

Probabilistic risk aversion for generalized rank-dependent functions

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Abstract

Probabilistic risk aversion, defined through quasi-convexity in probabilistic mixtures, is a common useful property in decision analysis. We study a general class of non-monotone mappings, called the generalized rank-dependent functions, which includes the preference models of expected utilities, dual utilities, and rank-dependent utilities as special cases, as well as signed Choquet functions used in risk management. Our results fully characterize probabilistic risk aversion for generalized rank-dependent functions: This property is determined by the distortion function, which is precisely one of the two cases: those that are convex and those that correspond to scaled quantile-spread mixtures. Our result also leads to seven equivalent conditions for quasi-convexity in probabilistic mixtures of dual utilities and signed Choquet functions. As a consequence, although probabilistic risk aversion is quite different from the classic notion of strong risk aversion for generalized rank-dependent functions, these two notions coincide for dual utilities under an additional continuity assumption.

Keywords: quasi-convexity; risk aversion, signed Choquet functions; rank-dependent utilities; probabilistic mixtures

1 Introduction

Expected utility theory (von Neumann and Morgenstern (1947)), dual utility theory (Yaari (1987)), and rank-dependent utility theory (Quiggin (1982)) are among the most popular probabilistic preference models, and they are closely related to several large classes of law-based risk measures (McNeil et al. (2015); Föllmer and Schied (2016)).

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These decision models and risk measures can be equivalently formulated on either a set of distributions or a set of random variables. A popular operation on a set of distributions is a probabilistic mixture; for instance, the independence axiom of [von Neumann and Morgenstern \(1947\)](#) is formulated using probabilistic mixtures. Quasi-convexity in probabilistic mixtures is a useful property in decision models and risk measures, and it means that if the first distribution is preferred over the second one, then the first distribution is also preferred to a mixture of the two. The property is called *probabilistic risk aversion* by [Wakker \(1994\)](#) to distinguish it from other notions of risk aversion such as weak risk aversion or strong risk aversion in the sense of [Rothschild and Stiglitz \(1970\)](#).¹ In the context of optimal decision under ambiguity, e.g., [Gilboa and Schmeidler \(1989\)](#), this property is convenient for applying minimax theorems, allowing us to exchange the order of a maximum (representing an optimal action) and an infimum (representing a worst-case probability measure) under mild conditions. The property of having both quasi-convexity and quasi-concavity is called betweenness ([Dekel \(1986\)](#)), which was introduced to weaken the independence axiom; see [Wakker \(1994\)](#) for the importance of these properties in decision theory. In mathematical finance, betweenness for risk measures corresponds to the property of convex level sets, as studied by [Weber \(2006\)](#), [Ziegel \(2016\)](#) and [Wang and Wei \(2020\)](#).

It is well known that the expected utility model is linear with respect to probabilistic mixtures, thus both convex and concave (we omit “in probabilistic mixtures” below unless it is contrasted to another sense of convexity). Whereas convexity is well understood for dual utilities and rank-dependent utility models (e.g., [Wakker \(1994\)](#)), quasi-convexity is not completely characterized for these models, and the result remains unknown even if under an increasing monotonicity (in the weak sense) or continuity condition. Although being weaker, quasi-convexity is similar to convexity, and as far as we know, the equivalence results between quasi-convexity and convexity in the literature are all under a strict monotonicity condition (see [Wakker \(1994\)](#); [Wakker and Yang \(2021\)](#)). Nevertheless, if we remove this strict monotonicity, there are commonly used functionals in decision theory, statistics and risk management, such as left and right quantiles which are quasi-convex but not convex.

The main aim of this paper is to understand quasi-convexity for a large class of mappings, called the *generalized rank-dependent functions*, which include dual utilities and rank-dependent utilities as special cases. Our main result is a full characterization of quasi-convexity for this class with a very weak assumption on the domain of the functions, that is, the domain contains all distributions supported on at least three nonindifferent outcomes. This characterization is built on a

¹Throughout the paper, we will mostly say “quasi-convexity” instead of “probabilistic risk aversion” to emphasize the mathematical essence of this property and to contrast it with other properties.

corresponding result on *signed Choquet functions*, a class of non-monotone and law-based mappings studied recently by Wang et al. (2020a,b), and the corresponding mappings without law-basedness were investigated earlier in Schmeidler (1986).

Although most preference models in decision theory are monotone (either with respect to some notions of stochastic dominance or other orders), there are three main advantages of working with non-monotone mappings, justifying the relevance of the study in this paper. First, signed Choquet functions include many popular non-monotone objects in risk management, such as the mean-median deviation, the Gini deviation, the inter-quantile range, and the inter-Expected Shortfall range; see the examples in Wang et al. (2020b). Note that variability measures in the sense of Furman et al. (2017) are never monotone with respect to first-order stochastic dominance. Second, removing monotonicity from the analysis allows us to have a deeper understanding of the essence of important properties, such as quasi-convexity, by disentangling monotonicity from them. The third advantage concerns technical convenience and unification. With monotonicity relaxed, results on convexity and concavity, or those on maxima and minima, are symmetric; we only need to analyze one of them, and the other is clear automatically. This is particularly helpful when we switch between the world of risk measures (a smaller value is preferred) and that of utilities (a larger value is preferred). That being said, it is not our intention to argue against monotonicity in decision making; opening up the discussions on non-monotone mappings indeed helps to better understand monotone ones.

We begin by collecting definitions and some preliminaries in Section 2. In Section 3, we focus on two important models in decision theory, dual utilities and rank-dependent utilities, by presenting a full characterization of their quasi-convexity (Theorem 1). This result implies, in particular, that for a dual utility with a continuous distortion function, strong risk aversion in the sense of Rothschild and Stiglitz (1970) is equivalent to probabilistic risk aversion. We discover a new risk functional, called *min-quantile mixture*, as the only possible form of dual utilities which are quasi-convex, other than the ones with convex distortion functions. To highlight the class of min-quantile mixtures, we use some properties to pin down it (Proposition 2). Our main technical result (Theorem 2) in Section 4 establishes a characterization of all quasi-convex generalized rank-dependent functions, more general than those treated in Theorem 1. The characterization only depends on the distortion functions. The class turns out to contain slightly more than those with convex distortion functions: A signed Choquet function is quasi-convex in probabilistic mixtures if and only if it is either convex in probabilistic mixtures or it is a *scaled quantile-spread mixture* (more general than min-quantile mixtures). Based on our main result, a unifying equivalence result on signed Choquet functions

(Theorem 3) is presented: If a distortion function is continuous, then quasi-convexity is equivalent to six other equivalent conditions. In Section 5, we give proofs of our main result which relies on delicate technical analysis and use some results of Debreu and Koopmans (1982), Wakker (1994), Wang et al. (2020b) and Wang and Wei (2020). The technical challenges may explain why the result was not available before, given the prominence of both concepts of quasi-convexity and rank-dependent utilities. Some implications of Theorem 2 for quasi-concavity and quasi-linearity are also reported in this section. In Section 6, we present a conflict between convexity in probabilistic mixtures and convexity in risk pooling among the class of constant-additive mappings. Section 7 concludes the paper.

2 Preliminaries

In this section, we present some background on convexity, quasi-convexity, rank-dependent utilities, and generalized rank-dependent functions.

2.1 Convexity and quasi-convexity

In this paper, the term “distribution” represents a probability measure over a set of outcomes which is the real line \mathbb{R} . Let \mathcal{M} be a set of distributions, and we always assume that it is convex throughout the paper. A mapping $\rho : \mathcal{M} \rightarrow \mathbb{R}$ is *p-convex* if

$$\rho(\lambda F + (1 - \lambda)G) \leq \lambda\rho(F) + (1 - \lambda)\rho(G) \quad \text{for all } F, G \in \mathcal{M} \text{ and } \lambda \in [0, 1],$$

and it is *p-quasi-convex* if

$$\rho(\lambda F + (1 - \lambda)G) \leq \max\{\rho(F), \rho(G)\} \quad \text{for all } F, G \in \mathcal{M} \text{ and } \lambda \in [0, 1].$$

As usual, p-concavity and p-quasi-concavity are defined by using $\rho(\lambda F + (1 - \lambda)G) \geq \lambda\rho(F) + (1 - \lambda)\rho(G)$ and $\rho(\lambda F + (1 - \lambda)G) \geq \min\{\rho(F), \rho(G)\}$, respectively, in the formulation above. *P-quasi-linearity* of a functional ρ means that it is both p-quasi-convex and p-quasi-concave. The reason that we emphasize “p” (which stands for “probabilistic”) for these properties will be explained soon, as another form of convexity and concavity will appear and be contrasted.

Quasi-convexity is an ordinal property, whereas convexity is not. Indeed, for a preference \succeq on \mathcal{M} numerically represented by ρ , i.e., $F \succeq G \iff \rho(F) \geq \rho(G)$, quasi-convexity of ρ corresponds

to the following property of \succeq (see e.g., Wakker and Yang (2019, 2021)),

$$\left(F \succeq G \implies F \succeq \lambda F + (1 - \lambda)G \right) \quad \text{for all } F, G \in \mathcal{M} \text{ and } \lambda \in [0, 1].$$

This property is known as probabilistic risk aversion by Wakker (1994). Intuitively, it means that the decision maker with preference \succeq dislikes combining two equally favourable distributions via a random draw, which generally induces additional randomness.

Convexity or concavity of a mapping ρ is commonly used in risk management, where the value of ρ , typically representing a monetary value, is primitive; see e.g., Föllmer and Schied (2016). Quasi-convexity or quasi-concavity of ρ is commonly used in decision theory, where the preference relation \succeq is the primitive; see e.g., Wakker (2010) and Cerreia-Vioglio et al. (2011). To unify both literature, we will formulate all properties on the numerical representation ρ .

Let \mathcal{X} be a set of random variables in a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that, first, \mathcal{X} is *law-based*, that is, if $X \in \mathcal{X}$ and Y has the same distribution of X , then $Y \in \mathcal{X}$, and second, $\mathcal{M} = \{\mathbb{P} \circ X^{-1} : X \in \mathcal{X}\}$, where X^{-1} is the set-valued inverse of X . That is, the set of distributions of all random variables in \mathcal{X} is exactly \mathcal{M} . To guarantee the existence of \mathcal{X} for all \mathcal{M} , we assume that the probability space is nonatomic.² A mapping ρ from \mathcal{M} to \mathbb{R} can be equivalently formulated as a mapping ρ from \mathcal{X} to \mathbb{R} via $\rho(X) := \rho(F)$ where F is the distribution of X ; here we slightly abuse the notation to use ρ to represent both, and this should be clear from the context. Such a mapping ρ on \mathcal{X} is *law-based*; that is, if X, Y have the same distribution, then $\rho(X) = \rho(Y)$. We need both versions of the same mapping to make some interesting contrasts. When \mathcal{X} is a convex set, a mapping ρ is *o-convex* (where “o” stands for “outcome”) on \mathcal{X} if

$$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y) \quad \text{for all } X, Y \in \mathcal{X} \text{ and } \lambda \in [0, 1],$$

and it is *o-quasi-convex* if

$$\rho(\lambda X + (1 - \lambda)Y) \leq \max\{\rho(X), \rho(Y)\} \quad \text{for all } X, Y \in \mathcal{X} \text{ and } \lambda \in [0, 1].$$

O-concavity and o-quasi-concavity are defined similarly. These properties are common for risk measures (Artzner et al. (1999); Föllmer and Schied (2016)). Moreover, Dong-Xuan et al. (2024) extended the study of o-quasi-convexity to the space of sequences of bounded random variables, further illustrating the relevance of these concepts in various frameworks.

