \geqslant 0,\end{aligned}$$

and thus $W_p(F^{C_i},\widetilde{G}) \leqslant W_p(F,\widetilde{G}) \leqslant \varepsilon$. Moreover, (8) and the above argument lead to

$$(W_p(F,\widetilde{G}))^p - (W_p(F^{\mathcal{I}},\widetilde{G}))^p = \sum_{i\in K}(W_p(F,\widetilde{G}))^p - (W_p(F^{C_i},\widetilde{G}))^p \geqslant 0.$$

Hence, $W_p(F^{\mathcal{I}},\widetilde{G}) \leqslant W_p(F,\widetilde{G}) \leqslant \varepsilon$.

To show the converse statement in Remark 3, suppose that $\mathcal{M}(\widetilde{G},\varepsilon)$ is closed under concentration within $\mathcal{I}$ and suppose for contradiction that $\widetilde{G}^{-1}$ is not a constant on some $C \in \mathcal{I}$. Take

$F \in \mathcal{M}(\widetilde{G}, \varepsilon)$ such that $F^{-1} = \widetilde{G}^{-1} + \varepsilon$. Thus $(F^C)^{-1} = (\widetilde{G}^C)^{-1} + \varepsilon$. It follows that

$$\begin{aligned}(W_p(F,\widetilde{G}))^p - (W_p(F^C,\widetilde{G}))^p &= \int_C \left|F^{-1}(u) - \widetilde{G}^{-1}(u)\right|^p \mathrm{d}u - \int_C \left|(F^C)^{-1}(u) - \widetilde{G}^{-1}(u)\right|^p \mathrm{d}u \\ &= C\varepsilon^p - \int_C \left|(\widetilde{G}^C)^{-1}(u) - \widetilde{G}^{-1}(u) + \varepsilon\right|^p \mathrm{d}u \\ &< C\varepsilon^p - \left|\int_C (\widetilde{G}^C)^{-1}(u) - \widetilde{G}^{-1}(u) + \varepsilon \,\mathrm{d}u\right|^p = 0,\end{aligned}$$

where the inequality follows from Jensen's inequality and the strict sign is due to the fact that $(\widetilde{G}^C)^{-1} - \widetilde{G}^{-1}$ is not a constant on C. Hence, we have $\varepsilon = W_p(F,\widetilde{G}) < W_p(F^C,\widetilde{G})$ and thus $F^C \notin \mathcal{M}(\widetilde{G}, \varepsilon)$. This leads to a contradiction and hence $\widetilde{G}^{-1}$ must be a constant on each interval in $\mathcal{I}$. $\square$

Proof of the claim in Example 12. For $\varepsilon \geqslant 0$, $\mathbf{w} \in [0,\infty)^n$, $p > 1$, $a > 1$ and $\mathbf{Z} \in (\mathcal{L}^p)^n$, by Theorem 7 of Mao et al. (2022), the uncertainty set

$$\{F_{\mathbf{w}^\top \mathbf{X}} \in \mathcal{M}_p : F_{\mathbf{X}} \in \mathcal{M}^n(F_{\mathbf{Z}}, \varepsilon)\} = \mathcal{M}(F_{\mathbf{w}^\top \mathbf{Z}}, \varepsilon\|\mathbf{w}\|_b),$$

where b is the conjugate of a (i.e., $1/a + 1/b = 1$). Note that $\mathbb{P}(\mathbf{w}^\top \mathbf{Z} = 0) \geqslant p_0$ and the quantile function of $\mathbf{w}^\top \mathbf{Z}$ is equal to 0 on $(0, p_0]$. It follows from Example 11 that the set $\mathcal{M}(F_{\mathbf{w}^\top \mathbf{Z}}, \varepsilon\|\mathbf{w}\|_b)$ is closed under concentration within $\{(0,t)\}$ for all $t \leqslant p_0$. $\square$

Proof of the claim in Example 13. We will show that the set of distributions,

$$\mathcal{M} = \{F_{X - V(X) + g(\mathbb{E}[V(X)])} \in \mathcal{M}_1 : V \in \mathcal{V}\},$$

is closed under concentration within $\{(p,1)\}$ for all $p \in [p_0, 1)$. For each $V \in \mathcal{V}$ and a standard uniform random variable U, we write $a = \mathbb{E}[F^{-1}_{X-V(X)}(U) | U \in (p,1)]$. Since $F_X^{-1}(p) \geqslant l$, we can take

$$W(x) = V(x)\mathbb{1}_{\{x \leqslant F_X^{-1}(p)\}} + (x-a)\mathbb{1}_{\{x > F_X^{-1}(p)\}}, \quad x \in \mathbb{R}.$$

It follows that $W \in \mathcal{V}$. Noting that $a = \mathbb{E}[X - V(X) | X > F_X^{-1}(p)]$, we have

$$\begin{aligned}&X - W(X) + g(\mathbb{E}[W(X)]) \\ &= (X - V(X))\mathbb{1}_{\{X \leqslant F_X^{-1}(p)\}} + a\mathbb{1}_{\{X > F_X^{-1}(p)\}} + g\left(\mathbb{E}[V(X)\mathbb{1}_{\{X \leqslant F_X^{-1}(p)\}} + (X-a)\mathbb{1}_{\{X > F_X^{-1}(p)\}}]\right) \\ &= (X - V(X))\mathbb{1}_{\{X \leqslant F_X^{-1}(p)\}} + a\mathbb{1}_{\{X > F_X^{-1}(p)\}} + g(\mathbb{E}[V(X)]),\end{aligned}$$

which follows the same distribution as $F^{(p,1)}_{X-V(X)+g(\mathbb{E}[V(X)])}$. It follows that $\mathcal{M}$ is closed under concentration within $\{(p,1)\}$ for all $p \in [p_0, 1)$. $\square$

A.2 A few additional technical remarks mentioned in the paper

Remark 6 (on Theorem 1)**.** Using Theorem 1, if for some $\mathbf{a} \in A$, the set $\mathcal{M} := \{F_{f(\mathbf{a},\mathbf{X})} : F_{\mathbf{X}} \in \widetilde{\mathcal{M}}\}$ is closed under concentration for all intervals and $\sup\{\rho_{h^*}(f(\mathbf{a},\mathbf{X})) : F_{\mathbf{X}} \in \widetilde{\mathcal{M}}\} = \infty$, then $\sup\{\rho_h(f(\mathbf{a},\mathbf{X})) : F_{\mathbf{X}} \in \widetilde{\mathcal{M}}\} = \infty$. Thus, both objectives in the inner optimization of (1) are infinite for this $\mathbf{a}$, which can be excluded from the outer optimization over A. Verifying $\sup\{\rho_{h^*}(f(\mathbf{a},\mathbf{X})) : F_{\mathbf{X}} \in \widetilde{\mathcal{M}}\} = \infty$ is easier than verifying $\sup\{\rho_h(f(\mathbf{a},\mathbf{X})) : F_{\mathbf{X}} \in \widetilde{\mathcal{M}}\} = \infty$ since generally ρ_h is smaller than ρ_{h^*}.

