Cash-subadditive risk measures without quasi-convexity

Xia Han[*]   Qiuqi Wang[†]   Ruodu Wang[‡]   Jianming Xia[§]

January 25, 2025

Abstract

In the literature on risk measures, cash subadditivity was proposed to replace cash additivity, motivated by the presence of stochastic or ambiguous interest rates and defaultable contingent claims. Cash subadditivity has been traditionally studied together with quasi-convexity, in a way similar to cash additivity with convexity. In this paper, we study cash-subadditive risk measures without quasi-convexity. One of our major results is that a general cash-subadditive risk measure can be represented as the lower envelope of a family of quasi-convex and cash-subadditive risk measures. Representation results of cash-subadditive risk measures with some additional properties are also examined. The notion of quasi-star-shapedness, which is natural analogue of and star-shapedness, is introduced, and we obtain a corresponding representation result via the lower envelope of normalized, quasi-convex and cash-subadditive risk measures.

KEYWORDS: cash subadditivity, quasi-convexity, stochastic dominance, star-shapedness, Lambda-VaR

1 Introduction

The quantification of market risk for pricing, portfolio selection, and risk management purposes has long been a point of interest to researchers and practitioners in finance. Measures of risk have been widely adopted to assess the riskiness of financial positions and determine capital reserves. Value-at-risk (VaR) has been one of the most commonly adopted risk measures in industry but is criticized due to its fundamental deficiencies; for instance, it does not account for "tail risk" and it lacks for subadditivity or convexity; see e.g., Daníelsson et al. (2001) and McNeil et al. (2015). In light of this, the notion of coherent risk measures that satisfy a set of reasonable axioms (monotonicity, cash additivity, subadditivity and positive homogeneity) was introduced by Artzner et al. (1999) and extensively treated by Delbaen (2002). Convex risk measures were introduced by Frittelli and Rosazza Gianin (2002) and Föllmer and Schied (2002) with convexity replacing

---

[*]School of Mathematical Sciences and LPMC, Nankai University, China. Email: xiahan@nankai.edu.cn

[†]Maurice R. Greenberg School of Risk Science, Georgia State University, U.S.A. Email: qwang30@gsu.edu

[‡]Department of Statistics and Actuarial Science, University of Waterloo, Canada. Email: wang@uwaterloo.ca

[§]Key Laboratory of Random Complex Structures and Data Science (RCSDS), National Center for Mathematics and Interdisciplinary Sciences (NCMIS), Academy of Mathematics and Systems Science, Chinese Academy of Sciences, China. Email: xia@amss.ac.cn.

subadditivity and positive homogeneity. There have been many other developments in the past two decades in various directions; see Föllmer and Schied (2016) and the references therein.

A common feature of all above risk measures is that the axiom of cash additivity (also called cash invariance or translation invariance) is employed. The cash additivity axiom has been challenged, in particular, by El Karoui and Ravanelli (2009), in a relevant context. The main motivation for cash additivity is that the random losses should be discounted by a constant numéraire. Therefore, cash additivity fails as soon as there is any form of uncertainty about interest rates. For this reason, El Karoui and Ravanelli (2009) replaced cash additivity by cash subadditivity and provided a representation result for convex cash-subadditive risk measures. In this context, Cerreia-Vioglio et al. (2011) argued that quasi-convexity rather than convexity is the appropriate mathematical translation of the statement "diversification should not increase the risk" and introduced the notion of quasi-convex cash-subadditivie risk measures. Farkas et al. (2014) studied general risk measures to model defaultable contingent claims and discussed their relationship with cash-subadditive risk measures, and Arduca and Munari (2024) further studied risk measures beyond frictionless markets. For other related work on cash subadditivity and quasi-convexity, see Frittelli and Maggis (2011), Cont et al. (2013), Drapeau and Kupper (2013), Frittelli et al. (2014) and Munari (2015). In decision theory, the economic counterpart of quasi-convexity of risk measures is quasi-concavity of utility functions, which is classically associated to uncertainty aversion in the economics of uncertainty; see, e.g., Schmeidler (1989), Cerreia-Vioglio et al. (2011), Mastrogiacomo and Rosazza Gianin (2015) and Liebrich and Svindland (2019).

The main aim of this paper is a thorough understanding of cash-subadditive risk measures when quasi-convexity, or the stronger property of convexity, is absent. This class of risk measures is very broad and, with proper normalization, it contains a wide majority of risk measure or preference functional considered in the literature. By relaxing cash additivity, both the theory of risk measures and that of expected utility and rank-dependent utility (Quiggin (1982)) can be included within the same framework. For instance, the mapping $X \mapsto -\mathbb{E}[u(-X)]/m$ for any increasing utility function u with derivative bounded above by $m > 0$ belongs to this class (recall that the constant m does not matter when modeling utility preferences); the same holds true if $\mathbb{E}$ is replaced by a non-additive and normalized Choquet integral (Yaari (1987); Schmeidler (1989)). Here, the utility function u may not be convex or concave; see the recent discussions and examples in Müller et al. (2017) and Castagnoli et al. (2022) for non-convex and non-concave loss and utility functions.

Cash-additive risk measures without convexity have been actively studied in the recent literature. In particular, a few representation results were obtained by Mao and Wang (2020), Jia et al. (2020) and Castagnoli et al. (2022). As a common feature, such risk measures can be represented as the infimum over a collection of convex and cash-additive risk measures (see Table 1 below),[1] in contrast to the classic theory of convex risk measures where representations are typically based on a supremum. In a similar fashion, one of our main results states that a general cash-subadditive risk measure can be represented as the lower envelope of a family of quasi-convex cash-subadditive

---

[1]The risk measures studied by Jia et al. (2020) and Castagnoli et al. (2022) have similar representations; their differences are studied by Moresco and Righi (2022).

risk measures.

In addition to a representation of general cash-subadditive risk measures, we will also give implicit and explicit representations of cash-subadditive risk measures with additional properties including quasi-star-shapedness, normalization (that is, $\rho(x) = x$ for all real x) and SSD-consistency (that is, consistency with second-order stochastic dominance). In particular, similarly to the argument that convexity does not fit well with cash-subadditive risk measures, star-shapedness introduced by Castagnoli et al. (2022) is no longer a natural property beyond the framework of cash-additive risk measures. In this sense, we introduce the property of quasi-star-shapedness induced naturally from quasi-convexity, and obtain a representation result of cash-subadditive risk measures that are normalized and quasi-star-shaped. It turns out that the representation result also holds true if we change normalization to a weaker version which we call quasi-normalization.

We examine a few other problems studied by Mao and Wang (2020), now under a general framework of cash subadditivity. Apart from the major differences, it also turns out that some of results obtained by Mao and Wang (2020) hold under the extended framework of cash subadditivity. A comparison of our results and some results in the literature is summarized in Section 6.

The new property of quasi-star-shapedness has a sound decision-theoretic foundation. Translating it into the setting of Anscombe and Aumann (1963), it means that the decision maker always prefers to replace part of an uncertain (random) payoff with an equally favourable certain (non-random) payoff. This property is a weaker requirement than the uncertainty aversion axiom studied by Maccheroni et al. (2006), which corresponds to quasi-convexity in our setting.

The rest of the paper is organized as follows. In Section 2, some preliminaries on risk measures are collected and the definition of cash-subadditive risk measures is given. Two new properties, quasi-star-shapedness and quasi-normalization, are introduced in Section 3 and we provide a few related results. In particular, we obtain a new formula (Theorem 3.1) for ΛVaR introduced by Frittelli et al. (2014), which is an example of quasi-star-shaped, quasi-normalized and cash-subadditive risk measures. Quasi-star-shapedness of other cash-subadditive risk measures is also discussed. In Section 4, representation results for general cash-subadditive risk measures are established. Section 5 contains representation results and other technical results on cash-subadditive risk measures with further properties including quasi-star-shapedness and SSD-consistency. Section 6 concludes the paper, and the appendix contains some further technical results and discussions that are not directly used in the main text. As the main message of this paper, most of the existing results on non-convex cash-additive risk measures have a nice parallel version for non-quasi-convex cash-subadditive risk measures, although they often require more sophisticated analysis to establish.

2   Cash-subadditive risk measures

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{M}_f$ be the set of finitely additive probabilities on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to P, and $\mathcal{M}$ represent the subset of $\mathcal{M}_f$ consisting of all its countably additive elements, i.e., probability measures. Let $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, P)$ be the set of all essentially bounded random variables on $(\Omega, \mathcal{F}, P)$, where P-a.s. equal random variables are

treated as identical.[2] Let a random variable $X \in \mathcal{X}$ represent the random loss faced by financial institutions in a fixed period of time; a positive value of $X \in \mathcal{X}$ represents a loss and a negative X represents a surplus; this sign convention is used by, e.g., McNeil et al. (2015). We write $X \stackrel{\mathrm{d}}{=} Y$ if two random variables $X, Y \in \mathcal{X}$ follow the same distribution under P. Throughout the paper, "increasing" and "decreasing" are in the nonstrict (weak) sense, $a \vee b$ (resp. $a \wedge b$) is the maximum (resp. minimum) between real numbers a and b, and $a_+ = a \vee 0$.