²A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is *nonatomic* if for each $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$ there exists $B \in \mathcal{F}$ contained in A such that $0 < \mathbb{P}(B) < \mathbb{P}(A)$.

For the same mapping ρ , o-convexity and p-convexity have different interpretations, and sometimes they conflict with each other. For instance, the variance is *o-convex and p-concave*. This conflict is indeed intuitive, because a mixture of two random losses, representing diversification, reduces variability, whereas a mixture of two distributions, representing throwing a die to determine between two models, increases variability.³ A mapping may be both o-convex and p-convex, and an example is the expected utility, $F \mapsto \int u dF$ for a convex function u ; this mapping is indeed o-convex and p-linear (i.e., both p-convex and p-concave). Nevertheless, Proposition 6 in Section 6 shows a conflict between o-convexity and p-convexity; that is, among continuous and constant-additive mappings (e.g., monetary risk measures of Föllmer and Schied (2016)), only a scaled expected value satisfies both properties.

We collect some notation used later in the paper. Denote by \mathcal{M}_c the set of all compactly supported distributions on \mathbb{R} , and \mathcal{X}_c , which is defined on a nonatomic probability space, is the set of all random variables having distributions in \mathcal{M}_c . Note that both \mathcal{M}_c and \mathcal{X}_c are convex. We use $\text{ess-sup } X$ and $\text{ess-inf } X$ to represent the essential supremum and the essential infimum of a random variable X on a probability space, respectively. Denote by δ_x the point-mass at $x \in \mathbb{R}$. The function $\mathbb{1}_A$ is the indicator function of an event A . Throughout, terms like “increasing” and “decreasing” are in the non-strict sense. All real-valued functions are tacitly assumed to be measurable. We say that a real-valued function is (strictly) monotone if it is (strictly) increasing or (strictly) decreasing. A functional $\rho : \mathcal{M} \rightarrow \mathbb{R}$ is monotone if $\rho(F) \leq \rho(G)$ for all $F, G \in \mathcal{M}$ such that $F \leq_{\text{FSD}} G$ where \leq_{FSD} represents the first-order stochastic dominance, i.e., $F \leq_{\text{FSD}} G$ means that $\int f dF \leq \int f dG$ for all increasing $f : \mathbb{R} \rightarrow \mathbb{R}$.

2.2 Generalized rank-dependent functions

We first formulate signed Choquet functions and dual utilities, and then introduce rank-dependent utilities and generalized rank-dependent functions. Signed Choquet functions are law-based mappings which are additive for comonotonic random variables (Theorem 1 of Wang et al. (2020b), based on Proposition 2 of Schmeidler (1986)), but not necessarily monotone. Denote the set of all *distortion functions* by \mathcal{H}^{BV} ,

$$\mathcal{H}^{\text{BV}} = \{h : [0, 1] \rightarrow \mathbb{R} \mid h \text{ is of bounded variation and } h(0) = 0\},$$

³In the risk measure literature, o-quasi-convexity on \mathcal{X} is argued by Cerreia-Vioglio et al. (2011) to better represent the consideration of diversification.

and its subset of increasing normalized distortion functions by \mathcal{H}^{DT} ,⁴

$$\mathcal{H}^{\text{DT}} = \{h : [0, 1] \rightarrow \mathbb{R} \mid h \text{ is increasing, } h(0) = 0 \text{ and } h(1) = 1\}.$$

A *signed Choquet function* $I_h : \mathcal{M} \rightarrow \mathbb{R}$ is defined as

$$I_h(F) = \int_0^\infty h \circ F((x, \infty)) \, dx + \int_{-\infty}^0 (h \circ F((x, \infty)) - h(1)) \, dx, \quad (1)$$

where $h \in \mathcal{H}^{\text{BV}}$ is its distortion function. If $h \in \mathcal{H}^{\text{DT}}$, then I_h is called a *dual utility* of Yaari (1987). As explained earlier, I_h is also formulated on \mathcal{X} , the set of random variables which have distributions in \mathcal{M} , via

$$I_h(X) = \int_0^\infty h(\mathbb{P}(X > x)) \, dx + \int_{-\infty}^0 (h(\mathbb{P}(X > x)) - h(1)) \, dx. \quad (2)$$

As a main subject of the paper, a *generalized rank-dependent function* is a mapping $R_{h,v} : \mathcal{M} \rightarrow \mathbb{R}$ defined by, for $h \in \mathcal{H}^{\text{BV}}$ and $v : \mathbb{R} \rightarrow \mathbb{R}$,

$$R_{h,v}(F) = \int_0^\infty h \circ F \circ v^{-1}((x, \infty)) \, dx + \int_{-\infty}^0 (h \circ F \circ v^{-1}((x, \infty)) - h(1)) \, dx, \quad (3)$$

where v^{-1} is the set-valued inverse of v . The function v is typically increasing in economic applications (e.g., a utility function). The signed Choquet function is a special case of $R_{h,v}$ with v being the identity, and if $h \in \mathcal{H}^{\text{DT}}$ and v is increasing, then $R_{h,v}$ corresponds to a *rank-dependent utility* of Quiggin (1993). Although rank-dependent utilities are well studied in decision theory (see e.g., Werner and Zank (2019); Eeckhoudt and Laeven (2022)), the class of generalized rank-dependent functions is newly introduced in this paper.

A v -transform maps the distribution of a random variable X to the distribution of $v(X)$.⁵ In other words, the distribution F is transformed to $F \circ v^{-1}$. The mapping $R_{h,v}$ can be formulated as a signed Choquet function under a v -transform, i.e.,

$$R_{h,v}(F) = I_h(F \circ v^{-1})$$

or equivalently formulated on \mathcal{X} via $R_{h,v}(X) = I_h(v(X))$. In addition to the expected utility or the rank-dependent utility models, a pricing functional for options can be seen as the (risk-neutral)

⁴DT stands for the dual theory of choice under risk of Yaari (1987).

⁵As shown by Liu et al. (2021), commutation with v -transforms essentially characterizes probability distortions among all distributional transforms.

expectation of a transformed asset price distribution.

We will encounter discrete distributions throughout the paper. An explicit representation of $R_{h,v}$ with discrete distributions is given below. For a discrete distribution F with the form $F = \sum_{i=1}^n p_i \delta_{x_i}$ where $v(x_1) \geq \dots \geq v(x_n)$, $p_1, \dots, p_n \geq 0$ and $\sum_{i=1}^n p_i = 1$, it holds that

$$R_{h,v}(F) = \sum_{i=1}^n (h(p_1 + \dots + p_i) - h(p_1 + \dots + p_{i-1}))v(x_i).$$

The following assumption on the set of distributions \mathcal{M} will be useful for our characterization results.

Assumption M. The set \mathcal{M} is a convex subset of \mathcal{M}_c and there exist three distinct points $x, y, z \in \mathbb{R}$ such that $\delta_x, \delta_y, \delta_z \in \mathcal{M}$.

Assumption **M** is very weak and harmless for any practical purpose. For a set \mathcal{M} satisfying Assumption **M**, denote by

$$\mathcal{V}_{\mathcal{M}} = \{v : \mathbb{R} \rightarrow \mathbb{R} \mid v(x), v(y) \text{ and } v(z) \text{ are distinct for some } \delta_x, \delta_y, \delta_z \in \mathcal{M}\}. \quad (4)$$

In the set $\mathcal{V}_{\mathcal{M}}$, we do not impose the continuity or the monotonicity on v , and we only require that v can take at least three distinct values on the points x, y, z . For instance, this requirement holds true if v is strictly increasing.

In what follows, we will take the convexity of h as the primary property as opposite to concavity to consider, as the preference represented by I_h is strongly risk averse in the sense of [Rothschild and Stiglitz \(1970\)](#) if h is increasing and convex (Theorem 2 of [Yaari \(1987\)](#)). Changing convexity to concavity makes no real mathematical difference since all results can be written for concavity via a sign change; recall that this is an advantage of working with non-monotone mappings such as generalized rank-dependent functions.

3 P-quasi-convexity of rank-dependent utilities

Given the importance of rank-dependent utilities in economics and finance, we first present results for this class, although these results find their more general versions in [Section 4](#).

3.1 P-quasi-convexity, concavity and linearity

Recall that rank-dependent utility is defined by [\(3\)](#) where $h \in \mathcal{H}^{\text{DT}}$ and $v : \mathbb{R} \rightarrow \mathbb{R}$ is increasing. The dual utility is a special case of $R_{h,v}$ when $h \in \mathcal{H}^{\text{DT}}$ and v is the identity.

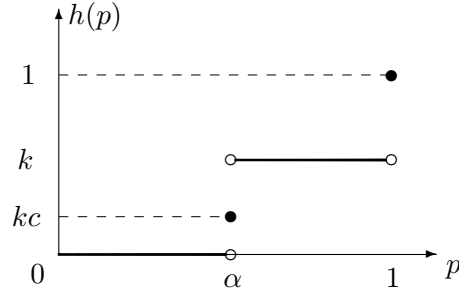


Figure 1: The distortion function of a min-quantile mixture.

We first introduce a few special cases of dual utilities and distortion functions in \mathcal{H}^{DT} that are important for understanding p-quasi-convexity. There is a one-to-one correspondence between dual utilities and distortion functions in \mathcal{H}^{DT} , and hence, it suffices to study the distortion functions. For a distribution F , the *left-* and *right-quantile* at level $\alpha \in (0, 1)$ are respectively defined by

$$Q_{\alpha}^{-}(F) = \inf\{x \in \mathbb{R} : F(x) \geq \alpha\} \quad \text{and} \quad Q_{\alpha}^{+}(F) = \inf\{x \in \mathbb{R} : F(x) > \alpha\}.$$

For using quantiles as preferences in decision theory, see [Rostek \(2010\)](#). At the level $\alpha = 0$ or 1 , left- and right-quantiles coincide, and they are defined by

$$\begin{aligned} Q_0^{+}(F) &= Q_0^{-}(F) = Q_0(F) = \inf\{x \in \mathbb{R} : F(x) > 0\}; \\ Q_1^{+}(F) &= Q_1^{-}(F) = Q_1(F) = \inf\{x \in \mathbb{R} : F(x) \geq 1\}. \end{aligned}$$

For some $c \in [0, 1]$ and $\alpha \in [0, 1]$, the *mixed quantile* is defined by $Q_{\alpha}^c = cQ_{\alpha}^{+} + (1 - c)Q_{\alpha}^{-}$. All mixed quantiles have *convex level sets* (CxLS), i.e., $\rho(F) = \rho(G) \implies \rho(\lambda F + (1 - \lambda)G) = \rho(F)$ for all $\lambda \in [0, 1]$ and $F, G \in \mathcal{M}$. This property and monotonicity together imply p-quasi-convexity and p-quasi-concavity.⁶ Finally, we introduce a new class of functionals, called the *min-quantile mixtures*, defined by

$$kQ_{\alpha}^c + (1 - k)Q_0, \quad \text{for some } \alpha, c, k \in [0, 1]. \quad (5)$$

A min-quantile mixture is a convex combination of the essential infimum and a mixed quantile. The distortion function of a min-quantile mixture has the form (see [Figure 1](#)):

⁶All signed Choquet functions with convex level sets are characterized by [Wang and Wei \(2020\)](#), which are slightly more than linear transformations of the quantiles and the mean.

$$h(p) = kc\mathbb{1}_{\{p=\alpha\}} + k\mathbb{1}_{\{\alpha < p < 1\}} + \mathbb{1}_{\{p=1\}}, \quad p \in [0, 1].$$

For strictly increasing distortion functions h , Wakker (1994) and Wakker and Yang (2021) showed that only the convex ones are possible for I_h to be p -quasi-convex. In the larger class of I_h with $h \in \mathcal{H}^{\text{DT}}$, the following result illustrates that the min-quantile mixtures are the only other choice satisfying p -quasi-convexity besides those with convex distortion functions.