Remark 7 (on Example 7)**.** Using Strassen's Theorem (e.g., Theorem 3.A.4 of Shaked and Shanthikumar (2007)), closedness under conditional expectation can equivalently be expressed using convex order. A set $\mathcal{M} \subset \mathcal{M}_1$ is closed under conditional expectation if and only if it holds that for $F \in \mathcal{M}$ and $G \leqslant_{\mathrm{cx}} F$, we have $G \in \mathcal{M}$.

Remark 8 (on Proposition 3)**.** In Proposition 3, if $\mathcal{M}$ is closed under conditional expectation, $\mathcal{I}$ can be taken as an infinite set. However, $\mathcal{M}$ may not be closed under concentration within an infinite $\mathcal{I}$ if we only assume that $\mathcal{M}$ is closed under concentration for all intervals. Indeed, if we take $\mathcal{M}$ as the set of distributions obtained by some $F \in \mathcal{M}$ with finitely many concentrations, then clearly $\mathcal{M}$ is closed under concentration for all intervals. However, $F^{\mathcal{I}} \notin \mathcal{M}$ when $\mathcal{I}$ is an infinite collection of disjoint intervals. This also serves as a counter-example of the converse statement of Proposition 2 since $\mathcal{M}$ is closed under concentration for all intervals but not closed under conditional expectation.

B Proofs of all technical results

We present all proofs of technical results in this appendix. Throughout, we denote the set of discontinuity points of h (excluding 0 and 1) by

$$J_h = \{t \in (0,1) : h(t) \neq h(t^+) \text{ or } h(t) \neq h(t^-)\}. \tag{A.1}$$

Note that $\hat{h}(t)$ can be written as

$$\hat{h}(t) = \begin{cases} h(t^+) \vee h(t^-) \vee h(t), & t \in J_h, \\ h(t), & \text{otherwise}. \end{cases} \tag{A.2}$$

B.1 Proof of results in Section 2

Proof of Proposition 1. Note that $(\hat{h})^* = h^* = \hat{h} = h$ on 0 and 1. For all $t \in (0,1)$, since $(\hat{h})^*(t) \geqslant \hat{h}(t) \geqslant h(t)$, we have $(\hat{h})^*(t) \geqslant h^*(t)$. On the other hand, we have $h^*(t) \geqslant h(t^+)$ for $t \in (0,1)$. Indeed, if $h^*(t_0) < h(t_0^+)$ for some $t_0 \in (0,1)$, then we have $h^*(t_0 + \varepsilon) < h(t_0 + \varepsilon)$ for some $\varepsilon > 0$, which leads to a contradiction. Similarly, we have $h^*(t) \geqslant h(t^-)$ for $t \in (0,1)$. Together with $h^* \geqslant h$ on $(0,1)$, we have $h^* \geqslant \hat{h}$ on $(0,1)$, which implies that $h^* \geqslant (\hat{h})^*$ on $(0,1)$. Therefore, $(\hat{h})^* = h^*$ on $[0,1]$.

Next, we assert that the set $\{t \in [0,1] : \hat{h}(t) \neq h^*(t)\}$ is a union of disjoint sets that are not singletons. To show this assertion, assume that the converse is true. There exists $x \in (0,1)$, such that $\hat{h}(x) < h^*(x)$ and $\hat{h}(t) = h^*(t)$ on $t \in (x-\varepsilon, x) \cup (x, x+\varepsilon)$ for some $0 < \varepsilon \leqslant x \wedge (1-x)$. It is clear that $x \in J_h$. Since h^* is continuous on $(x-\varepsilon, x+\varepsilon)$, we have

$$\hat{h}(x) < h^*(x) = h^*(x^+) = \hat{h}(x^+).$$

This contradicts (A.2). Therefore, the set $\{t \in [0,1] : \hat{h}(t) \neq h^*(t)\}$ is the union of some disjoint intervals, denoted by $\cup_{l \in L} A_l$ for some $L \subset \mathbb{N}$. For all $l \in L$, we denote the left and right endpoints of A_l by a_l and b_l, respectively, with $a_l < b_l$. Define a function via linear interpolation

$$h^c(t) = \begin{cases} \hat{h}(a_l) + \frac{\hat{h}(b_l) - \hat{h}(a_l)}{b_l - a_l}(t - a_l), & t \in A_l,\ l \in L, \\ \hat{h}(t), & \text{otherwise.} \end{cases}$$

It is clear that $h^c \leqslant h^*$ and h^c is continuous on $(0,1)$. We will prove that $h^c = h^*$ on $\cup_{l \in L} A_l$. Suppose for the purpose of contradiction that $h^c \neq h^*$ on $\cup_{l \in L} A_l$. Since $h^c < h^*$ for some point in $\cup_{l \in L} A_l$, there exists $x_0 \in A_l$ for some $l \in L$ such that $h^c(x_0) < \hat{h}(x_0)$. Thus we can take a point $(x_1, \hat{h}(x_1)) \in (0,1) \times \mathbb{R}$ with $\hat{h}(x_1) > h^c(x_1)$, which has the largest perpendicular distance to the straight line $h^c(t) = \hat{h}(a_l) + \frac{\hat{h}(b_l) - \hat{h}(a_l)}{b_l - a_l}(t - a_l)$, namely,

$$x_1 = \underset{\substack{x \in A_l \\ \hat{h}(x) > h^c(x)}}{\arg\max} \frac{(b_l - a_l)\hat{h}(x) - (\hat{h}(b_l) - \hat{h}(a_l))x - (b_l - a_l)\hat{h}(a_l) + (\hat{h}(b_l) - \hat{h}(a_l))a_l}{\left((\hat{h}(b_l) - \hat{h}(a_l))^2 + (b_l - a_l)^2\right)^{1/2}}.$$

The existence of the maximizer x_1 is due to the upper semicontinuity of $\hat{h}$. There exists a function g with $g = h^*$ on $[0,1] \setminus A_l$ and $g(x_1) = \hat{h}(x_1)$, such that g is concave and $\hat{h} \leqslant g \leqslant h^*$ on $[0,1]$. Since $h^* > \hat{h}$ on A_l, we have $h^*(x_1) > \hat{h}(x_1) = g(x_1)$. Thus h^* cannot be the concave envelope of $\hat{h}$, which leads to a contradiction. Thus, $h^* = h^c$ on $\cup_{l \in L} A_l$. Since $h^* = \hat{h} = h^c$ on $(0,1) \setminus (\cup_{l \in L} A_l)$, we have $h^* = h^c$. Therefore, $\{t \in [0,1] : \hat{h}(t) \neq h^*(t)\}$ is a union of disjoint open intervals, and h^* is linear on each of the intervals. $\square$

B.2 Proofs of results in Section 3

Proof of Theorem 1. We will first show that, assuming that $\mathcal{M}$ is closed under concentration within $\mathcal{I}_h$, we have

$$\sup_{F_X \in \mathcal{M}} \rho_{\hat{h}}(X) = \sup_{F_X \in \mathcal{M}} \rho_{h^*}(X). \tag{A.3}$$

After proving (A.3), we show the three statements in Theorem 1 in the order (i), (ii), and (iii).