A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is called a *risk measure* if it satisfies:

Monotonicity: $\rho(X) \leqslant \rho(Y)$ for all $X, Y \in \mathcal{X}$ with $X \leqslant Y$.

Monotonicity is self-explanatory and common in the literature on risk management, e.g., Artzner et al. (1999). It means that if the loss increases for almost all scenarios $\omega \in \Omega$, then the capital requirement in order for the financial position to be acceptable should increase as well. The risk measure ρ is called a *monetary risk measure* if it further satisfies

Cash additivity: $\rho(X + m) = \rho(X) + m$ for all $X \in \mathcal{X}$ and $m \in \mathbb{R}$.

Cash additivity (also called cash invariance or translation invariance) intuitively means that the risk measure $\rho(X)$ is the amount of capital that needs to be added to the financial position X to make it acceptable. Cash additivity is a nice and simplifying mathematical property, but the class of cash-additive risk measures is too restricted to include some common functionals such as the expectation of a convex loss function. From the viewpoint of financial practice, the assumption of cash additivity of a risk measure may fail when uncertainty of interested rates is taken into account. In this sense, we consider the more general class of risk measures ρ, as argued by El Karoui and Ravanelli (2009), satisfying

Cash subadditivity: $\rho(X + m) \leqslant \rho(X) + m$ for all $X \in \mathcal{X}$ and $m \geqslant 0$.[3]

The assumption of cash subadditivity allows non-linear increase of the capital requirement as cash is added to the financial position but the increase should not exceed linear growth. Moreover, for a mapping $\rho : \mathcal{X} \to (-\infty, \infty]$, cash subadditivity of ρ implies that ρ is finite everywhere as soon as it is finite somewhere, and hence we can focus only on real-valued mappings.

Remark 2.1. Cash-subadditive risk measures are L^∞-continuous; namely, $\lim_{n \to \infty} \rho(X_n) = \rho(X)$ for any sequence $X_n \in \mathcal{X}$ satisfying ess-sup $(|X_n - X|) \to 0$ as $n \to \infty$. Clearly, for all $X, Y \in \mathcal{X}$, $X \leqslant Y + \|X - Y\|$. By monotonicity and cash subadditivity of ρ, we have $\rho(X) - \rho(Y) \leqslant \|X - Y\|$. Switching the roles of X and Y yields that ρ is 1-Lipschitz L^∞-continuous. Conversely, if a risk measure is 1-Lipschitz L^∞-continuous, then $\rho(X + m) - \rho(X) \leqslant \|m\| = m$ for all $m \geqslant 0$. Therefore, 1-Lipschitz L^∞-continuity and cash subadditivity are equivalent for risk measures; this observation is made in Proposition 2.1 of Cerreia-Vioglio et al. (2011).

Cash-subadditive risk measures are often studied in the literature together with convexity, or more generally, with quasi-convexity; see El Karoui and Ravanelli (2009), Cerreia-Vioglio et al. (2011) and Frittelli et al. (2014).

---

[2]As such, equalities and inequalities should be understood in a P-a.s. sense.
[3]An equivalent definition of cash subadditivity is $\rho(X + m) \geqslant \rho(X) + m$ for all $X \in \mathcal{X}$ and $m \leqslant 0$.

Convexity: $\rho(\lambda X + (1-\lambda)Y) \leqslant \lambda\rho(X) + (1-\lambda)\rho(Y)$ for all $X, Y \in \mathcal{X}$ and $\lambda \in [0,1]$.

Quasi-convexity: $\rho(\lambda X + (1-\lambda)Y) \leqslant \max\{\rho(X), \rho(Y)\}$ for all $X, Y \in \mathcal{X}$ and $\lambda \in [0,1]$.

As the main objective of this paper is to study cash-subadditive risk measures without quasi-convexity, we first note that the lack of quasi-convexity arises in many economically relevant contexts, such as aggregation of risk measures, non-convex utility functions, and risk mitigation. Castagnoli et al. (2022) argued with examples that many operations on a collection of convex risk measures lead to a non-convex one; the same applies in the context of quasi-convexity. Other than those built from operations, we provide a few simple examples of cash-subadditive risk measures in the literature, which are not cash-additive or quasi-convex.

The risk measure *Value-at-Risk* (VaR) is given by, for $t \in (0,1]$,

$$\mathrm{VaR}_t(X) = \inf\{x \in \mathbb{R} \mid P(X \leqslant x) \geqslant t\}, \quad X \in \mathcal{X}. \tag{2.1}$$

Note that $\mathrm{VaR}_1(X) = \mathrm{ess\text{-}sup}(X)$. VaR is one of the most popular risk measures used in the banking industry; see McNeil et al. (2015). The next example is a generalization of VaR introduced by Frittelli et al. (2014) without cash additivity.

Example 2.1 (Λ-Value-at-Risk)**.** The risk measure Λ-Value-at-Risk is defined as, for some function $\Lambda : \mathbb{R} \to [0,1]$ that is not constantly 0,

$$\Lambda\mathrm{VaR}(X) = \inf\{x \in \mathbb{R} : P(X \leqslant x) \geqslant \Lambda(x)\}, \quad X \in \mathcal{X}. \tag{2.2}$$

In particular, if Λ is a constant $t \in (0,1)$, then $\Lambda\mathrm{VaR} = \mathrm{VaR}_t$. Although Frittelli et al. (2014) mainly studied increasing Λ, the recent work of Burzoni et al. (2017) and Bellini and Peri (2022) has shown that using a decreasing Λ leads to many advantages, including, robustness, elicitability, and an axiomatic characterization. For this reason, we assume that Λ is a decreasing function in this paper. Since for $c \geqslant 0$, $\Lambda\mathrm{VaR}(X + c) = \Lambda^*\mathrm{VaR}(X) + c$ where $\Lambda^*(t) = \Lambda(t + c) \leqslant \Lambda(t)$ for $t \in \mathbb{R}$, we obtain $\Lambda\mathrm{VaR}(X + c) \leqslant \Lambda\mathrm{VaR}(X) + c$, and therefore $\Lambda\mathrm{VaR}$ is cash subadditive; we can check that it is not cash additive in general.[4] Moreover, $\Lambda\mathrm{VaR}$ is generally not quasi-convex either, as the following argument illustrates. For any decreasing $\Lambda : \mathbb{R} \to (0, 1/3]$ and a standard normal random variable X, we have $\Lambda\mathrm{VaR}(X) = \Lambda\mathrm{VaR}(-X) \leqslant z_{1/3} < 0$, where $z_{1/3}$ is the $1/3$-quantile of the standard normal distribution. Hence, $\Lambda\mathrm{VaR}(0) = 0 > \max\{\Lambda\mathrm{VaR}(X), \Lambda\mathrm{VaR}(-X)\}$ violating quasi-convexity.

Example 2.2 (Expected insured loss)**.** Suppose that an insurance contract pays $f(X)$ for an insurable loss X (often non-negative), where f is an increasing function on $\mathbb{R}$ that is 1-Lipschitz and $f(x) = 0$ for $x \leqslant 0$.[5] A typical example is $f(x) = (x - d)_+ \wedge \ell$ for some $\ell > d > 0$, which

---

[4]Note that $\Lambda\mathrm{VaR}(X)$ is not necessarily cash subadditive when Λ is increasing. For instance, take $\Lambda : x \mapsto (3/4)\mathbb{1}_{\{x>1/2\}} + (1/2)\mathbb{1}_{\{x\leqslant 1/2\}}$. Let X be uniformly distributed on $[0,1]$ and take $\varepsilon > 0$. We have $\Lambda\mathrm{VaR}(X) = 1/2$ and $\Lambda(X + \varepsilon) = 3/4 + \varepsilon$, and this clearly violates cash subadditivity. Moreover, such $\Lambda\mathrm{VaR}$ is not even continuous with respect to L^∞-norm, which is a basic requirement for risk measures interpreted as capital reserve.

[5]A function $f : \mathbb{R} \to \mathbb{R}$ is called *1-Lipschitz* if $|f(x) - f(y)| \leqslant |x - y|$ for all $x, y \in \mathbb{R}$.

represents an insurance contract with deductible d and limit ℓ. The expected losses to the policy holder and to the insurer are given by, respectively,

$$\rho_{\mathrm{ph}}(X) = \mathbb{E}[X - f(X)] \quad \text{and} \quad \rho_{\mathrm{in}}(X) = \mathbb{E}[f(X)].$$

It is straightforward to check that ρ_{ph} and ρ_{in} are both monotone and cash subadditive, but generally neither cash additive nor quasi-convex. In particular, ρ_{in} (resp. ρ_{ph}) is concave if f is concave (resp. convex). For a related example in finance, take $f : x \mapsto x_+$ and fix a probability measure Q representing a pricing measure in a financial market. The put option premium on the insolvency of a firm with future asset value $-X$ is defined as $\mathbb{E}_Q[X_+]$, which is convex and cash subadditive but not cash additive; see Jarrow (2002) and El Karoui and Ravanelli (2009) for a connection between the put option premium and risk measures.