Theorem 1. *Suppose that Assumption M holds, $h \in \mathcal{H}^{\text{DT}}$, and $v : \mathbb{R} \rightarrow \mathbb{R}$ is increasing. The following statements are equivalent.*

- (i) h is convex or I_h is a min-quantile mixture.
- (ii) I_h is p -quasi-convex on \mathcal{M} .
- (iii) $R_{h,v}$ is p -quasi-convex on \mathcal{M} for some $v \in \mathcal{V}_{\mathcal{M}}$.
- (iv) $R_{h,v}$ is p -quasi-convex on \mathcal{M} for all functions v .

The proof of the above theorem follows from the more general result in Theorem 2 in the next section, which studies all generalized rank-dependent functions. In (iii), the condition that v can take three different values is essential. Note that if v is a constant function, then $R_{h,v}$ is p -quasi-convex regardless of h . Section 4 has more discussions on the role of this condition.

Next, we connect two notions of risk aversion. A functional $\rho : \mathcal{M} \rightarrow \mathbb{R}$ is *concave-order monotone* if $\rho(F) \leq \rho(G)$ for all $F, G \in \mathcal{M}_c$ such that $F \leq_{cv} G$ where \leq_{cv} represents concave order between distributions, i.e., $F \leq_{cv} G$ means that $\int f dF \leq \int f dG$ for all concave $f : \mathbb{R} \rightarrow \mathbb{R}$. Concave-order monotonicity of ρ is equivalent to *strong risk aversion* of the preference represented by ρ in the sense of Rothschild and Stiglitz (1970).

Since a min-quantile mixture does not have a continuous distortion function, Theorem 1 implies that the only class of continuous distortion functions yielding p -quasi-convexity is that of the convex ones. Yaari (1987) showed that a dual utility is strongly risk averse if and only if the distortion function is convex. The next corollary directly follows from this and Theorem 1.

Corollary 1. *For a dual utility on \mathcal{M}_c with a continuous distortion function, probabilistic risk aversion is equivalent to strong risk aversion.*

Wakker (1994) showed that the conclusion of Corollary 1 holds for strictly increasing h . Continuity of h is quite natural, which is implied by the axioms of Yaari (1987). From a decision theoretical standpoint, a continuous distortion function h is empirically plausible as in, e.g., Equation (6) of Tversky and Kahneman (1992) in the framework of cumulative prospect theory. For

rank-dependent utilities, strong risk aversion further requires v to be concave (Chew et al. (1987)), and hence the conclusion of Corollary 1 does not hold.

The following characterizations of p-quasi-concavity and p-quasi-linearity can be obtained in a similar way to Theorem 1. Their more general versions are Corollaries 4 and 5 in Section 5.

Proposition 1. *Suppose that Assumption M holds, and let $v \in \mathcal{V}_{\mathcal{M}}$ be increasing and $h \in \mathcal{H}^{\text{DT}}$.*

(i) *The following are equivalent: $R_{h,v}$ is p-quasi-concave; I_h is p-quasi-concave; h is concave or $I_h = kQ_{\alpha}^c + (1 - k)Q_1$ for some $k, \alpha, c \in [0, 1]$.*

(ii) *The following are equivalent: $R_{h,v}$ is p-quasi-linear; I_h is p-quasi-linear; I_h is one of the forms: $I_h = \mathbb{E}$; $I_h = cQ_1 + (1 - c)Q_0$ for some $c \in [0, 1]$; $I_h = Q_{\alpha}^c$ for some $c \in [0, 1]$ and $\alpha \in (0, 1)$.*

Remark 1. For $h \in \mathcal{H}^{\text{DT}}$, I_h is monotone and translation invariant.⁷ By Lemma 2.2 of Bellini and Bignozzi (2015), I_h is p-quasi-linearity if and only if it has CxLS. The class of dual utilities I_h satisfying CxLS is characterized by Kou and Peng (2016, Theorem 2), which are those in Proposition 1 (ii).

3.2 The min-quantile mixtures

As the min-quantile mixtures are the only dual utilities satisfying p-quasi-convexity among besides convex ones, we can find several properties that identity the min-quantile mixtures. For this, some terminology and properties are needed. We say that random variables X and Y are *comonotonic* if there exists $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that for all $\omega, \omega' \in \Omega_0$,

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0.$$

For a mapping $\rho : \mathcal{X}_c \rightarrow \mathbb{R}$ (also treated as a mapping from \mathcal{M}_c to \mathbb{R}), we say that ρ is *comonotonic-additive* if for any comonotonic random variables $X, Y \in \mathcal{X}_c$, $\rho(X + Y) = \rho(X) + \rho(Y)$; ρ is *p-locally indifferent* if there exist $A, B \in \mathcal{F}$ such that $\mathbb{P}(A) \neq \mathbb{P}(B)$ and $\rho(\mathbf{1}_A) = \rho(\mathbf{1}_B) > \rho(0)$.

Comonotonic additivity is popular in both the literature of decision theory (e.g., Yaari (1987); Schmeidler (1989)) and that of risk measures (e.g., Kusuoka (2001)), whereas p-local indifference is less known. Intuitively, p-local indifference means that ρ may be indifferent between two events A and B even if their probabilities are different. This property is called “local” because it only states the existence of such a pair of events, but says nothing about general pairs. Although it

⁷A functional is translation invariant if $\rho(X + c) = \rho(X) + c$ for all $X \in \mathcal{X}$ and $c \in \mathbb{R}$.

may be explained by the inability to assess the small difference between different probabilities, such a behaviour is empirically observed in behaviour experiments where only a few categories of probability are considered. For instance, [Piaget and Bärbel \(1975\)](#) showed that children of age 4–5 have only three levels of plausibility conception: certainly true, certainly untrue, uncertain. This is compatible with the min-quantile mixture whose distortion function has two jumps and is constant elsewhere. When the probability categories were related to verbal expressions, people often prefer to use only a few verbal categories of probability rather than a continuous scale; see [Wallsten et al. \(1986\)](#), [Wallsten and Budescu \(1995\)](#) and [Windschitl and Wells \(1996\)](#). [Cohen et al. \(1987\)](#) investigated the individual decision making under risk and under nonprobabilized uncertainty where they found that on the loss side, people appear to have coarser categories of plausibility than the continuum $[0, 1]$.

The next result characterizes the min-quantile mixtures.

Proposition 2. *A law-based functional $\rho : \mathcal{X}_c \rightarrow \mathbb{R}$ is monotone, comonotonic-additive, p -quasi-convex and p -locally indifferent with $\rho(1) = 1$ if and only if $\rho = kQ_\alpha^c + (1 - k)Q_0$ for some $c \in [0, 1]$ and $\alpha, k \in (0, 1]$, that is, ρ is a min-quantile mixture excluding the case Q_0 .*

Proof. Let us first verify sufficiency. Monotonicity, comonotonic-additivity and $\rho(1) = 1$ are trivial. The p -quasi-convexity follows from [Theorem 1](#). It remains to show that ρ is p -locally indifferent. Let A and B be such that $\mathbb{P}(A) = 1 - \alpha/2$ and $\mathbb{P}(A) = 1 - \alpha/3$. Then, we have $\rho(\mathbf{1}_A) = \rho(\mathbf{1}_B) = k > 0$. This completes the proof of sufficiency. To see necessity, it follows from [Wang et al. \(2020b, Theorem 1\)](#) that ρ is a signed Choquet function, i.e., $\rho(X) = \int_0^\infty h(\mathbb{P}(X > x)) dx + \int_{-\infty}^0 (h(\mathbb{P}(X > x)) - h(1)) dx$ with some $h \in \mathcal{H}^{BV}$. By [Wang et al. \(2020b, Lemma 2\)](#), monotonicity implies that h is increasing. Furthermore, $\rho(1) = 1$ means $h(1) = 1$. Hence, we have concluded $h \in \mathcal{H}^{DT}$. Applying [Theorem 1](#), we have $\rho = I_h$ for some convex $h \in \mathcal{H}^{DT}$ or $\rho = kQ_\alpha^c + (1 - k)Q_0$ for some $k, \alpha, c \in [0, 1]$. Since Q_0 also has a convex distortion function $h(t) = \mathbf{1}_{\{t=1\}}$, we only need to verify that I_h with convex h is not p -locally indifferent. P -local indifference implies $\rho(\mathbf{1}_A) = \rho(\mathbf{1}_B)$ for some A, B with $0 < \mathbb{P}(A) < \mathbb{P}(B) \leq 1$. Denote by $p = \mathbb{P}(A)$ and $q = \mathbb{P}(B)$. It holds that $h(p) = h(q) > 0$ with $0 < p < q \leq 1$, and this implies that h cannot be convex. Hence, we complete the proof. \square

We do not aim to promote the min-quantile mixtures as a plausible decision criterion or a new risk measures, as its interpretation and applications are similar to quantiles. Nevertheless, we find the class of min-quantile mixtures intriguing, both mathematically and decision-theoretically, as the only class within dual utilities that can accommodate both p -local indifference (inability

to distinguish probabilistic assessments) and probabilistic risk aversion (dislike of randomness), highlighting a conflict between the two properties.

4 Generalized rank-dependent functions

This section is the main part of the paper where we obtain a full characterization of p-quasi-convexity for generalized rank-dependent functions, as well as a unifying equivalence result for signed Choquet functions with strictly monotone or continuous distortion functions. We will mention Assumption **M** if it is needed in a result.

4.1 Invariance under transforms and three special distortion functions

Recall that the generalized rank-dependent function is a v -transform of a signed Choquet function. Similarly to Proposition 3.5 of Wang and Wei (2020), we first show that p-quasi-convexity is an invariance property under v -transforms.