For $h \in \mathcal{H}$, suppose that $\mathcal{M}$ is closed under concentration within $\mathcal{I}_h$. Take an arbitrary random variable Y with $F_Y \in \mathcal{M}$. Let $G = F_Y^{\mathcal{I}_h}$. For $h \in \mathcal{H}$, write functions $g(t) = 1 - \hat{h}(1-t)$ and $g_*(t) = 1 - h^*(1-t)$ for $t \in [0,1]$. By definition of $\mathcal{I}_h$, $g \neq g_*$ on each set in $\mathcal{I}_h$ and $g = g_*$ on other sets. For any $(a,b) \in \mathcal{I}_h$, we have $G^{-1}(t) = \frac{\int_a^b F_Y^{-1}(u)\,\mathrm{d}u}{b-a}$ for all $t \in (a,b]$ and $G^{-1+}(t) = \frac{\int_a^b F_Y^{-1}(u)\,\mathrm{d}u}{b-a}$

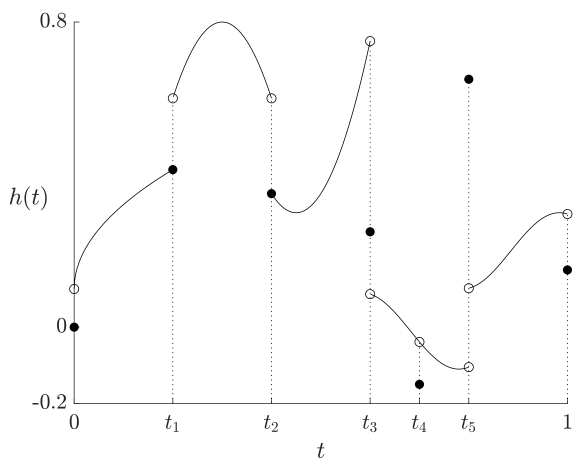
for all $t \in [a,b)$. Using the fact that g_* is linear on (a,b) and $g(t) = g_*(t)$ for $t = a, b$, we have

$$\begin{aligned}\int_{(a,b)} F_Y^{-1}(t)\,\mathrm{d}g_*(t) &= (g_*(b) - g_*(a))\frac{\int_a^b F_Y^{-1}(t)\,\mathrm{d}t}{b-a} \\ &= (g(b) - g(a))\frac{\int_a^b F_Y^{-1}(t)\,\mathrm{d}t}{b-a} \\ &= \int_{(a,b]} G^{-1}(t)\,\mathrm{d}g(t) + G^{-1+}(a)(g(a^+) - g(a)).\end{aligned} \tag{A.4}$$

Define the sets

$$J_+ = \{t \in J_h : \hat{h}(t^+) = \hat{h}(t) \neq \hat{h}(t^-)\}, \quad J_- = \{t \in J_h : \hat{h}(t^+) \neq \hat{h}(t) = \hat{h}(t^-)\},$$
$$\text{and} \quad J_0 = \{t \in J_h : \hat{h}(t^+) \neq \hat{h}(t) \neq \hat{h}(t^-)\}.$$

To better understand these sets, we recall Figure 1 (without concave envelopes) as Figure A.1 to demonstrate an example of a distortion function h, the corrresponding $\hat{h}$, the sets J_h, J_+, J_-, and J_0, and the sets $\hat{J}$, $\hat{J}_+$, $\hat{J}_-$, $\hat{J}^0_+$, and $\hat{J}^0_-$ (defined in the proof of (i) below).



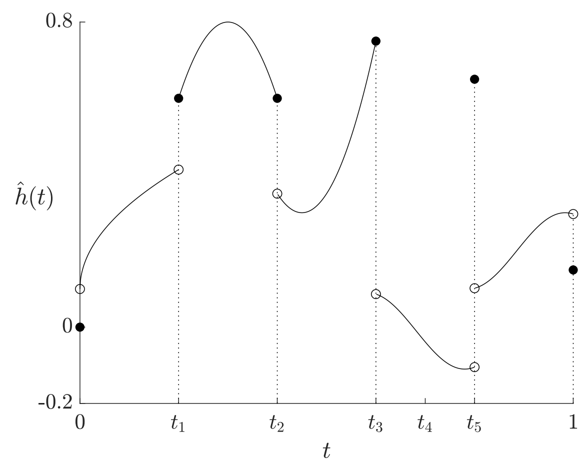


Figure A.1: An example of h (left) and $\hat{h}$ (right); in this figure, $J_h = \{t_1, t_2, t_3, t_4, t_5\}$, $J_+ = \{t_1\}$, $J_- = \{t_2, t_3\}$, and $J_0 = \{t_5\}$. Moreover, the sets we use in the proof of (i) are $\hat{J} = \{t_1, t_2, t_3, t_4\}$, $\hat{J}_+ = \{t_1, t_4\}$, $\hat{J}_- = \{t_2, t_3\}$, $\hat{J}^0_+ = \{t_4\}$, and $\hat{J}^0_- = \{t_3\}$

Note that for a random variable $Z_{\mathcal{I}_h} \sim F_Y^{\mathcal{I}_h}$, we have

$$\rho_{\hat{h}}(Z_{\mathcal{I}_h}) = \int_{(0,1]\setminus(J_+\cup J_0)} G^{-1}(t)\,\mathrm{d}g(t) + \sum_{t \in J_+\cup J_0\cup\{0\}} G^{-1+}(t)(g(t^+) - g(t)).$$

Hence using (A.4) and (8), we get

$$\begin{aligned}
&\rho_{h^*}(Y) - \rho_{\hat{h}}(Z_{\mathcal{I}_h}) \\
&= \int_0^1 F_Y^{-1}(t)\,\mathrm{d}g_*(t) + F_Y^{-1+}(0)(g_*(0^+) - g_*(0)) \\
&\quad - \int_{(0,1]\setminus(J_+\cup J_0)} G^{-1}(t)\,\mathrm{d}g(t) - \sum_{t\in J_+\cup J_0\cup\{0\}} G^{-1+}(t)(g(t^+) - g(t)) \\
&= \sum_{(a,b)\in\mathcal{I}_h}\left(\int_{(a,b)} F_Y^{-1}(t)\,\mathrm{d}g_*(t) - \int_{(a,b]} G^{-1}(t)\,\mathrm{d}g(t) - G^{-1+}(a)(g(a^+) - g(a))\right) = 0.
\end{aligned} \tag{A.5}$$

Since $\mathcal{M}$ is closed under concentration within $\mathcal{I}_h$, we have $F_Y^{\mathcal{I}_h} \in \mathcal{M}$ by definition. Thus we have

$$\rho_{h^*}(Y) = \rho_{\hat{h}}(Z_{\mathcal{I}_h}) \leqslant \sup_{F_X\in\mathcal{M}} \rho_{\hat{h}}(X),$$

which gives our desired equality (A.3) since $\rho_{h^*} = \rho_{(\hat{h})^*} \geqslant \rho_{\hat{h}}$.

Proof of (i): Using $h = \hat{h}$ and (A.3), we have $\sup_{F_X\in\mathcal{M}} \rho_h(X) = \sup_{F_X\in\mathcal{M}} \rho_{h^*}(X)$.

Proof of (ii): We will prove (ii) in two main steps. First, we show that (ii) holds if $\mathcal{I}_h$ is finite and h has finitely many discontinuity points. Next, we discuss general h.