Example 2.3 (Certainty equivalent with discount factor ambiguity). Consider the following α-maxmin expected utility (α-MEU, Marinacci (2002); Ghirardato et al. (2004)) with a profit-loss adjustment:

$$\alpha \min_{Q_1 \in \mathcal{Q}_1} \mathbb{E}_{Q_1}[\mathrm{e}^{\gamma X}] + (1 - \alpha) \max_{Q_2 \in \mathcal{Q}_2} \mathbb{E}_{Q_2}[\mathrm{e}^{\gamma X}], \quad X \in \mathcal{X}, \ \alpha \in [0, 1], \ \gamma > 0$$

with the loss function $x \mapsto \mathrm{e}^{\gamma x}$, where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two nonempty, weak*-compact and convex sets of finitely additive probabilities. The α-MEU model is of interest to the applied decision-theoretic literature as it can model deviations from the pure pessimism that the MEU model (corresponding to $\alpha = 0$) inherently embodies. The suggestion that decision makers are not purely pessimistic is supported by a vast amount of empirical literature, see e.g., Trautmann and Van De Kuilen (2015) for a review.

The certainty equivalent of the above α-MEU with a stochastic ambiguous discount factor D is given by

$$\rho(X) = \sup_{D \in \mathcal{I}} \left\{ \frac{1}{\gamma} \log \left(\alpha \min_{Q_1 \in \mathcal{Q}_1} \mathbb{E}_{Q_1}[\mathrm{e}^{\gamma DX}] + (1 - \alpha) \max_{Q_2 \in \mathcal{Q}_2} \mathbb{E}_{Q_2}[\mathrm{e}^{\gamma DX}] \right) \right\}, \quad X \in \mathcal{X},$$

where $\mathcal{I}$ is a set of random variables taking values in $[0, 1]$. For any $X \in \mathcal{X}$ and $m \geqslant 0$, we have

$$
\begin{aligned}
\rho(X + m) &= \sup_{D \in \mathcal{I}} \left\{ \frac{1}{\gamma} \log \left(\alpha \min_{Q_1 \in \mathcal{Q}_1} \mathbb{E}_{Q_1}[\mathrm{e}^{\gamma D(X+m)}] + (1 - \alpha) \max_{Q_2 \in \mathcal{Q}_2} \mathbb{E}_{Q_2}[\mathrm{e}^{\gamma D(X+m)}] \right) \right\} \\
&\leqslant \sup_{D \in \mathcal{I}} \left\{ \frac{1}{\gamma} \log \left(\alpha e^{\gamma m} \min_{Q_1 \in \mathcal{Q}_1} \mathbb{E}_{Q_1}[\mathrm{e}^{\gamma DX}] + (1 - \alpha) e^{\gamma m} \max_{Q_2 \in \mathcal{Q}_2} \mathbb{E}_{Q_2}[\mathrm{e}^{\gamma DX}] \right) \right\} \\
&= \sup_{D \in \mathcal{I}} \left\{ \frac{1}{\gamma} \log \left(\alpha \min_{Q_1 \in \mathcal{Q}_1} \mathbb{E}_{Q_1}[\mathrm{e}^{\gamma DX}] + (1 - \alpha) \max_{Q_2 \in \mathcal{Q}_2} \mathbb{E}_{Q_2}[\mathrm{e}^{\gamma DX}] \right) \right\} + m = \rho(X) + m.
\end{aligned}
$$

Therefore, ρ is a cash-subadditive risk measure, and it is generally not cash additive. Moreover, because of the presence of both minimum and maximum in α-MEU, quasi-convexity does not hold for ρ.

Example 2.4 (Risk measures based on eligible risky assets)**.** Take an acceptance set $\mathcal{A} \subseteq \mathcal{X}$ and a reference asset $S = (S_0, S_T) \in \mathcal{X}^2$, where S_0 is a constant representing the initial price of the asset, and S_T is a positive terminal payoff. Define the mapping $\rho_{\mathcal{A},S}$ as in Farkas et al. (2014) by

$$\rho_{\mathcal{A},S}(X) = \inf\left\{ m \in \mathbb{R} : X - \frac{m}{S_0}S_T \in \mathcal{A} \right\}, \quad X \in \mathcal{X}. \tag{2.3}$$

The quantity $\rho_{\mathcal{A},S}(X)$ represents the "minimal" amount of capital we have to raise and invest, at inception, in the asset S to meet the acceptability constraint specified by $\mathcal{A}$. By Proposition 5.1 of Farkas et al. (2014), we have that $\rho_{\mathcal{A},S}$ is cash subadditive under the assumption of $P(S_T < S_0) = 0$ (e.g., the bond can only default on the interest payment).[6] In this case, assuming that $\mathcal{A}$ is closed, $\rho_{\mathcal{A},S}$ is convex if and only if $\mathcal{A}$ is convex (see Lemma 2.5 of Farkas et al. (2014)).[7] In general, such risk measures are not quasi-convex unless they are also convex.

Although $\rho_{\mathcal{A},S}$ is not cash additive, by definition it is affine along S, that is, $\rho_{\mathcal{A},S}(X + \lambda S_T) = \rho_{\mathcal{A},S}(X) + \lambda S_0$ for all $\lambda \in \mathbb{R}$ and $X \in \mathcal{X}$. Conversely, we can check that all risk measures that are affine along S can be identified as $\rho_{\mathcal{A},S}$, where $\mathcal{A} = \{X \in \mathcal{X} : \rho(X) \leqslant 0\}$. Therefore, all results on cash-subadditive risk measures apply to such risk measures as long as $P(S_T < S_0) = 0$.

To account for more than one eligible asset, we consider two assets for an example. Fix two acceptance sets $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathcal{X}$, and two assets $S^1 = (S_0^1, S_T^1)$ and $S^2 = (S_0^2, S_T^2)$. Moreover, define the set

$$\mathcal{P}_0\left(S^1, S^2\right) := \left\{ \frac{m}{S_0^2}S_T^2 - \frac{m}{S_0^1}S_T^1; \ m \in \mathbb{R} \right\}.[8]$$

By Proposition 3.2.12 of Munari (2015), the inf-convolution of $\rho_{\mathcal{A}_1, S^1}$ and $\rho_{\mathcal{A}_2, S^2}$ has the following representation

$$\rho_{\mathcal{A}_1, S^1} \square \rho_{\mathcal{A}_2, S^2} = \rho_{\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{P}_0(S^1, S^2), S^1}.[9]$$

Thus, $\rho_{\mathcal{A}_1, S^1} \square \rho_{\mathcal{A}_2, S^2}$ is still cash subadditive; this can certainly be extended to a finite number of assets. For more details on inf-convolutions and their applications to the theory of risk measures, we refer to Barrieu and El Karoui (2005) and Filipović and Svindland (2008).

Some other relevant properties for a risk measure ρ are collected below, which will be used throughout the paper; we refer to Föllmer and Schied (2016) for a comprehensive treatment of properties of risk measures.

Normalization: $\rho(t) = t$ for all $t \in \mathbb{R}$.

---

[6]Beyond the sufficiency of the condition $\mathbb{P}(S_T < S_0) = 0$, for the acceptance set $\mathcal{A}$ chosen based on the common risk measures ES or VaR, if $\rho_{\mathcal{A},S}$ is cash subadditive, then $\mathbb{P}(S_T < S_0)$ needs to be sufficiently small; see Corollary 5.3 and Proposition 5.5 of Farkas et al. (2014). In particular, $P(S_T < S_0) = 0$ is a necessary condition for cash subadditivity with a VaR-based acceptance set.

[7]When the asset is not liquidly traded, we can define $\rho_{\mathcal{A},S,\pi}(X) = \inf\{\pi(m) \in \mathbb{R} : X - mS_T \in \mathcal{A}\}, X \in \mathcal{X}$, where $\pi : \mathbb{R} \to \mathbb{R}$ is some non-linear increasing function. Then $\rho_{\mathcal{A},S,\pi}$ is quasiconvex if and only if $\mathcal{A}$ is convex because the composition of convex and increasing functions leads to $\rho_{\mathcal{A},S}$ being quasi-convex.

[8]This set consists of the payoffs of all "portfolios" we can form at zero cost by combining the assets S^1 and S^2.

[9]For a fixed position $X \in \mathcal{X}$, the inf-convolution of $f_1 : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ and $f_2 : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ is the map $f_1 \square f_2 : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ defined by $f_1 \square f_2(X) := \inf\{f_1(Y) + f_2(X - Y); Y \in \mathcal{X}\}$. As such, $f_1 \square f_2(X)$ is the "minimal" total required capital across all possible allocations of the aggregated position X.

Law invariance: $\rho(X) = \rho(Y)$ for all $X, Y \in \mathcal{X}$ with $X \stackrel{\mathrm{d}}{=} Y$.