Proposition 3. *For $v : \mathbb{R} \rightarrow \mathbb{R}$ and a convex set of distributions \mathcal{M} , define $\mathcal{M}^v = \{F \circ v^{-1} : F \in \mathcal{M}\}$. Then, $R_{h,v}$ is p-quasi-convex on \mathcal{M} if and only if I_h is p-quasi-convex on \mathcal{M}^v .*

Proof. For any $\lambda \in (0, 1)$ and $F, G \in \mathcal{M}$, denote by $F' = F \circ v^{-1} \in \mathcal{M}^v$, $G' = G \circ v^{-1} \in \mathcal{M}^v$, and we have

$$(\lambda F + (1 - \lambda)G) \circ v^{-1} = \lambda(F \circ v^{-1}) + (1 - \lambda)(G \circ v^{-1}) = \lambda F' + (1 - \lambda)G',$$

which follows directly from the definition of probability measures. Since

$$R_{h,v}(\lambda F + (1 - \lambda)G) = I_h(\lambda F' + (1 - \lambda)G') \quad \text{and} \quad \max\{R_{h,v}(F), R_{h,v}(G)\} = \max\{I_h(F'), I_h(G')\},$$

we obtain

$$R_{h,v}(\lambda F + (1 - \lambda)G) \leq \max\{R_{h,v}(F), R_{h,v}(G)\} \iff I_h(\lambda F' + (1 - \lambda)G') \leq \max\{I_h(F'), I_h(G')\}.$$

It follows that p-quasi-convexity of I_h on \mathcal{M}^v is equivalent to that of $R_{h,v}$ on \mathcal{M} , noting that F, G can be arbitrarily chosen from \mathcal{M} and F', G' can be arbitrarily chosen from \mathcal{M}^v . This completes the proof. \square

The signed Choquet function I_h in the above proposition can be replaced by any functionals ρ , and $\rho_v(F) := \rho(F \circ v^{-1})$ should take place of $R_{h,v}$ simultaneously. To characterize p-quasi-convexity

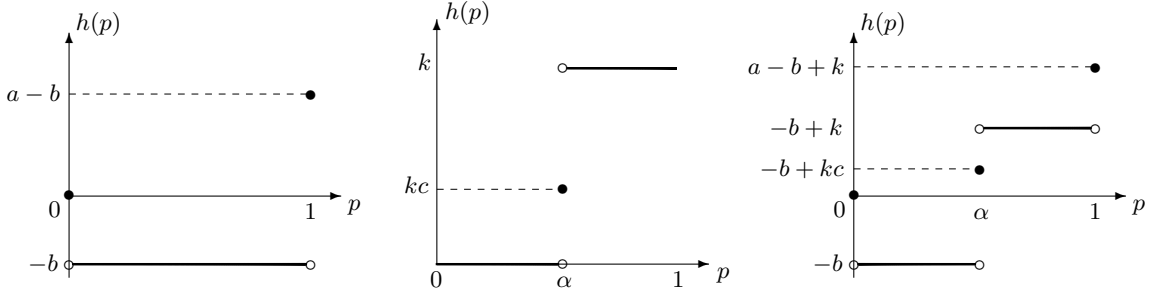


Figure 2: Distortion functions of an asymmetric spread (left), a scaled mixed quantile (middle) and scaled quantile-spread mixture (right) with $\alpha, c \in (0, 1)$ and $k > 0$.

of generalized rank-dependent functions, Proposition 3 implies that we can alternatively consider the characterization of the same property in the class of signed Choquet functions. As a result, p-quasi-convexity of $R_{h,v}$ only depends on the distortion function h . More precisely, we will show below that the property of h determines p-quasi-convexity of $R_{h,v}$ under Assumption M if $v \in \mathcal{V}_{\mathcal{M}}$ (Theorem 2), but the situation is different if v can only take two values (Proposition 5). It is trivial that p-quasi-convexity holds for all $h \in \mathcal{H}^{\text{BV}}$ if v is a constant function.

We define a few special cases of signed Choquet functions that will appear later in our characterization result. An *asymmetric (negative) spread* is the mapping

$$S_{a,b} = aQ_0 - bQ_1,$$

where $a, b \geq 0$. Note that $-S_{1,1} = Q_1 - Q_0$ is the usual spread of the support of the distribution. We omit “negative” below for simplicity and call $S_{a,b}$ an asymmetric spread. The *scaled quantile-spread mixture* is given by

$$S_{a,b} + kQ_{\alpha}^c, \quad \text{for some } a, b \geq 0, \alpha, c \in [0, 1], k \in \mathbb{R}. \quad (6)$$

A scaled quantile-spread mixture is the sum of an asymmetric spread $S_{a,b}$ and a scaled mixed quantile kQ_{α}^c , and this class includes mixed quantiles, asymmetric spreads, and min-quantile mixtures as special cases. Note that we allow $\alpha = 0$ or $\alpha = 1$ in (6), which leads to $aQ_0 - bQ_1$ where $a \geq 0$ or $b \geq 0$, but it does not include the case $a, b < 0$. Figure 2 reports an example of the distortion functions of an asymmetric spread, a scaled mixed quantile and a scaled quantile-spread mixture.

We denote by \mathcal{H}^{CX} the set of all convex distortion functions $h \in \mathcal{H}^{\text{BV}}$ and by \mathcal{H}^{QSM} the set of distortion functions of scaled quantile-spread mixtures. The functions in \mathcal{H}^{QSM} have the form

$$h(p) = -b\mathbb{1}_{\{0 < p < \alpha\}} + (-b + kc)\mathbb{1}_{\{p = \alpha\}} + (-b + k)\mathbb{1}_{\{\alpha < p < 1\}} + (a - b + k)\mathbb{1}_{\{p = 1\}}, \quad p \in [0, 1], \quad (7)$$

for some parameters in (6). The asymmetric spread corresponds to $a, b > 0$, $\alpha = 1$ and $k = 0$, the mixed quantile corresponds to $a = b = 0$, $\alpha, c \in (0, 1)$ and $k = 1$, and a min-quantile mixture in (5) corresponds to $b = 0$, $k \in [0, 1]$ and $a = 1 - k$.

The next lemma says that the asymmetric spread $S_{a,b}$ can be safely added to any I_h without changing its p-quasi-convexity, thus highlighting the special role of $S_{a,b}$.

Lemma 1. *Let \mathcal{M} be a convex set of distributions. Suppose that $h, \tilde{h} \in \mathcal{H}^{\text{BV}}$ satisfies $I_h = S_{a,b} + I_{\tilde{h}}$ for some $a, b \geq 0$. If $I_{\tilde{h}}$ is p-quasi-convex on \mathcal{M} , then I_h is also p-quasi-convex on \mathcal{M} .*

Proof. Suppose that $I_{\tilde{h}}$ is p-quasi-convex. For $F, G \in \mathcal{M}$ and $\lambda \in (0, 1)$, we have $Q_1(\lambda F + (1 - \lambda)G) = \max\{Q_1(F), Q_1(G)\}$ and $Q_0(\lambda F + (1 - \lambda)G) = \min\{Q_0(F), Q_0(G)\}$. Hence, by the p-quasi-convexity of $I_{\tilde{h}}$, we have

$$\begin{aligned} I_h(\lambda F + (1 - \lambda)G) &= aQ_0(\lambda F + (1 - \lambda)G) - bQ_1(\lambda F + (1 - \lambda)G) + I_{\tilde{h}}(\lambda F + (1 - \lambda)G) \\ &\leq \min\{aQ_0(F), aQ_0(G)\} - \max\{bQ_1(F), bQ_1(G)\} + \max\{I_{\tilde{h}}(F), I_{\tilde{h}}(G)\} \\ &\leq \max\{aQ_0(F) - bQ_1(F) + I_{\tilde{h}}(F), aQ_0(G) - bQ_1(G) + I_{\tilde{h}}(G)\} \\ &= \max\{I_h(F), I_h(G)\}, \end{aligned}$$

which implies the p-quasi-convexity of I_h . □

Using Lemma 1, we can verify that scaled quantile-spread mixtures are p-quasi-convex. Hence, both convex distortion functions and distortion functions in (7) lead to p-quasi-convexity of I_h . Combining with Proposition 3, the same conclusion also holds for $R_{h,v}$. We summarize these results in the following proposition. Later we will show that they are the only possibilities.

Proposition 4. *If $h \in \mathcal{H}^{\text{CX}} \cup H^{\text{QSM}}$ and $v : \mathbb{R} \rightarrow \mathbb{R}$, then I_h and $R_{h,v}$ are both p-quasi-convex on \mathcal{M}_c .*

Proof. Note that $\{F \circ v^{-1} : F \in \mathcal{M}_c\} \subseteq \mathcal{M}_c$. It follows from Proposition 3 that p-quasi-convexity of I_h on \mathcal{M}_c implies p-quasi-convexity of $R_{h,v}$ on \mathcal{M}_c . Therefore, we only need to consider the case of I_h . By the definition of I_h in (1), it is obvious that I_h is p-quasi-convex if h is convex. Suppose that $h \in H^{\text{QSM}}$ which admits a representation as $I_h = S_{a,b} + kQ_\alpha^c$ with $a, b \geq 0$, $\alpha, c \in [0, 1]$ and $k \in \mathbb{R}$. Since the mapping kQ_α^c is monotone and has convex level sets (e.g., Wang and Wei (2020)), we know that kQ_α^c is both p-quasi-convex and p-quasi-concave, which in turn implies that I_h is p-quasi-convex by applying Lemma 1. Hence, we complete the proof. □

4.2 The main result and a corollary on unbounded space

The following theorem, which is the main technical result of the paper, establishes that the only possible generalized rank-dependent functions or signed Choquet functions with p -quasi-convexity are the ones with distortion functions in Proposition 4, i.e., those with a convex distortion function or a distortion function in (7).

Theorem 2. *Suppose that Assumption M holds and $h \in \mathcal{H}^{\text{BV}}$. The following statements are equivalent.*

- (i) $h \in \mathcal{H}^{\text{CX}} \cup \mathcal{H}^{\text{QSM}}$.
- (ii) I_h is p -quasi-convex on \mathcal{M} .
- (iii) $R_{h,v}$ is p -quasi-convex on \mathcal{M} for some $v \in \mathcal{V}_{\mathcal{M}}$.
- (iv) $R_{h,v}$ is p -quasi-convex on \mathcal{M} for all functions v .

Because of Theorem 2, we will denote by $\mathcal{H}^{\text{QCX}} = \mathcal{H}^{\text{CX}} \cup \mathcal{H}^{\text{QSM}}$, which is the set of all $h \in \mathcal{H}^{\text{BV}}$ for $R_{h,v}$ and I_h to be p -quasi-convex. Besides the class of convex distortion functions, the only other choices are $h \in \mathcal{H}^{\text{QSM}}$. One can immediately observe that all distortion functions in \mathcal{H}^{QSM} are neither continuous nor strictly monotone.

The following proposition illustrates that if the domain of $R_{h,v}$ is chosen to be the set of all two-point distributions on two specific points such that the values of v are distinct on these two points, then p -quasi-convexity of $R_{h,v}$ is equivalent to quasi-convexity of h . Hence, there are more cases of p -quasi-convex $R_{h,v}$ than in Theorem 2.