Finite case: Here we prove (9) under the case where $\mathcal{I}_h$ is finite and h has finitely many discontinuity points (i.e. J_h in (A.1) is a finite set). Suppose that $\mathcal{M}$ is closed under concentration for all intervals, it directly implies that $\mathcal{M}$ is closed under concentration within $\mathcal{I}_h$ by Proposition 3. Therefore, (A.3) holds for all $h \in \mathcal{H}$. Next, we need to show that $\sup_{F_X\in\mathcal{M}} \rho_h(X) = \sup_{F_X\in\mathcal{M}} \rho_{\hat{h}}(X)$. Define

$$\hat{J} = \{t \in J_h : \hat{h}(t) \neq h(t)\},\quad \hat{J}_+ = \{t \in \hat{J} : \hat{h}(t) = \hat{h}(t^+)\},\quad \text{and}\quad \hat{J}_- = \hat{J}\setminus\hat{J}_+.$$

For $n > 0$, write intervals

$$A_s^n = \begin{cases} (1-s-1/\sqrt{n}, 1-s+1/n), & s\in\hat{J}_-, \\ (1-s-1/n, 1-s+1/\sqrt{n}), & s\in\hat{J}_+. \end{cases}$$

Let $\mathcal{I}^n = \{A_s^n : s \in \hat{J}\}$. Note that $h \in \mathcal{H}$ has finitely many discontinuity points. Thus the intervals in $\mathcal{I}^n$ are disjoint when n is large enough. For all $F_Y \in \mathcal{M}$ and $Y \sim F_Y$, we define

$$Z_{\mathcal{I}^n} = F_Y^{-1}(U)\mathbb{1}_{\{U\notin\bigcup_{s\in\hat{J}} A_s^n\}} + \sum_{s\in\hat{J}} \mathbb{E}[F_Y^{-1}(U)|U\in A_s^n]\mathbb{1}_{\{U\in A_s^n\}}.$$

It follows that $Z_{\mathcal{I}^n} \sim F_Y^{\mathcal{I}^n}$ and the right-quantile function of $Z_{\mathcal{I}^n}$, denoted by G_n^{-1+}, is given by the right-continuous adjusted version of

$$F_Y^{-1+}(t)\mathbb{1}_{\{t\notin\bigcup_{s\in\hat{J}} A_s^n\}} + \sum_{s\in\hat{J}} \frac{\int_{A_s^n} F_Y^{-1}(u)\,\mathrm{d}u}{\lambda(A_s^n)}\mathbb{1}_{\{t\in A_s^n\}},\quad t\in(0,1).$$

Thus we get

$$\lim_{n\to\infty} G_n^{-1+}(1-t) = \begin{cases} F_Y^{-1}(1-t), & t \in \hat{J}_-, \\ F_Y^{-1+}(1-t), & \text{otherwise.} \end{cases}$$

Similarly, if we denote the left-quantile function of $Z_{\mathcal{I}^n}$ by G_n^{-1}, then G_n^{-1} is given by the left-continuous version of

$$F_Y^{-1}(t)\mathbb{1}_{\{t\notin \bigcup_{s\in\hat{J}} A_s^n\}} + \sum_{s\in\hat{J}} \frac{\int_{A_s^n} F_Y^{-1}(u)\,\mathrm{d}u}{\lambda(A_s^n)}\mathbb{1}_{\{t\in A_s^n\}}.$$

It follows that

$$\lim_{n\to\infty} G_n^{-1}(1-t) = \begin{cases} F_Y^{-1+}(1-t), & t \in \hat{J}_+, \\ F_Y^{-1}(1-t), & \text{otherwise.} \end{cases}$$

Define, further, the sets

$$\hat{J}_+^0 = \{t \in \hat{J}_+ : h(t) \neq h(t^-)\} \ \text{ and } \ \hat{J}_-^0 = \{t \in \hat{J}_- : h(t) \neq h(t^+)\}.$$

For $u \in [0,1]$, write

$$\begin{aligned}
h_-(u) &= \sum_{t\in\hat{J}_-} (h(t)-h(t^-))\mathbb{1}_{\{u\geqslant t\}}, \quad h_-^0(u) = \sum_{t\in\hat{J}_-^0} (h(t^+)-h(t))\mathbb{1}_{\{u>t\}}, \\
h_+(u) &= \sum_{t\in\hat{J}_+} (h(t^+)-h(t))\mathbb{1}_{\{u>t\}}, \quad h_+^0(u) = \sum_{t\in\hat{J}_+^0} (h(t)-h(t^-))\mathbb{1}_{\{u\geqslant t\}}, \\
\hat{h}_-(u) &= \sum_{t\in\hat{J}_-} (h(t^+)-h(t^-))\mathbb{1}_{\{u>t\}}, \quad \hat{h}_+(u) = \sum_{t\in\hat{J}_+} (h(t^+)-h(t^-))\mathbb{1}_{\{u\geqslant t\}}, \\
\text{and } \ h_0(u) &= h(u)-h_+(u)-h_-(u)-h_+^0(u)-h_-^0(u) = \hat{h}(u)-\hat{h}_+(u)-\hat{h}_-(u).
\end{aligned}$$

Note that $|Z_{\mathcal{I}^n} - F_Y^{-1}(U)| = 0$ when $U \notin \bigcup_{s\in\hat{J}} A_s^n$ and $0,1 \in [0,1]\setminus\bigcup_{s\in\hat{J}} A_s^n$. We have $|Z_{\mathcal{I}^n} - F_Y^{-1}(U)| < \infty$. Therefore, by the dominated convergence theorem,

$$\begin{aligned}
&\lim_{n\to\infty} (\rho_{h_-}(Z_{\mathcal{I}^n}) + \rho_{h_-^0}(Z_{\mathcal{I}^n})) \\
&= \lim_{n\to\infty} \int_0^1 G_n^{-1+}(1-u)\,\mathrm{d}h_-(u) + \lim_{n\to\infty} \int_0^1 G_n^{-1}(1-u)\,\mathrm{d}h_-^0(u) \\
&= \sum_{t\in\hat{J}_-} F_Y^{-1}(1-t)(h(t)-h(t^-)) + \sum_{t\in\hat{J}_-^0} F_Y^{-1}(1-t)(h(t^+)-h(t)) \\
&= \sum_{t\in\hat{J}_-\setminus\hat{J}_-^0} F_Y^{-1}(1-t)(h(t)-h(t^-)) + \sum_{t\in\hat{J}_-^0} F_Y^{-1}(1-t)(h(t)-h(t^-)+h(t^+)-h(t)) \\
&= \sum_{t\in\hat{J}_-\setminus\hat{J}_-^0} F_Y^{-1}(1-t)(h(t^+)-h(t^-)) + \sum_{t\in\hat{J}_-^0} F_Y^{-1}(1-t)(h(t^+)-h(t^-)) = \rho_{\hat{h}_-}(Y).
\end{aligned}$$

Similarly, we get $\lim_{n\to\infty}(\rho_{h_+}(Z_{\mathcal{I}^n}) + \rho_{h_+^0}(Z_{\mathcal{I}^n})) = \rho_{\hat{h}_+}(Y)$. On the other hand, it is clear that