Normalization here is more general than the traditional definition of $\rho(0) = 0$ in Föllmer and Schied (2016), meaning that the risk of any constant equals itself. The two definitions are equivalent if ρ is cash additive. Monetary, convex and positively homogeneous risk measures are called coherent by Artzner et al. (1999).[10]

Next, we define the two most important notions of stochastic dominance in decision theory, the first-order stochastic dominance (FSD) and the second-order stochastic dominance (SSD). Given two random variables $X, Y \in \mathcal{X}$, we denote by $X \succeq_1 Y$, if $\mathbb{E}[f(X)] \geqslant \mathbb{E}[f(Y)]$ for all increasing functions $f : \mathbb{R} \to \mathbb{R}$, and denote by $X \succeq_2 Y$, if $\mathbb{E}[f(X)] \geqslant \mathbb{E}[f(Y)]$ for all increasing convex functions $f : \mathbb{R} \to \mathbb{R}$. Consistency with respect to FSD or SSD is defined as monotonicity in these partial orders.

FSD-consistency: $\rho(X) \geqslant \rho(Y)$ for all $X, Y \in \mathcal{X}$ whenever $X \succeq_1 Y$.

SSD-consistency: $\rho(X) \geqslant \rho(Y)$ for all $X, Y \in \mathcal{X}$ whenever $X \succeq_2 Y$.

It is well known that either FSD-consistency or SSD-consistency implies law invariance. For monetary risk measures, SSD-consistency is characterized by Theorem 3.1 of Mao and Wang (2020).

Finally, the notion of comonotonicity is useful for some results in this paper. A random vector $(X_1, \ldots, X_n) \in \mathcal{X}^n$ is called *comonotonic* if there exists a random variable $Z \in \mathcal{X}$ and increasing functions $f_1, \ldots, f_n$ on $\mathbb{R}$ such that $X_i = f_i(Z)$ almost surely for all $i = 1, \ldots, n$.

3 Quasi-star-shapedness, quasi-normalization, and Lambda VaR

In this section, we discuss two new properties that are specific to cash-subadditive risk measures without quasi-convexity, and they will be used in the representation results in Section 5.1.

3.1 Quasi-star-shapedness and quasi-normalization

In the context of cash-additive risk measures, Castagnoli et al. (2022) studied a weaker property than convexity:

Star-shapedness: $\rho(\lambda X) \leqslant \lambda \rho(X) + (1 - \lambda)\rho(0)$ for all $X \in \mathcal{X}$ and $\lambda \in [0, 1]$,

and formulated star-shapedness via $\rho(\lambda X) \leqslant \lambda \rho(X)$ for $\lambda \in [0, 1]$ with the extra normalization $\rho(0) = 0$. Star-shapedness is discussed in Artzner et al. (1999) and it has a natural economic motivation that additional liquidity risk may arise if a position is multiplied by a factor larger than 1. In case $\rho(0) \neq 0$, it is more natural to define star-shapedness via our formulation, which means convexity at 0 (has also been called "positive superhomogeneity" for obvious mathematical reasons), thus weaker than convexity. In the context of the cash-additive risk measures, we introduce the corresponding property for cash-subadditive risk measures:

---

[10]The functional ρ is said to be positively homogeneous if $\rho(\lambda X) = \lambda \rho(X)$ for all $X \in \mathcal{X}$ and $\lambda \geqslant 0$.

Quasi-star-shapedness: $\rho(\lambda X+(1-\lambda)t) \leqslant \max\{\rho(X), \rho(t)\}$ for all $X \in \mathcal{X}$, $t \in \mathbb{R}$ and $\lambda \in [0,1]$.

Since quasi-star-shapedness is new to the literature on risk measures, it may need some explanation. As explained by Castagnoli et al. (2022), star-shapedness reflects the consideration of liquidity risk, in a way similar to (but weaker than) convexity which reflects the consideration of diversification. For cash-additive risk measures, star-shapedness is equivalent to $\rho(\lambda X + (1-\lambda)t) \leqslant \lambda\rho(X) + (1-\lambda)\rho(t)$ for all $X \in \mathcal{X}$, $t \in \mathbb{R}$ and $\lambda \in [0,1]$; indeed, it means that ρ has convexity at each constant. This reformulation of star-shapedness implies our quasi-star-shapedness, which means that ρ has quasi-convexity at each constant. Obviously, quasi-star-shapedness is weaker than quasi-convexity.

Quasi-star-shapedness has a sound decision-theoretic interpretation, which we explain in Proposition 3.1 below. For a risk measure $\rho : \mathcal{X} \to \mathbb{R}$, the preference associated with ρ is a binary relation $\succeq$ on $\mathcal{X}$ defined by, for all $X, Y \in \mathcal{X}$, $X \succeq Y \iff \rho(X) \leqslant \rho(Y)$. The equivalence relation of $\succeq$ is denoted by $\simeq$. In other words, $\succeq$ represents the preference of an agent favouring less risk evaluated via ρ.

Proposition 3.1. *An L^∞-continuous risk measure $\rho : \mathcal{X} \to \mathbb{R}$ satisfies quasi-star-shapedness if and only if its associated preference $\succeq$ satisfies, for $X \in \mathcal{X}$, $t \in \mathbb{R}$ and $\lambda \in [0,1]$,*

$$X \simeq t \implies \lambda X + (1-\lambda)t \succeq X. \tag{3.1}$$

Proof. By definition of $\succeq$, (3.1) is equivalent to

$$\rho(X) = \rho(t) \implies \rho(\lambda X + (1-\lambda)t) \leqslant \rho(X), \tag{3.2}$$

which is clearly implied by quasi-star-shapedness. Hence, "only-if" statement holds true. To show the "if" statement, take arbitrary $X \in \mathcal{X}$ and $t \in \mathbb{R}$. If $\rho(X) \leqslant \rho(t)$, then we take $s \geqslant 0$ such that $\rho(X + s) = \rho(t)$. Such s exists since $s \mapsto \rho(X + s)$ is continuous and $X + s \geqslant t$ for s large enough. Using monotonicity of ρ and (3.2), we have

$$\rho(\lambda X + (1-\lambda)t) \leqslant \rho(\lambda(X + s) + (1-\lambda)t) \leqslant \rho(t) = \max\{\rho(X), \rho(t)\}$$

for each $\lambda \in [0,1]$. If $\rho(X) > \rho(t)$, then we take $s \geqslant 0$ such that $\rho(X) = \rho(t + s)$. Such s exists since $s \mapsto \rho(t + s)$ is continuous and $t + s \geqslant X$ for s large enough. Using monotonicity of ρ and (3.2), we have

$$\rho(\lambda X + (1-\lambda)t) \leqslant \rho(\lambda X + (1-\lambda)(t + s)) \leqslant \rho(X) = \max\{\rho(X), \rho(t)\}$$

for each $\lambda \in [0,1]$. Hence, quasi-star-shapedness holds. $\qquad\square$

Remark 3.1. Proposition 3.1 requires L^∞-continuity, which is a weak property satisfied by essentially all risk measures in the literature. The result in Proposition 3.1 holds true with the same proof if L^∞-continuity is replaced by the property of *solvability* in decision theory: For each $X \in \mathcal{X}$, there exists $t \in \mathbb{R}$ such that $\rho(X) = \rho(t)$.

Proposition 3.1 gives the following decision-theoretic interpretation of quasi-star-shapedness. Suppose that the preference $\succeq$ of an agent satisfies (3.1). If a random loss X is seen as equally favourable as a constant loss t, then $\lambda X + (1-\lambda)t$ is weakly preferred to X. That is, a combination of random X and constant t reduces the riskiness of X. In contrast, quasi-convexity requires the above relation to hold for random Y in place of constant t. Indeed, in the setting of Anscombe and Aumann (1963) where X and Y represent acts with uncertainty (thus, they are not necessarily $\mathbb{R}$-valued), the property, for $X, Y \in \mathcal{X}$ and $\lambda \in [0, 1]$,

$$X \simeq Y \implies \lambda X + (1 - \lambda)Y \succeq X, \tag{3.3}$$

is the uncertainty aversion axiom of Maccheroni et al. (2006), and it corresponds to quasi-convexity of the risk measure ρ in our setting. It is clear that (3.1) is weaker than (3.3) as the riskiness of X is only reduced when combined with an equally favourable constant loss, instead of an arbitrary equally favourable loss Y.

The difference between quasi-star-shapedness and quasi-convexity, or between (3.1) and (3.3), can also be explained via considerations for the dependence between pooled risks. For a law-invariant ρ and two losses X and Y with fixed distributions, the dependence structure of X and Y affects $\rho(\lambda X + (1 - \lambda)Y)$ but not $\rho(X)$ or $\rho(Y)$, and hence quasi-convexity imposes inequalities over all dependence structures. Such an issue does not appear for $\lambda X + (1 - \lambda)t$ as dependence is irrelevant between a random variable X and a constant t. Hence, relaxing quasi-convexity to quasi-star-shapedness gives rise to more flexibility on preferences over dependence. In particular, under quasi-convexity, comonotonicity is the worst-case dependence in risk aggregation; see Lemmas 5.1 and 5.2. This is not the case for quasi-star-shapedness, since VaR_t for $t \in (0, 1)$ is quasi-star-shaped but it does not take comonotonicity as the worst-case dependence.

Next, we discuss the issue of normalization. The risk measures in Examples 2.2 and 2.1 are not necessarily normalized. In general, cash-subadditive risk measures may not have the range of the entire real line. Hence, normalization may also need to be weakened in our setting of cash subadditive risk measures, which we define as follows.