Proposition 5. *Let $x, y \in \mathbb{R}$ satisfying $x \neq y$, and define $\mathcal{M} = \{p\delta_x + (1-p)\delta_y : p \in [0, 1]\}$. Suppose that $h \in \mathcal{H}^{\text{BV}}$ and $v : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $v(x) \neq v(y)$. Then $R_{h,v}$ is p -quasi-convex on \mathcal{M} if and only if h is quasi-convex.*

Proof. The p -quasi-convexity of $R_{h,v}$ on \mathcal{M} is equivalent to

$$R_{h,v}((\lambda p + (1-\lambda)q)\delta_x + (\lambda(1-p) + (1-\lambda)(1-q))\delta_y) \leq \max\{R_{h,v}(p\delta_x + (1-p)\delta_y), R_{h,v}(q\delta_x + (1-q)\delta_y)\}$$

for any $p, q, \lambda \in [0, 1]$. Under some algebra calculations, this is equivalent to

$$h(\lambda p + (1-\lambda)q) \leq \max\{h(p), h(q)\}$$

for any $p, q, \lambda \in [0, 1]$. Hence, we obtain the desired equivalence. \square

Proposition 5 illustrates that Assumption M on \mathcal{M} and the constraint $v \in \mathcal{V}_{\mathcal{M}}$ are needed for a nontrivial characterization of p-quasi-convexity. Analogously to Proposition 5, one can check that for h quasi-convex and v only taking two values, $R_{h,v}$ is p-quasi-convex on any convex set of distributions.

Below we present a corresponding result to Theorem 2 that the generalized rank-dependent functions are defined on a general space that contains a distribution of unbounded random variables.

Corollary 2. *Suppose that Assumption M holds. Define a convex set \mathcal{M}' such that $\mathcal{M} \subseteq \mathcal{M}'$ and \mathcal{M}' contains the distribution of a random variable, denoted by X , that is unbounded both from below and from above. Let $h \in \mathcal{H}^{\text{BV}}$ and $v \in \mathcal{V}_{\mathcal{M}}$ satisfying $\text{ess-inf } v(X) = -\infty$ and $\text{ess-sup } v(X) = +\infty$. Then $R_{h,v} : \mathcal{M}' \rightarrow \mathbb{R}$ is p-quasi-convex if and only if h is convex and continuous or h is the distortion function of mixed quantiles $I_h = kQ_{1-\alpha}^c$ for some $\alpha \in (0, 1)$, $c \in [0, 1]$ and $k \in \mathbb{R}$.*

In the case of signed Choquet function (v is the identity), Corollary 2 formalizes the observation that, since asymmetric spreads are not finite-valued on a general space with the distribution of unbounded random variables, we are left with only the class of convex distortion functions and those that correspond to scaled mixed quantiles.

4.3 A unifying equivalence

In this section, we present a unifying equivalence result on signed Choquet functions to illustrate the power of our main result. In what follows, the space is assumed to be \mathcal{M}_c or \mathcal{X}_c , and I_h is *o-superadditive* if $I_h(X + Y) \geq I_h(X) + I_h(Y)$ for all $X, Y \in \mathcal{X}_c$.

Theorem 3. *If $h \in \mathcal{H}^{\text{BV}}$ is continuous or strictly monotone, then the following are equivalent: (i) h is convex; (ii) I_h is concave-order monotone; (iii) I_h is o-superadditive; (iv) I_h is o-concave; (v) I_h is o-quasi-concave; (vi) I_h is p-convex; (vii) I_h is p-quasi-convex.*

Theorem 3 can be broken down to a few pieces. The first six properties, (i)-(vi) in Theorem 3, are shown to be equivalent by Wang et al. (2020b, Theorem 3) without any assumption on $h \in \mathcal{H}^{\text{BV}}$.⁸ This result does not include p-quasi-convexity. Indeed, p-quasi-convexity is not always equivalent to the above six conditions. In particular, any left or right quantile is both p-quasi-convex and p-quasi-concave, but it is not o-concave or o-convex. Nevertheless, p-quasi-convexity and p-convexity are not too far away from each other. Wakker (1994, Theorem 24) showed that (i) and (vii) in Theorem 3 are equivalent if $h \in \mathcal{H}^{\text{DT}}$ is strictly increasing.⁹ It is obvious that the

⁸Wang et al. (2020b) used the narrative of risk measures, where results are stated for concave h ; we seamlessly translate these results by a sign change.

⁹Under the same assumption, Wakker and Yang (2021, Corollary 8) showed the equivalence of (v) quasi-concavity and (vii) p-quasi-convexity.

normalization $h(1) = 1$ does not matter, and thus \mathcal{H}^{DT} can be safely replaced by \mathcal{H}^{BV} in the above result. Note that $I_h = -I_{-h}$ for all $h \in \mathcal{H}^{\text{BV}}$. Hence, for any strictly decreasing $h \in \mathcal{H}^{\text{BV}}$, (i) and (vii) are also equivalent. Combining these arguments, we obtain the equivalence in Theorem 3 under the assumption that h is strictly monotone. From a technical standpoint, strict monotonicity rules out any non-trivial considerations for signed Choquet functions over dual utilities (up to a sign change). The remaining part of Theorem 3, concerning continuous h , relies on Theorem 2.

Another result about the equivalence of (i)-(vii) is based on L^q , $q \in [1, \infty)$, i.e., the normed space of random variables with finite q -th moment.

Corollary 3. *Let $h \in \mathcal{H}^{\text{BV}}$, $q \in [1, \infty)$. For a norm-continuous mapping $I_h : L^q \rightarrow \mathbb{R}$, (i)-(vii) in Theorem 3 are equivalent.*

Corollary 3 follows from Corollary 2 and the fact that mixed quantiles are not norm-continuous. Note that the same cannot be said on L^∞ since all signed Choquet functions are norm-continuous on L^∞ (e.g., Wang et al. (2020b, Theorem 1)). Different from the other results, Corollary 3 imposes a continuity condition on I_h , instead of h , to guarantee the equivalence of (i)-(vii).

5 Proofs of Section 4 and further consequences

This section is dedicated to a proof of Theorem 2, which includes Theorem 1 as a special case, and it further leads to the results in Corollaries 2 and 3. Moreover, characterizations of p -quasi-concavity and p -quasi-linearity, which are presented later, also follow from Theorem 2.

5.1 Proofs of Theorem 2 and its corollaries in Section 4

Note that (i) \Rightarrow (ii), (iii), (iv) are proved in Proposition 4, and it is obvious that (iv) \Rightarrow (iii). Next, we will show the assertion that (ii) \Rightarrow (i) implies (iii) \Rightarrow (i), and this means that it suffices to verify (ii) \Rightarrow (i) for the completeness of the proof of Theorem 2. To see this assertion, suppose that (ii) \Rightarrow (i) holds. Recall the set \mathcal{M} that satisfies Assumption M. We define the set of all v -transforms in \mathcal{M} as

$$\mathcal{M}^v = \{F \circ v^{-1} : F \in \mathcal{M}\} \supseteq \{p\delta_{v(x)} + (q-p)\delta_{v(y)} + (1-q)\delta_{v(z)} : 0 \leq p \leq q \leq 1\}.$$

It holds that \mathcal{M}^v satisfies Assumption M. By Proposition 3, if (iii) holds, then I_h is p -quasi-convex on \mathcal{M}^v . Using the result that (ii) \Rightarrow (i) holds (note that the three distinct points in Assumption M can be chosen arbitrarily, and now let them be $v(x), v(y), v(z)$), we arrive at (i). Hence, (iii) \Rightarrow (i) is verified.

In the following, we aim to prove (ii) \Rightarrow (i) where several technical lemmas are needed. When we mention Assumption **M**, $x, y, z \in \mathbb{R}$ represent the distinct points defined in Assumption **M**, and we assume without loss of generality that $x > y > z$. Suppose that \mathcal{M} satisfies Assumption **M** and $h \in \mathcal{H}^{\text{BV}}$. We define a bivariate function as follows

$$\pi(p, q) = (x - y)h(p) + (y - z)h(q), \quad (p, q) \in T_2 := \{(a, b) \in [0, 1] : a \leq b\}. \quad (8)$$

The first lemma shows that the p-quasi-convexity of I_h on \mathcal{M} implies that $\pi(p, q)$ is quasi-convex on T_2 .

Lemma 2. *Suppose that \mathcal{M} satisfies Assumption **M**. For $h \in \mathcal{H}^{\text{BV}}$, if I_h is p-quasi-convex on \mathcal{M} , then the function π defined in (8) is quasi-convex on T_2 . In particular, h is quasi-convex on $[0, 1]$.*

Proof. For any $0 \leq p \leq q \leq 1$, we have

$$\begin{aligned} I_h(p\delta_x + (q - p)\delta_y + (1 - q)\delta_z) &= xh(p) + y(h(q) - h(p)) + z(h(1) - h(q)) \\ &= zh(1) + (x - y)h(p) + (y - z)h(q) \\ &= zh(1) + \pi(p, q). \end{aligned}$$

One can verify that the p-quasi-convexity of I_h implies

$$\pi(\lambda p_1 + (1 - \lambda)p_2, \lambda q_1 + (1 - \lambda)q_2) \leq \max\{\pi(p_1, q_1), \pi(p_2, q_2)\} \quad (9)$$

for any $\lambda, p_1, p_2, q_1, q_2 \in [0, 1]$ with $p_1 \leq q_1$ and $p_2 \leq q_2$. This is equivalent to the quasi-convexity of π on T_2 . Let $p = 0$ in (8), the quasi-convexity of π implies the quasi-convexity of h on $[0, 1]$. This completes the proof. \square

The second lemma is a generalization of Lemma 26 of [Wakker \(1994\)](#), who considered the strictly increasing distortion functions, to the set \mathcal{H}^{BV} and the domain \mathcal{M} that satisfies Assumption **M**. For $h \in \mathcal{H}^{\text{BV}}$ and $0 \leq p < q \leq 1$ such that $h(p) \neq h(q)$, we define $\lambda_h(p, q)$ as

$$\lambda_h(p, q) = \frac{h(p)/2 + h(q)/2 - h(p/2 + q/2)}{|h(q) - h(p)|}. \quad (10)$$

The value of this function can be interpreted as a measure of the local convexity of h .

Lemma 3. *Let $h \in \mathcal{H}^{\text{BV}}$ and $0 \leq p < q \leq s < t \leq 1$. Suppose that \mathcal{M} satisfies Assumption **M** and I_h is p-quasi-convex on \mathcal{M} . If $h(p) \neq h(q)$, $h(s) \neq h(t)$ and $|h(q) - h(p)|(x - y) = |h(t) - h(s)|(y - z)$, then we have $\lambda_h(p, q) + \lambda_h(s, t) \geq 0$.*

Proof. Let $0 \leq p < q \leq s < t \leq 1$ and $h(p) \neq h(q)$ and $h(s) \neq h(t)$. By Lemma 2, it holds that h is quasi-convex on $[0, 1]$. Thus, one can check that only one of the following three cases will happen. Case 1: $h(p) < h(q) \leq h(s) < h(t)$; Case 2: $h(p) > h(q) \geq h(s) > h(t)$; Case 3: $h(p) > h(q)$ and $h(s) < h(t)$.