$\lim_{n\to\infty} \rho_{h_0}(Z_{\mathcal{I}^n}) = \rho_{h_0}(Y)$. Therefore, we have

$$\begin{aligned}\lim_{n\to\infty} \rho_h(Z_{\mathcal{I}^n}) &= \lim_{n\to\infty} (\rho_{h_-}(Z_{\mathcal{I}^n}) + \rho_{h_-^0}(Z_{\mathcal{I}^n}) + \rho_{h_+}(Z_{\mathcal{I}^n}) + \rho_{h_+^0}(Z_{\mathcal{I}^n}) + \rho_{h_0}(Z_{\mathcal{I}^n}))\\ &= \rho_{\hat{h}_-}(Y) + \rho_{\hat{h}_+}(Y) + \rho_{h_0}(Y) = \rho_{\hat{h}}(Y).\end{aligned}$$

Thus we have

$$\rho_{\hat{h}}(Y) = \lim_{n\to\infty} \rho_h(Z_{\mathcal{I}^n}) \leqslant \sup_{F_X\in\mathcal{M}} \rho_h(X). \tag{A.6}$$

Using (A.3) and (A.6), we get

$$\sup_{F_X\in\mathcal{M}} \rho_{h^*}(X) = \sup_{F_X\in\mathcal{M}} \rho_{\hat{h}}(X) \leqslant \sup_{F_X\in\mathcal{M}} \rho_h(X).$$

General case: We prove Theorem 1 for all general $h \in \mathcal{H}$ where $\mathcal{I}_h$ or the number of discontinuity points of h is countable.

1. If $\mathcal{I}_h$ is countable, it suffices to prove (A.3). We write $\mathcal{I}_h$ as the collection of (a_i, b_i) for $i \in \mathbb{N}$ and define $\mathcal{I}_1^n = \{(a_i, b_i) : i = 1, \dots, n\}$ for all $n \in \mathbb{N}$. Define the function

$$h_n(t) = \begin{cases} h^*(t), & t \in (1 - b_i, 1 - a_i),\ i = 1, \dots, n,\\ \hat{h}(t), & \text{otherwise.}\end{cases}$$

It is clear that for all $n \in \mathbb{N}$, the set $\{t \in [0, 1] : h_n(t) \neq \hat{h}(t)\}$ is a finite union of disjoint open intervals and h_n is linear on each of the intervals. For all random variables Y with $F_Y \in \mathcal{M}$, let random variable $Z_{\mathcal{I}_1^n} \sim F_Y^{\mathcal{I}_1^n}$. Similar to (A.3), we have

$$\rho_{h_n}(Y) = \rho_{\hat{h}}(Z_{\mathcal{I}_1^n}) \leqslant \sup_{F_X\in\mathcal{M}} \rho_{\hat{h}}(X), \quad \text{for all } n \in \mathbb{N}.$$

Note that $h_n(t) \uparrow h^*(t)$ as $n \to \infty$ for all $t \in (0, 1)$. By the monotone convergence theorem, we get $\rho_{h_n}(Y) \to \rho_{h^*}(Y)$ as $n \to \infty$. It follows that

$$\sup_{F_X\in\mathcal{M}} \rho_{\hat{h}}(X) \geqslant \rho_{h_n}(Y) \xrightarrow{n\to\infty} \rho_{h^*}(Y).$$

2. If $h \in \mathcal{H}$ has countably many discontinuity points, it suffices to prove (A.6). There exist series of finite sets $\{\hat{J}^m\}_{m\in\mathbb{N}} \subset \hat{J}$, such that $\hat{J}^m \to \hat{J}$ as $m \to \infty$. For all $m \in \mathbb{N}$, write

$$\hat{h}_m(t) = \begin{cases} \hat{h}(t), & t \in \hat{J}^m,\\ h(t), & \text{otherwise,}\end{cases}$$

and define

$$\hat{J}_+^m = \{t \in \hat{J}^m : \hat{h}_m(t) = \hat{h}_m(t^+)\}, \quad \text{and} \quad \hat{J}_-^m = \hat{J}^m \setminus \hat{J}_+^m.$$

For $n>0$, let $\mathcal{I}_2^{n,m}=\{B_s^{n,m}: i\in \hat{J}^m\}$ with

$$B_s^{n,m}=\begin{cases}(1-s-1/\sqrt{n},1-s+1/n), & s\in \hat{J}_-^m,\\ (1-s-1/n,1-s+1/\sqrt{n}), & s\in \hat{J}_+^m.\end{cases}$$

Following the same argument as (A.6), for all random variable Y with $F_Y\in\mathcal{M}$, we have

$$\sup_{F_X\in\mathcal{M}}\rho_h(X)\geqslant \rho_h(Z_{\mathcal{I}_2^{n,m}})\xrightarrow{n\to\infty}\rho_{\hat{h}_m}(Y),\quad \text{for all } m\in\mathbb{N},$$

where $Z_{\mathcal{I}_2^{n,m}}\sim F_Y^{\mathcal{I}_2^{n,m}}$. Moreover, we have $\hat{h}_m(t)\uparrow \hat{h}(t)$ for all $t\in[0,1]$ as $m\to\infty$. By the monotone convergence theorem, we have $\rho_{\hat{h}_m}(Y)\to\rho_{\hat{h}}(Y)$ as $m\to\infty$. Therefore, we have

$$\sup_{F_X\in\mathcal{M}}\rho_{\hat{h}}(X)\leqslant \sup_{F_X\in\mathcal{M}}\rho_h(X).$$

Proof of (iii): For all $h\in\mathcal{H}$, if $\mathcal{M}$ is closed under concentration within $\mathcal{I}_h$ and $h=\hat{h}$, we have $F_Y^{\mathcal{I}_h}\in\mathcal{M}$ by definition. Since $Z_{\mathcal{I}_h}\sim F_Y^{\mathcal{I}_h}$, (A.5) gives

$$\rho_{h^*}(Y)=\rho_{\hat{h}}(Z_{\mathcal{I}_h})=\rho_h(Z_{\mathcal{I}_h}).$$

Note that $\rho_h\leqslant\rho_{h^*}$ generally. Therefore, if $\max_{F_Y\in\mathcal{M}}\rho_{h^*}(Y)$ is attained by F_Y, then so is $\max_{F_Y\in\mathcal{M}}\rho_h(Y)$ by $F_Y^{\mathcal{I}_h}$. Obviously, these two quantities share a common maximizer $F_Y^{\mathcal{I}_h}$ because

$$\rho_{h^*}(Z_{\mathcal{I}_h})\leqslant\max_{F_Y\in\mathcal{M}}\rho_{h^*}(Y)=\max_{F_Y\in\mathcal{M}}\rho_h(Y)=\rho_h(Z_{\mathcal{I}_h})\leqslant\rho_{h^*}(Z_{\mathcal{I}_h}).$$

The proof is complete. $\square$

Proof of Theorem 2. Suppose for contradiction that $\mathcal{M}_{\text{opt}}$ is not closed under concentration within $\mathcal{I}_h$. There exists $F_Y\in\mathcal{M}_{\text{opt}}$, such that $F_Y^{\mathcal{I}_h}\notin\mathcal{M}_{\text{opt}}$. Define the set $\mathcal{Y}_h=\{(F_Y^{-1}(a),F_Y^{-1}(b)):(a,b)\in\mathcal{I}_h\}$. Since $F_Y^{\mathcal{I}_h}\notin\mathcal{M}_{\text{opt}}$, there exists an interval $(a,b)\in\mathcal{I}_h$, such that F_Y^{-1} is not constant on (a,b). Thus the Lebesgue measure $\lambda((F_Y^{-1}(a),F_Y^{-1}(b)))>0$. Since $h^*>h$ on (a,b),

$$\begin{aligned}\rho_{h^*}(Y)-\rho_h(Y)&=\int_{\mathbb{R}}(h^*(\mathbb{P}(Y>x))-h(\mathbb{P}(Y>x)))\,\mathrm{d}x\\&=\sum_{A\in\mathcal{Y}_h}\int_A(h^*(\mathbb{P}(Y>x))-h(\mathbb{P}(Y>x)))\,\mathrm{d}x>0.\end{aligned}\tag{A.7}$$