Quasi-normalization: $\rho(t) = t$ for all $t \in D_\rho$, where $D_\rho = \{\rho(X) \mid X \in \mathcal{X}\}$ is the range of ρ.

The risk measure $X \mapsto \mathbb{E}[\min\{X, d\}]$ in Example 2.2 satisfies quasi-normalization with range $(-\infty, d]$, and $\Lambda\mathrm{VaR}$ in Example 2.1 satisfies quasi-normalization with range $(-\infty, z]$ where $z = \inf\{x \in \mathbb{R} : \Lambda(x) = 0\}$ with the convention $\inf \emptyset = \infty$.

3.2 A new representation of Lambda VaR

The next result gives quasi-star-shapedness of $\Lambda\mathrm{VaR}$, complementing the fact observed by Castagnoli et al. (2022) that VaR is star-shaped. We also obtain, as a by-product, an alternative representation of $\Lambda\mathrm{VaR}$. In what follows, set $\mathrm{VaR}_0(X) = -\infty$ for any $X \in \mathcal{X}$, which follows from plugging $t = 0$ in (2.1).

Theorem 3.1. *Let* $\Lambda : \mathbb{R} \to [0,1]$ *be a decreasing function that is not constantly* 0. *The risk measure* $\Lambda\mathrm{VaR}$ *in* (2.2) *has the representation*

$$\Lambda\mathrm{VaR}(X) = \inf_{x \in \mathbb{R}} \left\{ \mathrm{VaR}_{\Lambda(x)}(X) \vee x \right\} = \sup_{x \in \mathbb{R}} \left\{ \mathrm{VaR}_{\Lambda(x)}(X) \wedge x \right\}, \quad X \in \mathcal{X}, \tag{3.4}$$

and moreover, $\Lambda\mathrm{VaR}$ *is quasi-star-shaped.*

Proof. Note that for $X \in \mathcal{X}$, $x \in \mathbb{R}$ and $t \in [0,1]$, $P(X \leqslant x) \geqslant t$ if and only if $\mathrm{VaR}_t(X) \leqslant x$. Moreover, since Λ is decreasing, the set $\{x \in \mathbb{R} : \mathrm{VaR}_{\Lambda(x)}(X) \leqslant x\}$ is an interval with right end-point ∞. By definition, for $X \in \mathcal{X}$,

$$
\begin{aligned}
\Lambda\mathrm{VaR}(X) &= \inf\{x \in \mathbb{R} : P(X \leqslant x) \geqslant \Lambda(x)\} \\
&= \inf\{x \in \mathbb{R} : \mathrm{VaR}_{\Lambda(x)}(X) \leqslant x\} \\
&= \inf\{\mathrm{VaR}_{\Lambda(x)}(X) \vee x : \mathrm{VaR}_{\Lambda(x)}(X) \leqslant x\} \geqslant \inf_{x \in \mathbb{R}} \left\{ \mathrm{VaR}_{\Lambda(x)}(X) \vee x \right\}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\Lambda\mathrm{VaR}(X) &= \inf\{x \in \mathbb{R} : \mathrm{VaR}_{\Lambda(x)}(X) \leqslant x\} \\
&= \sup\{x \in \mathbb{R} : \mathrm{VaR}_{\Lambda(x)}(X) > x\} \\
&= \sup\{\mathrm{VaR}_{\Lambda(x)}(X) \wedge x : \mathrm{VaR}_{\Lambda(x)}(X) > x\} \leqslant \sup_{x \in \mathbb{R}} \left\{ \mathrm{VaR}_{\Lambda(x)}(X) \wedge x \right\}.
\end{aligned}
$$

Since $\mathrm{VaR}_{\Lambda(x)}(X) \wedge x \leqslant \mathrm{VaR}_{\Lambda(y)}(X) \vee y$ for any $x, y \in \mathbb{R}$, we have

$$\Lambda\mathrm{VaR}(X) \leqslant \sup_{x \in \mathbb{R}} \left\{ \mathrm{VaR}_{\Lambda(x)}(X) \wedge x \right\} \leqslant \inf_{x \in \mathbb{R}} \left\{ \mathrm{VaR}_{\Lambda(x)}(X) \vee x \right\} \leqslant \Lambda\mathrm{VaR}(X),$$

thus showing (3.4). Next, verify that the mapping $X \mapsto \mathrm{VaR}_\alpha(X) \vee x$ is quasi-star-shaped for all $\alpha \in [0,1]$ and $x \in \mathbb{R}$. Note that if $\alpha = 0$ then it is trivial. If $\alpha > 0$, then $\mathrm{VaR}_\alpha(X) \vee x = \mathrm{VaR}_\alpha(X \vee x)$; see Lemma A.27 of Föllmer and Schied (2016). For all $X \in \mathcal{X}$, $t \in \mathbb{R}$ and $\lambda \in [0,1]$,

$$
\begin{aligned}
\mathrm{VaR}_\alpha(x \vee (\lambda X + (1 - \lambda)t)) &\leqslant \mathrm{VaR}_\alpha(\lambda(x \vee X) + (1 - \lambda)(x \vee t)) \\
&= \lambda\mathrm{VaR}_\alpha(x \vee X) + (1 - \lambda)\mathrm{VaR}_\alpha(x \vee t) \\
&\leqslant \max\{\mathrm{VaR}_\alpha(x \vee X), \mathrm{VaR}_\alpha(x \vee t)\}. \tag{3.5}
\end{aligned}
$$

Finally, we need to use Lemma 3.1 below, which states that the infimum of quasi-normalized, quasi-star-shaped and cash subadditive risk measures is quasi-star-shaped. Since $X \mapsto \mathrm{VaR}_{\Lambda(x)}(X) \vee x$ is quasi-normalized, quasi-star-shaped and cash subadditive for all $x \in \mathbb{R}$, $\Lambda\mathrm{VaR}$ is quasi-star-shaped by Lemma 3.1. $\qquad\square$

Remark 3.2. We note that $\Lambda\mathrm{VaR}$ is generally not star-shaped. For instance, take $\Lambda : x \mapsto \mathbb{1}_{\{x \leqslant 1\}}$. For this choice, we have $\Lambda\mathrm{VaR}(x) = x \wedge 1$ for $x \in \mathbb{R}$. It follows that $\Lambda\mathrm{VaR}(1) = 1 > 1/2 = \Lambda\mathrm{VaR}(2)/2 + \Lambda\mathrm{VaR}(0)/2$, and hence $\Lambda\mathrm{VaR}$ is not star-shaped. Indeed, any Λ with $\inf\{x \in \mathbb{R} : \Lambda(x) = 0\} = 1$ suffices for this example. Note that each $X \mapsto \mathrm{VaR}_\alpha(X) \vee x$ in the representation

11

(3.4) is star-shaped (including $\alpha = 0$); see (3.5). Therefore, the infimum of quasi-normalized, star-shaped and cash-subadditive risk measures is not necessarily star-shaped, in sharp contrast to the corresponding result on quasi-star-shaped risk measures in Lemma 3.1. This example shows that quasi-star-shapedness is more natural than, and genuinely different from, star-shapedness in the context of cash-subadditive risk measures.[11]

Theorem 3.1 can be applied to solve portfolio optimization problems with ΛVaR constraints. Let $\Lambda : \mathbb{R} \to [0, 1]$ be a decreasing function which is not constantly 0. In a portfolio optimization problem, one often maximizes an objective, e.g., an expected utility or an expected return, under the constraint that a risk measure does not exceed a certain level z (and often together with a budget constraint). For $X \in \mathcal{X}$, by Theorem 3.1, we have

$$\Lambda\mathrm{VaR}(X) \leqslant z \iff \inf_{x \in \mathbb{R}} \{\mathrm{VaR}_{\Lambda(x)}(X) \vee x\} \leqslant z \iff \inf_{x \leqslant z} \mathrm{VaR}_{\Lambda(x)}(X) \leqslant z \iff \mathrm{VaR}_{\Lambda(z)}(X) \leqslant z.$$

Therefore, optimization under a ΛVaR constraint below a constant level z is equivalent to that under a $\mathrm{VaR}_{\Lambda(z)}$ constraint below the same level z, which has been well studied in the risk management literature; see e.g., Basak and Shapiro (2001) and Basak et al. (2006).

3.3 A few useful technical results

The next lemma shows that quasi-normalization and quasi-star-shapedness are preserved under a minimum operation, a fact used in the proof of Theorem 3.1.