Let us first verify Cases 1 and 2. Suppose that $0 \leq p < q \leq s < t \leq 1$, and $h(p) < h(q) \leq h(s) < h(t)$ or $h(p) > h(q) \geq h(s) > h(t)$. All distributions in this proof are of the form

$$p\delta_{x_1} + \frac{q-p}{2}\delta_{x_2} + \frac{q-p}{2}\delta_{x_3} + (s-q)\delta_{x_4} + \frac{t-s}{2}\delta_{x_5} + \frac{t-s}{2}\delta_{x_6} + (1-t)\delta_{x_7} \quad (11)$$

with fixed probabilities and $x_1, \dots, x_7 \in \{x, y, z\}$ with $x_1 \geq \dots \geq x_7$. Hence, we will simply use $F = (x_1, \dots, x_7)$ to represent a distribution F , i.e., F has the form in (11). Define $F = (x, y, y, y, y, y, z)$ and $G = (x, x, x, y, z, z, z)$. One can verify that

$$I_h(F) - I_h(G) = (y - z)(h(t) - h(s)) - (x - y)(h(q) - h(p)).$$

Since $h(q) - h(p)$ and $h(t) - h(s)$ have the same sign, and $x > y > z$, the condition $|h(q) - h(p)|(x - y) = |h(t) - h(s)|(y - z)$ in the lemma implies $I_h(F) = I_h(G)$. By the p -quasi-convexity of I_h , we have $I_h((F + G)/2) \leq (I_h(F) + I_h(G))/2$ where $(F + G)/2 = (x, x, y, y, y, z, z)$. This leads to

$$\left(\frac{h(p) + h(q)}{2} - h\left(\frac{p+q}{2}\right) \right) (x - y) + \left(\frac{h(s) + h(t)}{2} - h\left(\frac{s+t}{2}\right) \right) (y - z) \geq 0.$$

Dividing by the positive factors in $|h(q) - h(p)|(x - y) = |h(t) - h(s)|(y - z)$, we have

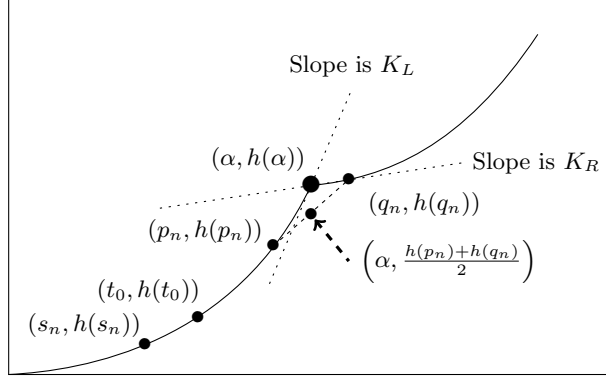
$$\frac{1}{|h(q) - h(p)|} \left(\frac{h(p) + h(q)}{2} - h\left(\frac{p+q}{2}\right) \right) + \frac{1}{|h(t) - h(s)|} \left(\frac{h(s) + h(t)}{2} - h\left(\frac{s+t}{2}\right) \right) \geq 0.$$

This gives $\lambda_h(p, q) + \lambda_h(s, t) \geq 0$. To see Case 3, we only need to take $F = (x, y, y, y, z, z, z)$ and $G = (x, x, x, y, y, y, z)$, and then, it follows from a similar proof of Cases 1 and 2 in the previous arguments. \square

The third Lemma is divided into three cases to consider nonconvex distortion functions. We will apply Lemmas 2, 3 and some results in Debreu and Koopmans (1982) to prove this lemma.

Lemma 4. *Let $h \in \mathcal{H}^{\text{BV}}$. Suppose that \mathcal{M} satisfies Assumption M and I_h is p -quasi-convex on \mathcal{M} . The following three statements hold.*

- (i) *If h is nonconvex on $(0, 1)$, then there is a point $\alpha \in (0, 1)$ such that h is a constant on both intervals $(0, \alpha)$ and $(\alpha, 1)$, and the constant values are different.*



Step 1. There exists only one nonconvexity kink α ; h is convex on $[0, \alpha)$ and on $(\alpha, 1]$.
 Step 2. Contradiction from being nonconstant on $(0, \alpha)$. Here, $p_n = \alpha - 1/n$, $q_n = \alpha + 1/n$,
 $|h(t_0) - h(s_n)|(x - y) = |h(p_n) - h(q_n)|(y - z)$, $K_L = h'_-(\alpha)$ and $K_R = h'_+(\alpha)$.

Figure 3: Graphic illustration of the proof of Lemma 4 (i).

(ii) If h is nonconvex on $[0, 1)$ and convex on $(0, 1)$, then $h(0+) > h(0) = 0$, $h(p) = h(0+)$ for all $p \in (0, 1)$ and $h(1-) \leq h(1)$.

(iii) If h is nonconvex on $(0, 1]$ and convex on $(0, 1)$, then $h(1-) > h(1)$, $h(p) = h(1-)$ for all $p \in (0, 1)$ and $h(0+) \leq h(0) = 0$.

Proof. By Lemma 2, the p -quasi-convexity of I_h implies the quasi-convexity of π defined in (8). Hence, it is sufficient to prove this lemma based on the quasi-convexity of π .

(i) We prove this statement in two steps (see Figure 3 for an illustration). First, we will show that there is at most one nonconvexity kink of h on $(0, 1)$. Second, we will show that h should be a constant both on the left and the right of the nonconvexity kink.

Step 1: Suppose that h is nonconvex on $(a, b) \subseteq (0, 1)$, and we will show that there is at most one nonconvexity kink on (a, b) . To see this, note that the function π defined in (8) is quasi-convex both on $(0, a) \times (a, b)$ and $(a, b) \times (b, 1)$. Since h is nonconvex on (a, b) , it follows from Debreu and Koopmans (1982, Theorem 2) that h is convex both on $(0, a)$ and $(b, 1)$. Applying this result to smaller and smaller subintervals of (a, b) , there is one point $\alpha \in (a, b)$ (the nonconvexity kink) such that h is convex both on $(0, \alpha)$ and $(\alpha, 1)$.

Step 2: We aim to verify that h is a constant on both intervals $(0, \alpha)$ and $(\alpha, 1)$, and this will complete the proof of (i). We only show that h is a constant on $(0, \alpha)$ as the proof of the other case is similar. Assume now by contradiction that h is nonconstant on $(c, d) \subseteq (0, \alpha)$. Let us consider the bivariate function π (see (8)) on $(c, d) \times (d, 1)$. Since α is a nonconvexity kink, we have that h is nonconstant on $(d, 1)$. Hence, h is nonconstant both on (c, d) and $(d, 1)$. Note that π is quasi-convex. It follows from Debreu and Koopmans (1982, Theorem 1) that h is continuous on

$(d, 1)$. Since $\alpha \in (d, 1)$ and h is convex on both $(0, \alpha)$ and $(\alpha, 1)$, it holds that h is continuous on $(0, 1)$. Now, recall $\lambda_h(p, q)$ defined in (10), and we rewrite it as

$$\lambda_h(p, q) = \frac{h(p)/2 + h(q)/2 - h(p/2 + q/2)}{|h(q) - h(p)|} = \frac{f(p, q) - g(p, q)}{2|f(p, q) + g(p, q)|}, \quad (12)$$

where

$$f(p, q) = \frac{h(q) - h(p/2 + q/2)}{(q - p)/2} \quad \text{and} \quad g(p, q) = \frac{h(p/2 + q/2) - h(p)}{(q - p)/2}.$$

By the convexity of h on $(0, \alpha)$ and also noting that h is nonconstant on $(0, \alpha)$, there exists t_0 in $(0, \alpha)$ such that h has a nonzero left derivative at t_0 . Therefore, it follows from (12) that

$$\lambda_h(s, t_0) \rightarrow 0 \quad \text{as } s \uparrow t_0. \quad (13)$$

Denote $K_L = h'_-(\alpha)$ and $K_R = h'_+(\alpha)$ by the left and right derivative of h at α . Since α is a point of nonconvex kink, we have $\infty \geq K_L > K_R \geq -\infty$. We assert that $K_L K_R \geq 0$, and this implies $|K_L + K_R| > 0$. Otherwise, we have $K_L > 0$ and $K_R < 0$ which implies h is not quasi-convex on $[0, 1]$, and this contradicts to Lemma 2. Take $p_n = \alpha - 1/n$ and $q_n = \alpha + 1/n$ for $n \in \mathbb{N}$. Below we aim to verify the assertion that

$$\lim_{n \rightarrow \infty} \lambda_h(p_n, q_n) < 0. \quad (14)$$

To see this, by (12), we have

$$\lambda_h(p_n, q_n) = \frac{a_n - b_n}{2|a_n + b_n|},$$

where

$$a_n = n(h(\alpha + 1/n) - h(\alpha)) \quad \text{and} \quad b_n = n(h(\alpha) - h(\alpha - 1/n)).$$

Note that $a_n \rightarrow K_R$ and $b_n \rightarrow K_L$ as n tends to infinity. If $\infty > K_L > K_R > -\infty$, then

$$\lambda_h(p_n, q_n) = \frac{a_n - b_n}{2|a_n + b_n|} \rightarrow \frac{K_R - K_L}{2|K_R + K_L|} < 0.$$

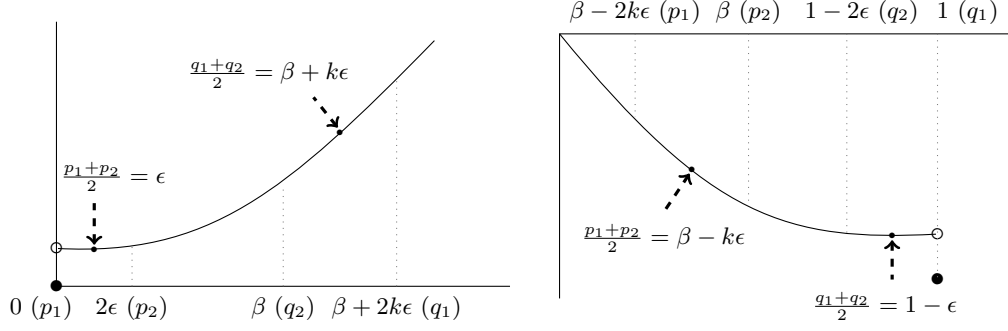


Figure 4: Contradiction from h being nonconstant on $(0, 1)$ in the proof of (ii) and (iii) of Lemma 4: the Left panel illustrates (ii) and the right panel illustrates (iii).