On the other hand, we have

$$\rho_{h^*}(Y)\leqslant\sup_{F_X\in\mathcal{M}}\rho_{h^*}(X)=\sup_{F_X\in\mathcal{M}}\rho_h(X)=\rho_h(Y)\leqslant\rho_{h^*}(Y),$$

which leads to a contradiction to (A.7). Therefore, $\mathcal{M}_{\text{opt}}$ is closed under concentration within $\mathcal{I}_h$. $\square$

Proof of Proposition 2. We first prove that closedness under conditional expectation implies closedness under concentration for all intervals. For all random variables $Y \in \mathcal{L}^1$ and intervals $C \subset [0,1]$, define

$$X = F_Y^{-1}(U)\mathbb{1}_{\{U \notin C\}} + \mathbb{E}[F_Y^{-1}(U)|U \in C]\mathbb{1}_{\{U \in C\}},$$

where $U \sim \mathrm{U}[0,1]$. The distribution of X is the concentration F_Y^C. For all $\sigma(X)$-measurable random variables Z, we have that $Z|\{U \in C\}$ is constant. Hence,

$$\begin{aligned}
\mathbb{E}[XZ] &= \mathbb{E}[ZF_Y^{-1}(U)\mathbb{1}_{\{U \notin C\}} + Z\mathbb{E}[F_Y^{-1}(U)|U \in C]\mathbb{1}_{\{U \in C\}}] \\
&= \mathbb{E}[ZF_Y^{-1}(U)\mathbb{1}_{\{U \notin C\}}] + \mathbb{E}[\mathbb{E}[ZF_Y^{-1}(U)|U \in C]\mathbb{1}_{\{U \in C\}}] \\
&= \mathbb{E}[ZF_Y^{-1}(U)\mathbb{1}_{\{U \notin C\}}] + \mathbb{E}[ZF_Y^{-1}(U)|U \in C]\mathbb{P}(U \in C) \\
&= \mathbb{E}[ZF_Y^{-1}(U)\mathbb{1}_{\{U \notin C\}}] + \mathbb{E}[ZF_Y^{-1}(U)\mathbb{1}_{\{U \in C\}}] = \mathbb{E}[ZF_Y^{-1}(U)].
\end{aligned}$$

It follows that $\mathbb{E}[Y|X] = \mathbb{E}[F_Y^{-1}(U)|X] = X$, $\mathbb{P}$-almost surely. If a set of distributions, $\mathcal{M}$, is closed under conditional expectation and $F_Y \in \mathcal{M}$, then $F_{\mathbb{E}[Y|X]} \in \mathcal{M}$, which implies that $F_Y^C = F_X \in \mathcal{M}$. Thus $\mathcal{M}$ is also closed under concentration for all intervals.

For counter-examples showing that the converse statement does not hold in general, see Example 6 and Remark 8. □

Proof of Proposition 3. (i) Suppose that $\mathcal{M}$ is closed under concentration for all intervals and $\mathcal{I}$ is a finite. Using (7), we can see that $F^{\mathcal{I}}$ is the resulting distribution obtained by sequentially applying finitely many C-concentrations to F over all $C \in \mathcal{I}$. We thus have $F^{\mathcal{I}} \in \mathcal{M}$ for all $F \in \mathcal{M}$.

(ii) Suppose that $\mathcal{M}$ is closed under conditional expectation and $F \in \mathcal{M}$. We define

$$X = F^{-1}(U)\mathbb{1}_{\{U \notin \bigcup_{C \in \mathcal{I}} C\}} + \sum_{C \in \mathcal{I}} \mathbb{E}[F^{-1}(U)|U \in C]\mathbb{1}_{\{U \in C\}},$$

whose left-quantile function is given by (8) according to (7). Following similar argument to the proof of Proposition 2, for all $\sigma(X)$-measurable random variables Z, we have

$$\begin{aligned}
\mathbb{E}[XZ] &= \mathbb{E}[ZF^{-1}(U)\mathbb{1}_{\{U \notin \bigcup_{C \in \mathcal{I}} C\}} + \sum_{C \in \mathcal{I}} Z\mathbb{E}[F^{-1}(U)|U \in C]\mathbb{1}_{\{U \in C\}}] \\
&= \mathbb{E}[ZF^{-1}(U)\mathbb{1}_{\{U \notin \bigcup_{C \in \mathcal{I}} C\}}] + \sum_{C \in \mathcal{I}} \mathbb{E}[\mathbb{E}[ZF^{-1}(U)|U \in C]\mathbb{1}_{\{U \in C\}}] \\
&= \mathbb{E}[ZF^{-1}(U)\mathbb{1}_{\{U \notin \bigcup_{C \in \mathcal{I}} C\}}] + \sum_{C \in \mathcal{I}} \mathbb{E}[ZF^{-1}(U)\mathbb{1}_{\{U \in C\}}] = \mathbb{E}[ZF^{-1}(U)].
\end{aligned}$$

Thus $\mathbb{E}[F^{-1}(U)|X] = X$, $\mathbb{P}$-almost surely, which implies that $F^{\mathcal{I}} = F_X \in \mathcal{M}$. □

B.3 Proofs of results in Section 4

Proof of Theorem 3. To prove the first statement, according to the proof of Theorem 1, it suffices to show that for all increasing $h \in \mathcal{H}$, $\mathbf{X} \in (\mathcal{L}^1)^n$ and $\mathscr{G} \subset \mathscr{F}$, $\rho_h(\mathbb{E}[f(\mathbf{a}, \mathbf{X})|\mathscr{G}]) \leqslant \rho_h(f(\mathbf{a}, \mathbb{E}[\mathbf{X}|\mathscr{G}]))$, which holds directly by Jensen's inequality and monotonicity of ρ_h. The second statement holds by Theorem 1. The last statement follows from $\rho_h(\mathbb{E}[f(\mathbf{a},\mathbf{X})|\mathscr{G}]) = \rho_h(f(\mathbf{a},\mathbb{E}[\mathbf{X}|\mathscr{G}]))$ and using Theorem 1. $\square$