Lemma 3.1. *The infimum of quasi-normalized, quasi-star-shaped, and cash-subadditive risk measures (assuming it is real-valued) is again quasi-normalized, quasi-star-shaped and cash-subadditive.*

Proof. Let $\mathcal{C}$ be a class of quasi-normalized, quasi-star-shaped and cash-subadditive risk measures, and denote by $\rho = \inf_{\psi \in \mathcal{C}} \psi$. It is obvious that ρ is cash subadditive and monotone. It remains to show that ρ is quasi-normalized and quasi-star-shaped. Denote by $d = \inf D_\rho$, $u = \sup D_\rho$, $d_\psi = \inf D_\psi$ and $u_\psi = \sup D_\psi$ for $\psi \in \mathcal{C}$. For any $X \in \mathcal{X}$ and $\psi \in \mathcal{C}$, if $u < d_\psi$, then $\rho(X) \leqslant u < \psi(X)$. Hence, we can write

$$\rho(X) = \inf_{\psi \in \mathcal{C}'} \psi(X) \text{ where } \mathcal{C}' = \{\psi \in \mathcal{C} \mid u \geqslant d_\psi\}.$$

Note that $d \leqslant d_\psi \leqslant u \leqslant u_\psi$ for each $\psi \in \mathcal{C}'$. Moreover, by monotonicity and quasi-normalization of ψ, for any $\psi \in \mathcal{C} \cup \{\rho\}$, we have

$$t \leqslant u_\psi \implies \psi(t) = t \vee d_\psi, \tag{3.6}$$

$$t \geqslant d_\psi \implies \psi(t) = t \wedge u_\psi. \tag{3.7}$$

We first show that ρ is quasi-normalized. Take a constant $t \in (d, u)$. Since $t < u \leqslant u_\psi$, by (3.6), we have $\psi(t) \geqslant t$ for all $\psi \in \mathcal{C}'$. Hence, $\rho(t) = \inf_{\psi \in \mathcal{C}'} \psi(t) \geqslant t$. Moreover, since $t > d$ and $d = \inf_{\psi \in \mathcal{C}'} d_\psi$, there exists $\psi \in \mathcal{C}'$ such that $d_\psi < t$. By (3.7), we get $\psi(t) \leqslant t$. Hence,

---

[11]On the other hand, for an increasing Λ, $\Lambda\mathrm{VaR}(X)$ is in general not quasi-star-shaped. A counter-example can be built using a Bernoulli random variable; see Example A.1 in Appendix A.5.

$\rho(t) = \inf_{\psi \in \mathcal{C}'} \psi(t) \leqslant t$. Thus, we obtain $\rho(t) = t$ for $t \in (d, u)$. It remains to verify $\rho(d) = d$ (resp. $\rho(u) = u$) if $d > -\infty$ (resp. $u < \infty$). This follows from the fact that a cash-subadditive risk measure is L^∞-continuous. Therefore, $\rho(t) = t$ for $t \in D_\rho$, and thus ρ is quasi-normalized.

Next, we show that ρ is quasi-star-shaped. For $X \in \mathcal{X}$, $t \in \mathbb{R}$ and $\lambda \in [0, 1]$, quasi-star-shapendess of $\psi \in \mathcal{C}'$ gives

$$\rho(\lambda X + (1 - \lambda)t) = \inf_{\psi \in \mathcal{C}'} \psi(\lambda X + (1 - \lambda)t) \leqslant \inf_{\psi \in \mathcal{C}'} \max\{\psi(X), \psi(t)\}. \tag{3.8}$$

If $t \geqslant u$, then $\rho(t) = u$ and

$$\rho(\lambda X + (1 - \lambda)t) \leqslant u = \rho(t).$$

If $t < u$, then $\psi(t) = t \vee d_\psi$ for $\psi \in \mathcal{C}'$ and $\rho(t) = t \vee d$. It follows that

$$\inf_{\psi \in \mathcal{C}'} \max\{\psi(X), \psi(t)\} = \inf_{\psi \in \mathcal{C}'} \max\{\psi(X), t, d_\psi\}$$

$$= \inf_{\psi \in \mathcal{C}'} \max\{\psi(X), t\} = \max\left\{\inf_{\psi \in \mathcal{C}'} \psi(X), t\right\} \leqslant \max\{\rho(X), \rho(t)\}.$$

Using (3.8) and combining both cases, we obtain $\rho(\lambda X + (1 - \lambda)t) \leqslant \max\{\rho(X), \rho(t)\}$ for all $\lambda \in [0, 1]$, $X \in \mathcal{X}$ and $t \in \mathbb{R}$ and thus ρ is quasi-star-shaped. $\qquad\square$

Finally, we show that in the classic setting of cash-additive risk measures, we do not need to distinguish between each of normalization, star-shapedness and covexity and their quasi-versions. This result further illustrates that quasi-star-shapedness is a natural property to consider for cash-subadditive risk measures.

Proposition 3.2. *For cash-additive risk measures,*

  (i) *normalization is equivalent to quasi-normalization;*

 (ii) *star-shapedness is equivalent to quasi-star-shapedness;*

(iii) *convexity is equivalent to quasi-convexity.*

In contrast, for cash-subadditive risk measures, none of the above equivalence holds true.

Proof. The statements on normalization are straightforward. Those on convexity are well known and can be checked with acceptance sets; see e.g., Proposition 2.1 and Example 2.2 of Cerreia-Vioglio et al. (2011). We only show the statements on star-shapedness.

(a) For cash-subadditive risk measures, the fact that these star-shapedness and quasi-star-shapedness are not necessarily equivalent is illustrated in Remark 3.2.

(b) Suppose that a cash-additive risk measure ρ is star-shaped. Cash additivity and star-shapedness yield that, for all $X \in \mathcal{X}$, $t \in \mathbb{R}$ and $\lambda \in [0, 1]$,

$$\rho(\lambda X + (1 - \lambda)t) = \rho(\lambda X) + (1 - \lambda)t \leqslant \lambda\rho(X) + (1 - \lambda)\rho(t) \leqslant \max\{\rho(X), \rho(t)\},$$

13

which implies that ρ is quasi-star-shaped.

(c) Suppose that a cash-additive risk measure ρ is quasi-star-shaped. Let $\tilde{\rho} = \rho - \rho(0)$, and hence $\tilde{\rho}$ is normalized. The acceptance set of $\tilde{\rho}$ is given by

$$\mathcal{A}_{\tilde{\rho}} = \{X \in \mathcal{X} : \tilde{\rho}(X) \leqslant 0\}.$$

Note that $0 \in \mathcal{A}_{\tilde{\rho}}$ and $\tilde{\rho}$ is quasi-star-shaped. Therefore, for any $X \in \mathcal{A}_{\tilde{\rho}}$ and $\lambda \in [0,1]$, we have $\tilde{\rho}(\lambda X) \leqslant \max\{\tilde{\rho}(X), \rho(0)\} \leqslant 0$. Hence, $\lambda X \in \mathcal{A}_{\tilde{\rho}}$, and thus the set $\mathcal{A}_{\tilde{\rho}}$ is star-shaped. By Proposition 2 of Castagnoli et al. (2022), we know that $\tilde{\rho}$ is star-shaped. In turn, this implies that ρ is star-shaped. $\qquad\square$

3.4 Quasi-star-shapedness of other cash-subadditive risk measures

In addition to ΛVaR, we discuss quasi-star-shapedness of other risk measures in the examples in Section 2. First, we consider the expected loss $\rho : X \mapsto \mathbb{E}[f(X)]$ in Example 2.2 and obtain a characterization result stronger than the second statement of Lemma 5.2 in Cerreia-Vioglio et al. (2011). Note that 1-Lipschitz continuity of f is equivalent to cash subadditivity of ρ, but for this result we only need continuity.

Proposition 3.3. *Suppose that f is continuous and increasing function on $\mathbb{R}$. The expected loss $\rho : X \mapsto \mathbb{E}[f(X)]$ on $\mathcal{X}$ is quasi-star-shaped if and only if f is convex.*

Proof. The "if" statement is straightforward, as convexity is stronger than quasi-star-shapedness.

Below, we will show the "only if" statement. For convexity, it suffices to show $f((a+b)/2) \leqslant (f(a) + f(b))/2$ for all $a < b$ since f is continuous. We prove this in a few steps. First, we show that if f has a positive derivative, then the conclusion holds true. In the second step, we show that f is either strictly increasing or first a constant then strictly increasing. In the third step, we show that the conditions in Step 2 is sufficient to use the conclusion in Step 1.

Step 1. Define the set

$$\mathcal{D} = \{(a, b, t) \in \mathbb{R}^3 : \ a < t < b, \ f(a) + f(b) = 2f(t), \ f \text{ is differentiable at } t \text{ and } f'(t) > 0\}.$$

For $(a, b, t) \in \mathcal{D}$, take a random variable X with distribution specified by $\mathbb{P}(X = a) = \mathbb{P}(X = b) = 1/2$. Quasi-star-shapedness implies for all $\lambda \in (0, 1)$,

$$\frac{1}{2}(f(\lambda a + (1-\lambda)t) + f(\lambda b + (1-\lambda)t)) = \mathbb{E}[f(\lambda X + (1-\lambda)t)] \leqslant \max\{\mathbb{E}[f(X)], f(t)\} = f(t).$$

Hence, we have

$$\frac{1}{t-a} \cdot \frac{f(\lambda b + (1-\lambda)t) - f(t)}{\lambda(b-t)} \leqslant \frac{1}{b-t} \cdot \frac{f(t) - f(\lambda a + (1-\lambda)t)}{\lambda(t-a)}.$$

14

Letting $\lambda \to 0$ yields

$$\frac{f'(t)}{t-a} \leqslant \frac{f'(t)}{b-t}.$$

It follows from $f'(t) > 0$ that $a + b \leqslant 2t$. By the monotonicity of f, we have

$$\frac{1}{2}(f(a) + f(b)) = f(t) \geqslant f\left(\frac{a+b}{2}\right). \tag{3.9}$$

This completes the proof of Step 1.