If $K_L = +\infty$ and $K_R \in \mathbb{R}$, then

$$\lambda_h(p_n, q_n) = \frac{a_n - b_n}{2|a_n + b_n|} = \frac{a_n/b_n - 1}{2|a_n/b_n + 1|} \rightarrow -\frac{1}{2}.$$

If $K_L \in \mathbb{R}$ and $K_R = -\infty$, then

$$\lambda_h(p_n, q_n) = \frac{a_n - b_n}{2|a_n + b_n|} = \frac{1 - b_n/a_n}{2|1 + b_n/a_n|} \rightarrow -\frac{1}{2}.$$

The case of $K_L = \infty$ and $K_R = -\infty$ cannot happen as h should be quasi-convex on $[0, 1]$ by Lemma 2. Therefore, we have verified (14). Recall the definition of t_0 in (13). Define a sequence $\{s_n\}_{n \in \mathbb{N}}$ such that $s_n < t_0$ for all $n \in \mathbb{N}$, $s_n \rightarrow t_0$ and $|h(t_0) - h(s_n)|(x - y) = |h(p_n) - h(q_n)|(y - z)$ for large enough n . Because h is continuous on $(0, 1)$, such a sequence exists. Noting that $t_0 < \alpha$, we have $s_n < t_0 < p_n < q_n$ for large enough n . By Lemma 3, we obtain $\lambda_h(s_n, t_0) + \lambda_h(p_n, q_n) \geq 0$ for large enough n . However, by combining (13) and (14), we have $\lambda_h(s_n, t_0) + \lambda_h(p_n, q_n) < 0$ for large enough n , and this yields a contradiction. Therefore, we conclude that h should be a constant on both intervals $(0, \alpha)$ and $(\alpha, 1)$, and this completes the proof of (i).

(ii): Suppose that h is nonconvex on $[0, 1)$ and convex on $(0, 1)$. This implies that $h(0+) > h(0) = 0$. It follows from Lemma 2 that h is quasi-convex on $[0, 1]$. Combining this with $h(0+) > h(0)$, it holds that h is increasing on $[0, 1]$, which implies $h(1-) \leq h(1)$. It remains to verify that $h(p) = h(0+)$ for all $p \in (0, 1)$. Assume now by contradiction that h is nonconstant on $(0, 1)$. Since h is convex on $(0, 1)$, there exists $\beta \in (0, 1)$ such that h is strictly increasing on $[\beta, 1)$. Let $\epsilon > 0$ and $k = 2(x - y)/(y - z)$, and denote by $p_1 = 0$, $p_2 = 2\epsilon$, $q_1 = \beta + 2k\epsilon$ and $q_2 = \beta$ such that $p_2 \leq q_2$ and $q_1 < 1$ (see the left panel of Figure 4). We calculate the following items:

$$A_1 := \pi(p_1, q_1) = (y - z)h(\beta + 2k\epsilon),$$

$$A_2 := \pi(p_2, q_2) = (x - y)h(2\epsilon) + (y - z)h(\beta),$$

and

$$A := \pi\left(\frac{p_1 + p_2}{2}, \frac{q_1 + q_2}{2}\right) = (x - y)h(\epsilon) + (y - z)h(\beta + k\epsilon).$$

On the one hand, since $h(\epsilon) > h(0+) > 0$ for any $\epsilon \geq 0$ and h is continuous on $(0, 1)$, we have $A > A_1$ for small enough $\epsilon > 0$. On the other hand,

$$\begin{aligned} A_2 - A &= (x - y)(h(2\epsilon) - h(\epsilon)) - (y - z)(h(\beta + k\epsilon) - h(\beta)) \\ &= \epsilon(x - y) \left(\frac{h(2\epsilon) - h(\epsilon)}{\epsilon} - \frac{2(h(\beta + k\epsilon) - h(\beta))}{k\epsilon} \right) < 0, \end{aligned}$$

where the inequality holds because h is convex on $(0, 1)$ and strictly decreasing on $[\beta, 1)$. Therefore, we conclude that $A > \max\{A_1, A_2\}$ which contradicts the quasi-convexity of π . This completes the proof of (ii).

(iii): The proof of this statement is similar to (ii). Suppose that h is nonconvex on $(0, 1]$ and convex on $(0, 1)$. This implies $h(1-) > h(1)$. By Lemma 2, we know that h is quasi-convex on $[0, 1]$. Combining with $h(1-) > h(1)$, we conclude that h is decreasing on $[0, 1]$. Hence, we have $h(0+) \leq h(0)$. It remains to verify that h is a constant on $(0, 1)$. Assume now by contradiction that h is nonconstant on $(0, 1)$. Then, there exists $\beta \in (0, 1)$ such that h is strictly decreasing on $(0, \beta]$. Let $\epsilon > 0$ and $k = 2(y - z)/(x - y)$, and denote by $p_1 = \beta - 2k\epsilon$, $p_2 = \beta$, $q_1 = 1$ and $q_2 = 1 - 2\epsilon$ such that $p_2 \leq q_2$ and $p_1 > 0$ (see the right panel Figure 4). We calculate the following items:

$$A_1 := \pi(p_1, q_1) = (x - y)h(\beta - 2k\epsilon) + (y - z)h(1),$$

$$A_2 := \pi(p_2, q_2) = (x - y)h(\beta) + (y - z)h(1 - 2\epsilon),$$

and

$$A := \pi\left(\frac{p_1 + p_2}{2}, \frac{q_1 + q_2}{2}\right) = (x - y)h(\beta - k\epsilon) + (y - z)h(1 - \epsilon).$$

On the one hand, since $h(1 - \epsilon) \geq h(1-) > h(1)$ for any $\epsilon > 0$ and h is continuous on $(0, 1)$, we have $A > A_1$ for small enough $\epsilon > 0$. On the other hand,

$$\begin{aligned} A_2 - A &= (x - y)(h(\beta) - h(\beta - k\epsilon)) - (y - z)(h(1 - \epsilon) - h(1 - 2\epsilon)) \\ &= \epsilon(y - z) \left(\frac{2(h(\beta) - h(\beta - k\epsilon))}{k\epsilon} - \frac{h(1 - \epsilon) - h(1 - 2\epsilon)}{\epsilon} \right) < 0, \end{aligned}$$

where the inequality holds because h is convex on $(0, 1)$ and strictly increasing on $(0, \beta]$. Therefore,

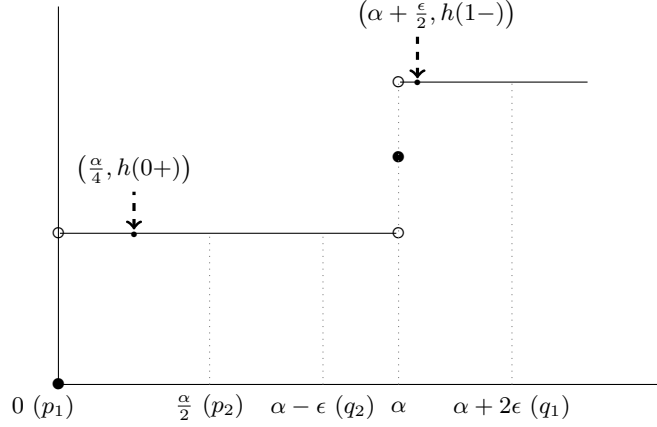


Figure 5: Contradiction from $h(0+) > 0$.

we conclude that $A > \max\{A_1, A_2\}$ which implies π is not quasi-convex, a contradiction. Hence, we complete the proof of (iii). \square

In the following, we give the proof of (ii) \Rightarrow (i) in Theorem 2, and this will complete the proof of Theorem 2.

Proof of (ii) \Rightarrow (i) in Theorem 2. It suffices to verify that for nonconvex $h \in \mathcal{H}^{\text{BV}}$, the p-quasi-convexity of I_h implies $h \in \mathcal{H}^{\text{QSM}}$. To see this, we divide h into three cases as shown in Lemma 4: Case 1. h is nonconvex on $(0, 1)$; Case 2. h is nonconvex on $[0, 1)$ and convex on $(0, 1)$; Case 3. h is nonconvex on $(0, 1]$ and convex on $(0, 1)$.

By Lemma 4, if h is the form of Case 2 or Case 3, then one can check that $h \in \mathcal{H}^{\text{QSM}}$. Thus, it remains to consider Case 1.

Suppose that h is nonconcave on $(0, 1)$. By Lemma 4 (i), there exists $\alpha \in (0, 1)$ such that h is a constant both on $(0, \alpha)$ and $(\alpha, 1)$ and $h(\alpha-) \neq h(\alpha+)$. Hence, h can be represented as

$$h(p) = h(0+)\mathbb{1}_{\{0 < p < \alpha\}} + h(\alpha)\mathbb{1}_{\{p = \alpha\}} + h(1-)\mathbb{1}_{\{\alpha < p < 1\}} + h(1)\mathbb{1}_{\{p = 1\}}, \quad p \in [0, 1].$$

In order to show that $h \in \mathcal{H}^{\text{QSM}}$, we need to verify two assertions: (a) $h(0+) \leq 0$ and $h(1-) \leq h(1)$; (b) $(h(\alpha) - h(0+))(h(1-) - h(\alpha)) \geq 0$. Both assertions will be proved by counter-evidence. For (a), we assume by contradiction that $h(0+) > 0$ or $h(1-) > h(1)$. We only consider the case of $h(0+) > 0$ as the case of $h(1-) > h(1)$ is similar. By Lemma 2, we know that h is quasi-convex, and combining with $h(0+) > 0$, it holds that h is increasing on $[0, 1]$ which implies $h(0+) = h(\alpha-) < h(\alpha+) = h(1-)$. Let $p_1 = 0$, $p_2 = \alpha/2$, $q_1 = \alpha + 2\epsilon$ and $q_2 = \alpha - \epsilon$ such that $\epsilon > 0$, $p_2 \leq q_2 < q_1 < 1$ (see Figure 5). Recall the function π defined in (8), we have

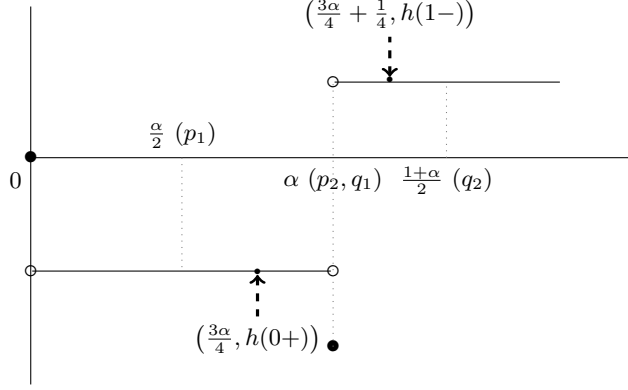


Figure 6: Contradiction from $h(\alpha) < \min\{h(0+), h(1-)\}$.

$$\pi(p_1, q_1) = (x - y)h(0) + (y - z)h(1-), \quad \pi(p_2, q_2) = (x - z)h(0+),$$

and

$$\pi\left(\frac{p_1 + p_2}{2}, \frac{q_1 + q_2}{2}\right) = (x - y)h(0+) + (y - z)h(1-).$$