Proof of Theorem 4. (i) For all $\mathbf{X} = (X_1,\dots,X_n) \in (\mathcal{L}^1)^n$, take a comonotonic $\widetilde{\mathbf{X}} = (\widetilde{X}_1,\dots,\widetilde{X}_n) \in (\mathcal{L}^1)^n$ such that $\widetilde{X}_i \stackrel{\mathrm{d}}{=} X_i$ for all $i = 1,\dots,n$. It follows that $\mathbb{E}[g(\mathbf{X})] \leqslant \mathbb{E}[g(\widetilde{\mathbf{X}})]$ for all supermodular functions $g: \mathbb{R}^n \to \mathbb{R}$ due to Theorem 5 of Tchen (1980). By Proposition 2.2.5 of Simchi-Levi et al. (2005), we have $f(\mathbf{a},\mathbf{X}) \leqslant_{\mathrm{icx}} f(\mathbf{a},\widetilde{\mathbf{X}})$. Moreover, there exists a standard uniform random variable U such that $\widetilde{X}_i = F^{-1}_{\widetilde{X}_i}(U)$ for all $i = 1,\dots,n$ and $f(\mathbf{a},\widetilde{\mathbf{X}}) = F^{-1}_{f(\mathbf{a},\widetilde{\mathbf{X}})}(U)$ almost surely (Denneberg, 1994). Take

$$f(\mathbf{a},\widetilde{\mathbf{X}})^{\mathcal{I}_h} = F^{-1}_{f(\mathbf{a},\widetilde{\mathbf{X}})}(U)\mathbb{1}_{\{U\notin\bigcup_{C\in\mathcal{I}_h}C\}} + \sum_{C\in\mathcal{I}_h}\mathbb{E}[F^{-1}_{f(\mathbf{a},\widetilde{\mathbf{X}})}(U)|U\in C]\mathbb{1}_{\{U\in C\}} \sim F^{\mathcal{I}_h}_{f(\mathbf{a},\widetilde{\mathbf{X}})}.$$

It follows that $f(\mathbf{a},\widetilde{\mathbf{X}})^{\mathcal{I}_h} = \mathbb{E}[f(\mathbf{a},\widetilde{\mathbf{X}})|\mathscr{G}]$, where $\mathscr{G} = \sigma(U\mathbb{1}_{\{U\notin\bigcup_{C\in\mathcal{I}_h}C\}})$. Similarly, $\widetilde{X}_i^{\mathcal{I}_h} = \mathbb{E}[\widetilde{X}_i|\mathscr{G}]$ for all $i = 1,\dots,n$, where

$$\widetilde{X}_i^{\mathcal{I}_h} = F^{-1}_{\widetilde{X}_i}(U)\mathbb{1}_{\{U\notin\bigcup_{C\in\mathcal{I}_h}C\}} + \sum_{C\in\mathcal{I}_h}\mathbb{E}[F^{-1}_{\widetilde{X}_i}(U)|U\in C]\mathbb{1}_{\{U\in C\}} \sim F^{\mathcal{I}_h}_{\widetilde{X}_i}.$$

Since f is supermodular and positively homogeneous, we have by Theorem 3 of Marinacci and Montrucchio (2008) that $f(\mathbf{a},\mathbf{X})$ is concave in $\mathbf{X}$. By Jensen's inequality, we have

$$f(\mathbf{a},\widetilde{\mathbf{X}})^{\mathcal{I}_h} = \mathbb{E}[f(\mathbf{a},\widetilde{\mathbf{X}})|\mathscr{G}] \leqslant f(\mathbf{a},\mathbb{E}[\widetilde{\mathbf{X}}|\mathscr{G}]) = f(\mathbf{a},\widetilde{X}_1^{\mathcal{I}_h},\dots,\widetilde{X}_n^{\mathcal{I}_h}).$$

Thus we have

$$\begin{aligned}\rho_{h^*}(f(\mathbf{a},\mathbf{X})) \leqslant \rho_{h^*}(f(\mathbf{a},\widetilde{\mathbf{X}})) = \rho_h(f(\mathbf{a},\widetilde{\mathbf{X}})^{\mathcal{I}_h}) &\leqslant \rho_h(f(\mathbf{a},\widetilde{X}_1^{\mathcal{I}_h},\dots,\widetilde{X}_n^{\mathcal{I}_h}))\\ &\leqslant \sup_{F_1\in\mathcal{F}_1,\dots,F_n\in\mathcal{F}_n}\ \sup_{F_{\mathbf{Y}}\in\mathcal{D}(F_1,\dots,F_n)} \rho_h(f(\mathbf{a},\mathbf{Y})),\end{aligned}$$

where the first inequality follows from Theorem 4.A.3 of Shaked and Shanthikumar (2007) and Theorem 5 of Wang et al. (2020a) and the second equality is by the proof of Theorem 1. Combined with the fact that

$$\sup_{F_1\in\mathcal{F}_1,\dots,F_n\in\mathcal{F}_n}\ \sup_{F_{\mathbf{X}}\in\mathcal{D}(F_1,\dots,F_n)} \rho_h(f(\mathbf{a},\mathbf{X})) \leqslant \sup_{F_1\in\mathcal{F}_1,\dots,F_n\in\mathcal{F}_n}\ \sup_{F_{\mathbf{X}}\in\mathcal{D}(F_1,\dots,F_n)} \rho_{h^*}(f(\mathbf{a},\mathbf{X})),$$

we have (18) holds.

(ii) Suppose that the supremum of the right-hand side of (18) is attained by some $F_1 \in \mathcal{F}_1,\dots,F_n \in \mathcal{F}_n$ and $F_{\mathbf{X}} \in \mathcal{D}(F_1,\dots,F_n)$. For comonotonic $(\widetilde{X}_1,\dots,\widetilde{X}_n)$ such that $\widetilde{X}_i \sim F_i$ for all $i = 1,\dots,n$, using the argument in (i),

$$\rho_{h^*}(f(\mathbf{a},\mathbf{X})) \leqslant \rho_h(f(\mathbf{a},\widetilde{X}_1^{\mathcal{I}_h},\dots,\widetilde{X}_n^{\mathcal{I}_h})),$$

where $(\widetilde{X}_1^{\mathcal{I}_h},\dots,\widetilde{X}_n^{\mathcal{I}_h})$ is comonotonic and $\widetilde{X}_i^{\mathcal{I}_h} \sim F_i^{\mathcal{I}_h}$ for all $i = 1,\dots,n$. Similarly to the proof of

Theorem 1 (iii), since $\rho_h \leqslant \rho_{h^*}$, we have the supremum of the left-hand side of (18) is attained by $F_1^{\mathcal{I}_h}, \dots, F_n^{\mathcal{I}_h}$ and $(\widetilde{X}_1^{\mathcal{I}_h}, \dots, \widetilde{X}_n^{\mathcal{I}_h})$, which also obtain the supremum of the right-hand side of (18) since

$$\begin{aligned}\rho_{h^*}(f(\mathbf{a}, \widetilde{X}_1^{\mathcal{I}_h}, \dots, \widetilde{X}_n^{\mathcal{I}_h})) &\leqslant \max_{F_1 \in \mathcal{F}_1, \dots, F_n \in \mathcal{F}_n} \max_{F_{\mathbf{X}} \in \mathcal{D}(F_1, \dots, F_n)} \rho_{h^*}(f(\mathbf{a}, \mathbf{X})) \\ &= \max_{F_1 \in \mathcal{F}_1, \dots, F_n \in \mathcal{F}_n} \max_{F_{\mathbf{X}} \in \mathcal{D}(F_1, \dots, F_n)} \rho_h(f(\mathbf{a}, \mathbf{X})) \\ &= \rho_h(f(\mathbf{a}, \widetilde{X}_1^{\mathcal{I}_h}, \dots, \widetilde{X}_n^{\mathcal{I}_h})) \leqslant \rho_{h^*}(f(\mathbf{a}, \widetilde{X}_1^{\mathcal{I}_h}, \dots, \widetilde{X}_n^{\mathcal{I}_h})).\end{aligned}$$

□

B.4 Proofs of results in Section 5 and related lemmas

In the following, we write q as the Hölder conjugate of p. The following lemma closely resembles Theorem 3.4 of Liu et al. (2020) with only an additional statement on the uniqueness of the quantile function of the maximizer.