Step 2. Next, we verify that for $a, b \in \mathbb{R}$ and $a < b$, $f(a) = f(b)$ implies $f(x) = f(a)$ for all $x < a$. We prove this by contradiction. Suppose that $f(a) = f(b)$ for some $a < b$ and f is not a constant on $(-\infty, a]$. Since f is increasing, there exists $x_0 \in (-\infty, a)$ such that $f(x_0 + \varepsilon) > f(x_0)$ for any $\varepsilon > 0$. Such an x_0 exists because we can always take $\tilde{x} \in (-\infty, a)$ with $f(\tilde{x}) < f(a)$ and choose $x_0 = \sup\{x \in (-\infty, a) : f(x) = f(\tilde{x})\} < a$ due to the continuity of f. As a result, it is clear that $f(x_0) < f(a)$. Take X with $\mathbb{P}(X = x_0) = \mathbb{P}(X = b) = 1/2$. The continuity of f guarantees $2f(t) = f(x_0) + f(b)$ for some $t \in (x_0, a)$. Using quasi-star-shapedness, we have for $\lambda \in (0, 1)$,

$$\frac{1}{2}(f(\lambda x_0 + (1-\lambda)t) + f(\lambda b + (1-\lambda)t)) = \mathbb{E}[f(\lambda X + (1-\lambda)t)] \leqslant \max\{\mathbb{E}[f(X)], f(t)\} = f(t).$$

Setting $\lambda = (a-t)/(b-t) \in (0, 1)$ yields

$$\frac{1}{2}(f(x_0 + (1-\lambda)(t-x_0)) + f(a)) \leqslant f(t) = \frac{1}{2}(f(x_0) + f(b)) = \frac{1}{2}(f(x_0) + f(a)).$$

Note that $f(x_0 + (1-\lambda)(t-x_0)) > f(x_0)$ because $t > x_0$, and this yields a contradiction. Hence, we have proved Step 2.

Step 3. As shown in Step 2, we know that there are two possible cases: (i) f is strictly increasing on $\mathbb{R}$; (ii) there exists $x_0 \in \mathbb{R}$ such that $f(x) = f(x_0)$ for $x \leqslant x_0$ and f is strictly increasing on $[x_0, \infty)$. In the first case, for $a, b \in \mathbb{R}$ and $a < b$, let $t = f^{-1}((f(a) + f(b))/2)$, where f^{-1} is the inverse function of f. Since f is strictly increasing and continuous, we have $t \in (a, b)$ and $2f(t) = f(a) + f(b)$. If f is differentiable at t, then it follows from Step 1 that $f((a+b)/2) \leqslant (f(a)+f(b))/2$. If f is not differentiable at t, there exists a sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq (a, b)$ such that $t_n \to t$ and f is differentiable at t_n for all $n \in \mathbb{N}$ because the monotonicity of f implies that f is differentiable almost everywhere. Define $b_n = f^{-1}(2f(t_n) - f(a))$. It is not difficult to verify that $b_n > t_n > a$ and $f(b_n) = 2f(t_n) - f(a)$ for all $n \in \mathbb{N}$, and $b_n \to b$. Using Step 1, we have $f((a+b_n)/2) \leqslant (f(a)+f(b_n))/2$ for all $n \in \mathbb{N}$, and the continuity yields $f((a+b)/2) \leqslant (f(a)+f(b))/2$. In the second case, the above argument gives $f((a + b)/2) \leqslant (f(a) + f(b))/2$ for $b > a \geqslant x_0$, and this is sufficient for the convexity of f. $\qquad\square$

Remark 3.3. Consider the expected utility model with an increasing continuous utility function u and the preference relation $\succeq$ on $\mathcal{X}$ given by $X \succeq Y \iff \mathbb{E}[u(X)] \geqslant \mathbb{E}[u(Y)]$. Here X and Y are

interpreted as financial surpluses. Using Propositions 3.1 and 3.3, u is concave if and only if

$$\text{for all } X \in \mathcal{X}, \; t \in \mathbb{R} \text{ and } \lambda \in [0,1]: \quad X \simeq t \implies \lambda X + (1-\lambda)t \succeq X.$$

This gives a weaker sufficient condition for risk aversion in the expected utility model than the usual one where t is replaced by a random variable Y; see Principi et al. (2024, Section 6) for a different relaxation of the usual condition.

Next, we consider the certainty equivalent ρ of α-MEU in Example 2.3. For technical tractability, we assume that the discount factor is deterministic. Such ρ is generally not quasi-convex or cash additive.

Proposition 3.4. *Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two nonempty, weak*-compact and convex sets of finitely additive probabilities. For $I \subseteq [0,1]$ and $\gamma > 0$, the risk measure*

$$\rho(X) = \sup_{r \in I} \left\{ \frac{1}{\gamma} \log \left(\alpha \min_{Q_1 \in \mathcal{Q}_1} \mathbb{E}_{Q_1}[\mathrm{e}^{\gamma r X}] + (1-\alpha) \max_{Q_2 \in \mathcal{Q}_2} \mathbb{E}_{Q_2}[\mathrm{e}^{\gamma r X}] \right) \right\}, \quad X \in \mathcal{X} \qquad (3.10)$$

is quasi-star-shaped.

Proof. Take any $X \in \mathcal{X}$, $t \in \mathbb{R}$ and $\lambda \in [0,1]$. Note that for any probability measure Q, we have

$$\mathbb{E}_Q[\mathrm{e}^{\gamma r \lambda X}] \leqslant \left(\mathbb{E}_Q[\mathrm{e}^{\gamma r X}] \right)^\lambda.$$

This gives $\rho(\lambda X) \leqslant \lambda \rho(X)$. We have

$$
\begin{aligned}
&\rho(\lambda X + (1-\lambda)t) \\
&= \sup_{r \in I} \left\{ \frac{1}{\gamma} \log \left(\alpha \min_{Q_1 \in \mathcal{Q}_1} \mathbb{E}_{Q_1}[\mathrm{e}^{\gamma r(\lambda X + (1-\lambda)t)}] + (1-\alpha) \max_{Q_2 \in \mathcal{Q}_2} \mathbb{E}_{Q_2}[\mathrm{e}^{\gamma r(\lambda X + (1-\lambda)t)}] \right) \right\} \\
&= \sup_{r \in I} \left\{ \frac{1}{\gamma} \log \left(\alpha \min_{Q_1 \in \mathcal{Q}_1} \mathbb{E}_{Q_1}[\mathrm{e}^{\gamma r \lambda X}] + (1-\alpha) \max_{Q_2 \in \mathcal{Q}_2} \mathbb{E}_{Q_2}[\mathrm{e}^{\gamma r \lambda X}] \right) + (1-\lambda)rt \right\} \\
&\leqslant \rho(\lambda X) + \sup_{r \in I} \left\{ (1-\lambda)rt \right\} \leqslant \lambda \rho(X) + (1-\lambda)\rho(t).
\end{aligned}
$$

Therefore, quasi-star-shapedness (and star-shapedness) holds. $\qquad\square$

We say a set $\mathcal{A} \subseteq \mathcal{X}$ is *star-shaped* if $\lambda X + (1-\lambda)t \in \mathcal{A}$ for all $\lambda \in [0,1]$, $X \in \mathcal{A}$ and $t \in \mathcal{A} \cap \mathbb{R}$. Next, we show that star-shapedness of the acceptance set $\mathcal{A}$ characterizes the quasi-star-shapedness of the risk measure $\rho_{\mathcal{A},S}$ in Example 2.4.

Proposition 3.5. *For a closed set $\mathcal{A} \subseteq \mathcal{X}$, the risk measure $\rho_{\mathcal{A},S}$ in (2.3) is quasi-star-shaped if and only if $\mathcal{A}$ is star-shaped.*

Proof. To show the "only if" statement, assume that $\rho_{\mathcal{A},S}$ is quasi-star-shaped. For all $X \in \mathcal{A}$, $t \in \mathcal{A} \cap \mathbb{R}$, and $\lambda \in [0,1]$, $\rho_{\mathcal{A},S}(\lambda X + (1-\lambda)t) \leqslant \max\{\rho_{\mathcal{A},S}(X), \rho_{\mathcal{A},S}(t)\} \leqslant 0$. Hence, $\lambda X + (1-\lambda)t \in \mathcal{A}$ which implies that $\mathcal{A}$ is star-shaped.