One can check that $\pi((p_1 + p_2)/2, (q_1 + q_2)/2) > \max\{\pi(p_1, q_1), \pi(p_2, q_2)\}$ which implies π is not quasi-convex, and this contradicts Lemma 2. Hence, we have $h(0+) \geq 0$. For (b), we assume by contradiction that $h(\alpha) < \min\{h(0+), h(1-)\}$ or $h(\alpha) > \max\{h(0+), h(1-)\}$. The case of $h(\alpha) > \max\{h(0+), h(1-)\}$ contradicts Lemma 2 as h should be quasi-convex. If $h(\alpha) < \min\{h(0+), h(1-)\}$, let $p_1 = \alpha/2$, $p_2 = q_1 = \alpha$ and $q_2 = (1 + \alpha)/2$ (see Figure 6). We have

$$\pi(p_1, q_1) = (x - y)h(0+) + (y - z)h(\alpha), \quad \pi(p_2, q_2) = (x - y)h(\alpha) + (y - z)h(1-)$$

and

$$\pi\left(\frac{p_1 + p_2}{2}, \frac{q_1 + q_2}{2}\right) = (x - y)h(0+) + (y - z)h(1-).$$

One can check that $\pi((p_1 + p_2)/2, (q_1 + q_2)/2) > \max\{\pi(p_1, q_1), \pi(p_2, q_2)\}$, which yields a contradiction to Lemma 2. Hence, we conclude that if h is the form of Case 1, then $h \in \mathcal{H}^{\text{QSM}}$. This completes the proof. \square

Proof of Corollary 2. The “if” statement is straightforward to verify. To show the “only if” statement, we first note that, since \mathcal{M}' contains \mathcal{M} which satisfies Assumption M, it follows from Theorem 2 that $h \in \mathcal{H}^{\text{QCX}}$. Since the distribution of X is in \mathcal{M}' , and X is unbounded from below and above, we know that $h \in \mathcal{H}^{\text{BV}}$ must be continuous at 0 and 1; otherwise $Q_1(v(X)) = \infty$ and $Q_0(v(X)) = -\infty$ would lead to $R_{h,v}(X) \notin \mathbb{R}$. This continuity condition for $h \in \mathcal{H}^{\text{QCX}}$ leads to the cases stated in the corollary. \square

Proof of Corollary 3. Based on Corollary 2, it suffices to show that mixed quantiles are not norm-continuous. To verify this fact, take $\alpha \in (0, 1)$ and $c \in [0, 1]$, and a uniform random variable U on $[0, 1]$. Let $X_n = a\mathbf{1}_{\{U \in [\alpha - \epsilon, \alpha + \epsilon]\}} + b\mathbf{1}_{\{U > \alpha + \epsilon\}}$ and $X = b\mathbf{1}_{\{U > \alpha + \epsilon\}}$ for some $0 < a < b$ with $a \neq cb$. It is clear that $Q_\alpha^c(X) = cb$, $Q_\alpha^c(X_n) = a$ for each $n \in \mathbb{N}$, and $X_n \rightarrow X$ in L^q . This example justifies the non-continuity of Q_α^c . \square

5.2 P-quasi-concavity and p-quasi-linearity

A considerable convenience to working with non-monotone $h \in \mathcal{H}^{\text{BV}}$ is that we can put a negative sign in front of h without leaving the class, which is not the case for $h \in \mathcal{H}^{\text{DT}}$. In particular, we have $R_{h,v} = -R_{-h,v}$ for $h \in \mathcal{H}^{\text{BV}}$. Therefore, $R_{h,v}$ is p-quasi-concave if and only if $R_{-h,v}$ is p-quasi-convex, and all results on p-quasi-convexity immediately translate into results on p-quasi-concavity.

Corollary 4. *Suppose that Assumption M holds. For $h \in \mathcal{H}^{\text{BV}}$ and $v \in \mathcal{V}_M$, $R_{h,v}$ is p-quasi-concave on \mathcal{M} if and only if $h \in (-\mathcal{H}^{\text{QCX}})$, that is, h is concave or $-h \in \mathcal{H}^{\text{QSM}}$.*

Note that p-quasi-linearity of a functional ρ means that it is both p-quasi-convex and p-quasi-concave. Combining Theorem 2 and Corollary 4, a characterization of generalized rank-dependent functions with p-quasi-linearity is obtained.

Corollary 5. *Suppose that Assumption M holds. For $h \in \mathcal{H}^{\text{BV}}$ and $v \in \mathcal{V}_M$, $R_{h,v}$ is p-quasi-linear on \mathcal{M} if and only if the signed Choquet function I_h has one of the following forms.*

- (i) $I_h = k\mathbb{E}$ for some $k \in \mathbb{R}$ where \mathbb{E} represents the expectation.
- (ii) $I_h = k(cQ_1 + (1 - c)Q_0)$ for some $k \in \mathbb{R}$ and $c \in [0, 1]$.
- (iii) $I_h = kQ_{1-\alpha}^c$ for some $k \in \mathbb{R}$, $c \in [0, 1]$ and $\alpha \in (0, 1)$.

Proof. By Theorem 2 and Corollary 4, we know that $R_{h,v}$ is p-quasi-linear if and only if $h \in \mathcal{H}^{\text{QCX}} \cap (-\mathcal{H}^{\text{QCX}})$. It is straightforward to check that $R_{h,v}$ is one of the three forms in the corollary. \square

Although p-quasi-linearity by definition is not related to the monotonicity of the distortion function, all three forms of the distortion functions in Corollary 5 are monotone. This is not surprising. Because p-quasi-linearity of I_h implies quasi-linearity of h (see Lemma 2 and use a parallel result for the case of p-quasi-concavity), and a quasi-linear univariate function must be monotone.

Remark 2. As a direct result of Corollary 5, only the three forms of I_h in the corollary are possible to make I_h p-quasi-linearity under Assumption M. This is not new as Wang and Wei (2020) showed a same result for I_h with CxLS (slightly weaker than p-quasi-linearity). Nevertheless, Wang and Wei (2020) worked on a set containing all three-point distributions, a stronger condition than ours as the points are fixed in Assumption M.

6 A conflict between o-convexity and p-convexity

In this section, we illustrate a conflict between o-convexity and p-convexity for constant-additive mappings; that is, a continuous and constant-additive mapping cannot be both o-convex and p-convex on \mathcal{X}_c or \mathcal{M}_c unless it is a multiple of the expectation. Recall that o-convexity of a mapping on \mathcal{M}_c or \mathcal{X}_c is defined as convexity on \mathcal{X}_c . A mapping $\rho : \mathcal{X}_c \rightarrow \mathbb{R}$ is *constant additive* if $\rho(X + c) = \rho(X) + \rho(c)$ for $X \in \mathcal{X}_c$ and $c \in \mathbb{R}$. On \mathcal{X}_c , continuity is with respect to the supremum-norm. Continuous and constant-additive mappings on \mathcal{X}_c include, but are not limited to, all signed Choquet functions and normalized monetary risk measures (Föllmer and Schied (2016)).

Proposition 6. *For a continuous and constant-additive mapping $\rho : \mathcal{X}_c \rightarrow \mathbb{R}$, the following are equivalent:*

- (i) ρ is o-convex and p-convex;
- (ii) ρ is o-concave and p-concave;
- (iii) $\rho = k\mathbb{E}$ for some $k \in \mathbb{R}$.

Proof. Note that (i) and (ii) are symmetric, and (iii) \Rightarrow (i) is trivial. It suffices to show the direction (i) \Rightarrow (iii). Since a p-convex mapping is necessarily law-based, we equivalently formula ρ on \mathcal{M}_c . Denote by δ_x the point-mass at $x \in \mathbb{R}$. Denote by $k = \rho(1)$. Note that $\rho(x) = kx$ for $x \in \mathbb{R}$ since ρ is constant additive and continuous. Let $F = \sum_{i=1}^n p_i \delta_{x_i}$ for some numbers $p_1, \dots, p_n \geq 0$ which add up to 1 and $\{x_1, \dots, x_n\} \subseteq \mathbb{R}$. Let $X \sim F$. By p-convexity of ρ , we have $\rho(F) \leq \sum_{i=1}^n p_i \rho(\delta_{x_i}) = \sum_{i=1}^n p_i k x_i = k\mathbb{E}[X]$. Since ρ is law-based, continuous and o-convex, it is convex-order monotone by e.g., the representation in Liu et al. (2020, Theorem 2.2). Thus, $\rho(X) \geq \rho(\mathbb{E}[X]) = k\mathbb{E}[X]$ for all $X \in \mathcal{X}_c$. Putting the above two inequalities together, we have $\rho(X) = k\mathbb{E}[X]$ for all finitely supported random variables $X \in \mathcal{X}_c$.

For a general $X \in \mathcal{X}_c$, let $X_n = \lfloor nX \rfloor / n$ for $n \in \mathbb{N}$, which is an approximation of X . It is clear that $X_n \rightarrow X$ as $n \rightarrow \infty$, and $|X_n - X| \leq 1/n$. Using continuity of ρ again, we have $\rho(X_n) \rightarrow \rho(X)$ as $n \rightarrow \infty$. Since $\rho(X_n) = k\mathbb{E}[X_n] \rightarrow k\mathbb{E}[X]$ as $n \rightarrow \infty$, we obtain $\rho(X) = k\mathbb{E}[X]$. \square

The same conclusion in Proposition 6 holds for $\rho : L^p \rightarrow \mathbb{R}$ for $p \in [1, \infty)$ following the same proof.

Remark 3. Constant additivity of ρ is essential for Proposition 6. A mapping $\rho : \mathcal{X}_c \rightarrow \mathbb{R}$ that is monotone, p-convex and o-convex does not need to be p-linear. For an example, take $\rho_1 : X \mapsto \mathbb{E}[f(X)]$ and $\rho_2 : X \mapsto \mathbb{E}[g(X)]$ where f and g are two increasing convex functions. Clearly, ρ_1 and ρ_2 are both o-convex and p-linear. Since convexity is preserved under a maximum operation, the mapping $\rho := \max\{\rho_1, \rho_2\}$ is o-convex and p-convex, but it is not p-linear unless $f \geq g$ or $g \geq f$. The reason that the proof does not work in this case is that, by letting $\ell(x) = \rho(x)$ for $x \in \mathbb{R}$, we can show using the argument above that $\ell(\mathbb{E}[X]) \leq \rho(X) \leq \mathbb{E}[\ell(X)]$, but this does not pin down ρ unless ℓ is linear.

7 Conclusion

Probabilistic risk aversion (i.e., p-quasi-convexity) is characterized for rank-dependent utilities (Theorem 1) and generalized rank-dependent functions (Theorem 2). A new class of functionals, the mean-quantile mixtures, is shown to be the only class of dual utilities that are p-quasi-convex and p-locally indifferent (Proposition 2). We have chosen to use p-convexity and o-concavity to present our main results, to be consistent with the literature on decision theory (e.g., Quiggin (1993); Wakker (2010)); by a simple sign change, we obtain corresponding results for p-concavity and o-convexity, a convention that is more common in the literature of risk management (e.g., McNeil et al. (2015); Föllmer and Schied (2016)). Based on the characterization for generalized rank-dependent functions, we obtain a unified result of signed Choquet functions (Theorem 3) containing seven equivalent conditions for p-quasi-convexity. Our results are formulated for the more general objects, namely, signed Choquet functions and generalized rank-dependent functions. The corresponding results for dual utilities and rank-dependent utilities are also new, and our results help to understand classic decision models by disentangling monotonicity from other important properties.

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