Lemma A.1. *For $h \in \mathcal{H}^*$, $m \in \mathbb{R}$, $v > 0$ and $p > 1$, we have*

$$\sup_{F_Y \in \mathcal{M}(p,m,v)} \rho_h(Y) = mh(1) + v[h]_q,$$

If $0 < [h]_q < \infty$, the above supremum is attained by a random variable X such that $F_X \in \mathcal{M}(p, m, v)$ with its quantile function uniquely determined by

$$\mathrm{VaR}_t(X) = m + v\phi_h^q(t), \quad t \in (0,1) \quad a.e. \tag{A.8}$$

If $[h]_q = 0$, the above maximum value is attained by any random variable X such that $F_X \in \mathcal{M}(p, m, v)$.

Proof. The only statement that is more than Theorem 3.4 of Liu et al. (2020) is the uniqueness of the quantile function (A.8). Without loss of generality, assume $m = 0$ and $v = 1$. Using the Hölder inequality

$$\begin{aligned}\sup_{F_Y \in \mathcal{M}(p,0,1)} \int_0^1 h'(t)\mathrm{VaR}_{1-t}(Y)\,\mathrm{d}t &= \sup_{F_Y \in \mathcal{M}(p,0,1)} \int_0^1 (h'(t) - c_{h,q})\mathrm{VaR}_{1-t}(Y)\,\mathrm{d}t \\ &\leqslant \sup_{F_Y \in \mathcal{M}(p,0,1)} \|h' - c_{h,q}\|_q \left(\int_0^1 |\mathrm{VaR}_{1-t}(Y)|^p\,\mathrm{d}t\right)^{1/p} = [h]_q.\end{aligned}$$

The maximum is attained by F_X only if the above inequality is an equality, which is equivalent to that the function $t \mapsto |\mathrm{VaR}_{1-t}(X)|^p$ is a multiple of $|h' - c_{h,q}|^q$. Therefore,

$$\mathrm{VaR}_t(X) = \frac{|h'(1-t) - c_{h,q}|^q}{h'(1-t) - c_{h,q}}[h]_q^{1-q} = \phi_h^q(t), \quad t \in (0,1) \quad \text{a.e.}$$

Hence, the quantile function of X is uniquely determined by (A.8). □

Lemma A.2. *For all $h \in \mathcal{H}$ with $h = \hat{h}$, $m \in \mathbb{R}$, $v > 0$ and $p > 1$, if $[h^*]_q < \infty$, we have*

$$\sup_{F_Y \in \mathcal{M}(p,m,v)} \rho_h(Y) = \sup_{F_Y \in \mathcal{M}(p,m,v)} \rho_{h^*}(Y) = mh(1) + v[h^*]_q,$$

and the above suprema are simultaneously attained by a random variable X such that $F_X \in \mathcal{M}(p,m,v)$ with

$$\mathrm{VaR}_t(X) = m + v\phi^q_{h^*}(t), \quad t \in (0,1) \text{ a.e.} \tag{A.9}$$

Proof. The statement directly follows from Theorem 1 and Lemma A.1. □

Proof of Theorem 5. Together with Theorem 1, Lemmas A.1 and A.2 give the statement in Theorem 5 on the supremum. Arguments for the infimum are symmetric. For instance, noting that $(-h)^* = -h_*$, Theorem 1 yields

$$\begin{aligned}\inf_{F_Y \in \mathcal{M}(p,m,v)} \rho_h(Y) &= -\sup_{F_Y \in \mathcal{M}(p,m,v)} \rho_{-h}(Y)\\ &= -\sup_{F_Y \in \mathcal{M}(p,m,v)} \rho_{(-h)^*}(Y)\\ &= -\sup_{F_Y \in \mathcal{M}(p,m,v)} \rho_{-h_*}(Y) = \inf_{F_Y \in \mathcal{M}(p,m,v)} \rho_{h_*}(Y).\end{aligned}$$

We omit the detailed arguments for the infimum in Theorem 5. □

Proof of Proposition 6. Note that $\rho_h \leqslant \rho_{h^*}$, which is implied by $h \leqslant h^*$ and (4). By Hölder's inequality, for any $Y \in \mathcal{L}^p$, using (14), we have

$$\begin{aligned}\int_0^1 h^{*\prime}(t)\mathrm{VaR}_{1-t}(Y)\,\mathrm{d}t &= \int_0^1 (h^{*\prime}(t) - c_{h^*,q})\mathrm{VaR}_{1-t}(Y)\,\mathrm{d}t + c_{h,q}\mathbb{E}[Y]\\ &\leqslant [h^*]_q \|Y\|_p + c_{h^*,q}\mathbb{E}[Y] < \infty.\end{aligned}$$

The other half of the statement is analogous. □

Proof of Corollary 1. We prove the first half (the suprema). The second half is symmetric to the first half. Theorem 5 and Lemma A.2 give

$$\sup_{F_Y \in \mathcal{M}(p,m,v)} \mathrm{VaR}_\alpha(Y) = \sup_{F_Y \in \mathcal{M}(p,m,v)} \mathrm{ES}_\alpha(Y) = m + v[h^*]_q.$$

By Lemma A.1, the corresponding random variable Z which attains $\mathrm{ES}_\alpha(Z) = m + v[h^*]_q$ has left-quantile function

$$F_Z^{-1}(t) = m + v\phi^q_{h^*}(t) = m + v\frac{\left|\frac{1}{1-\alpha}\mathbb{1}_{(\alpha,1]}(t) - 1\right|^q}{\frac{1}{1-\alpha}\mathbb{1}_{(\alpha,1]}(t) - 1}[h^*]_q^{1-q}, \quad t \in [0,1] \text{ a.e.}$$

Note that $\phi^q_{h^*}(t)$ only takes two values for $t \geqslant \alpha$ and $t < \alpha$, respectively. Thus Z is a bi-atomic random variable, and using $\mathbb{E}[Z] = m$, we have, for some $k_p > 0$,

$$\mathbb{P}\left(Z = m + \alpha k_p\right) = 1 - \alpha \quad \text{and} \quad \mathbb{P}\left(Z = m - (1-\alpha)k_p\right) = \alpha.$$

We note that the number k_p can be determined from $\mathbb{E}[|Z - m|^p] = v^p$, that is,

$$k_p = v\left(\alpha^p(1-\alpha) + (1-\alpha)^p\alpha\right)^{-1/p},$$

leading to

$$\sup_{F_Y \in \mathcal{M}(p,m,v)} \mathrm{VaR}_\alpha(Y) = \sup_{F_Y \in \mathcal{M}(p,m,v)} \mathrm{ES}_\alpha(Y) = m + v\alpha\left(\alpha^p(1-\alpha) + (1-\alpha)^p\alpha\right)^{-1/p},$$

and thus the desired equalities in the statement on suprema hold. $\square$