Next, we show the "if" statement. Suppose that $\mathcal{A}$ is star-shaped. For all $X \in \mathcal{X}$, $t \in \mathbb{R}$, and $\lambda \in [0,1]$, we have $-\rho_{\mathcal{A},S}(t)S_T/S_0 + t \in \mathcal{A} \cap \mathbb{R}$ and $-\rho_{\mathcal{A},S}(X)S_T/S_0 + X \in \mathcal{A}$. The star-shapedness of $\mathcal{A}$ implies that $\lambda X + (1-\lambda)t - \lambda\rho_{\mathcal{A},S}(X)S_T/S_0 - (1-\lambda)\rho_{\mathcal{A},S}(t)S_T/S_0 \in \mathcal{A}$. Thus,

$$\rho_{\mathcal{A},S}(\lambda X + (1-\lambda)t) = \inf\left\{m \in \mathbb{R} : \lambda X + (1-\lambda)t - \frac{m}{S_0}S_T \in \mathcal{A}\right\}$$

$$\leqslant \lambda\rho_{\mathcal{A},S}(X) + (1-\lambda)\rho_{\mathcal{A},S}(t) \leqslant \max\{\rho_{\mathcal{A},S}(X), \rho_{\mathcal{A},S}(t)\}. \qquad \square$$

Propositions 3.3–3.5 together illustrate that quasi-star-shapedness of cash-subadditive risk measures appears under natural conditions, and it can hold in relevant cases where quasi-convexity does not hold (as in the cases of Propositions 3.4–3.5).

4   Representation results on cash-subadditive risk measures

In this section, we present a representation result, Theorem 4.1, of general cash-subadditive risk measures, which illustrates that a cash-subadditive risk measure is the lower envelope of a family of quasi-convex cash-subadditive risk measures.

Theorem 4.1. *For a functional $\rho : \mathcal{X} \to \mathbb{R}$, the following statements are equivalent.*

  (i) ρ is a cash-subadditive risk measure.

  (ii) There exists a set $\mathcal{C}$ of quasi-convex cash-subadditive risk measures such that

$$\rho(X) = \min_{\psi \in \mathcal{C}} \psi(X), \quad \text{for all } X \in \mathcal{X}. \tag{4.1}$$

In order to prove Theorem 4.1, we need the following lemma, which will also be useful for a few other results.

Lemma 4.1. *If $\rho : \mathcal{X} \to \mathbb{R}$ is a risk measure, then $\rho(X) = \min_{Z \in \mathcal{X}} \rho_Z(X)$ for all $X \in \mathcal{X}$, where*

$$\rho_Z(X) = \inf\{\rho(Z+m) \mid m \in \mathbb{R}, \ Z+m \geqslant X\}, \quad X, Z \in \mathcal{X}.$$

Proof. For all $X, Z \in \mathcal{X}$, by the definition of ρ_Z, we have

$$\rho_Z(X) = \rho(Z + \text{ess-sup}(X-Z)).$$

Since ρ is monotone and $Z + \text{ess-sup}(X-Z) \geqslant X$, we have $\rho_Z(X) = \rho(Z + \text{ess-sup}(X-Z)) \geqslant \rho(X)$. Note that $\rho_X(X) = \rho(X + \text{ess-sup}(X-X)) = \rho(X)$. Thus, we have $\rho_Z(X) \geqslant \rho_X(X)$ and this gives $\min_{Z \in \mathcal{X}} \rho_Z(X) = \rho_X(X) = \rho(X)$. $\qquad \square$

Proof of Theorem 4.1. "(ii) $\Rightarrow$ (i)" is obvious. We now prove "(i) $\Rightarrow$ (ii)". Assume that ρ is a cash-subadditive risk measure. By Lemma 4.1, we have $\rho(X) = \min_{Z \in \mathcal{X}} \rho_Z(X)$ for all $X \in \mathcal{X}$,

where

$$\rho_Z(X) = \inf\{\rho(Z+m) \mid m \in \mathbb{R}, Z+m \geqslant X\} = \rho(Z + \text{ess-sup}(X - Z)), \quad X, Z \in \mathcal{X}.$$

It is clear that ρ_Z is monotonic. We show that ρ_Z is cash subadditive. Indeed, for all $m \geqslant 0$ and $X \in \mathcal{X}$, we have

$$\rho_Z(X + m) = \rho(Z + \text{ess-sup}(X + m - Z)) = \rho(Z + \text{ess-sup}(X - Z) + m)$$
$$\leqslant \rho(Z + \text{ess-sup}(X - Z)) + m = \rho_Z(X) + m.$$

Next, we show that ρ_Z is quasi-convex. To this end, we need to show that, for all $\alpha \in \mathbb{R}$, $X_1, X_2 \in \mathcal{X}$ and $\lambda \in [0, 1]$,

$$\rho_Z(X_i) \leqslant \alpha, \; i = 1, 2 \implies \rho_Z(\lambda X_1 + (1 - \lambda)X_2) \leqslant \alpha.$$

Assume that $\rho_Z(X_i) \leqslant \alpha$ for $i = 1, 2$. For all $\varepsilon > 0$ and $i = 1, 2$, there exists some $m_i \in \mathbb{R}$ such that $Z + m_i \geqslant X_i$ and $\rho(Z + m_i) \leqslant \rho_Z(X_i) + \varepsilon \leqslant \alpha + \varepsilon$. Thus we have

$$\lambda X_1 + (1 - \lambda)X_2 \leqslant Z + \lambda m_1 + (1 - \lambda)m_2.$$

It then follows that

$$\rho_Z(\lambda X_1 + (1 - \lambda)X_2) \leqslant \rho(Z + \lambda m_1 + (1 - \lambda)m_2) \leqslant \rho(Z + \max\{m_1, m_2\}) \leqslant \alpha + \varepsilon.$$

The arbitrariness of ε implies that $\rho_Z(\lambda X_1 + (1 - \lambda)X_2) \leqslant \alpha$. Therefore, ρ_Z is quasi-convex. Finally, $\{\rho_Z \mid Z \in \mathcal{X}\}$ is a desired family of quasi-convex cash-subadditive risk measures. □

The representation in Theorem 4.1 can be interpreted as that any cash-subadditive risk measure can be seen as a best-case representative from a collection of quasi-convex ones (which may be obtained through market competition, i.e., taking the cheapest price when risk measures are interpreted as price mechanisms), and this is similar to the situation in Castagnoli et al. (2022) in the context of monetary risk measures. As far as we are aware of, Theorem 4.1 is the first characterization result of cash-subadditive risk measures that are not necessarily quasi-convex. A connection between Theorem 4.1 and some results of Jia et al. (2020) on monetary risk measures are discussed in Appendix A.1. We also note that, by straightforward argument, an equivalent statement to Theorem 4.1 (ii) is

$$\rho(X) = \min\{\psi(X) \mid \psi \text{ is a quasi-convex cash-subadditive risk measure}, \; \psi \geqslant \rho\}, \quad X \in \mathcal{X}. \quad (4.2)$$

Note that (4.2) gives the largest set $\mathcal{C}$ of quasi-convex cash-subadditive risk measures for which the representation in Theorem 4.1 holds.

Remark 4.1. Using the same argument as for Theorem 4.1, a similar result holds for risk measures

without cash subadditivity; that is, a functional $\rho : \mathcal{X} \to \mathbb{R}$ is a risk measure if and only if

$$\rho(X) = \min\{\psi(X) \mid \psi \text{ is a quasi-convex risk measure, } \psi \geqslant \rho\}, \quad X \in \mathcal{X}.$$

Example 4.1 (ΛVaR). For a decreasing function $\Lambda : \mathbb{R} \to [0, 1]$ that is not constantly 0, by Theorem 3.1, the ΛVaR in (2.2) admits the representation $\Lambda\text{VaR}(X) = \inf_{x \in \mathbb{R}}\{\text{VaR}_{\Lambda(x)}(X) \vee x\}$, $X \in \mathcal{X}$. Since VaR commutes with continuous increasing transforms, we have

$$\Lambda\text{VaR}(X) = \inf_{x \in \mathbb{R}} \text{VaR}_{\Lambda(x)}(X \vee x). \tag{4.3}$$

Let $\mathcal{C}_t$ be the set of coherent risk measures dominating VaR_t for $t \in (0, 1)$. By Theorem 6.8 of Delbaen (2002), VaR_t has the representation

$$\text{VaR}_t(X) = \min_{\rho \in \mathcal{C}_t} \rho(X), \quad X \in \mathcal{X}. \tag{4.4}$$

For $x \in \mathbb{R}$, denote by $\tau_x : X \mapsto \tau(X \vee x)$ for $\tau \in \mathcal{C}_{\Lambda(x)}$ and by $\mathcal{C}_{\Lambda,x} = \{\tau_x : \tau \in \mathcal{C}_{\Lambda(x)}\}$. Using (4.3) and (4.4), we get the representation (4.1) for ΛVaR as

$$\Lambda\text{VaR}(X) = \min\left\{\rho(X) \;\middle|\; \rho \in \bigcup_{x \in \mathbb{R}} \mathcal{C}_{\Lambda,x}\right\}, \quad X \in \mathcal{X}. \tag{4.5}$$

We check that $\tau_x \in \mathcal{C}_{\Lambda,x}$ is cash subadditive and quasi-convex for any $x \in \mathbb{R}$. Indeed, τ_x is convex since it is the composition of a convex risk measure τ and a convex transform $y \mapsto y \vee x$. To see that it is cash subadditive, it suffices to note that $\tau_x(X + c) \leqslant \tau(X \vee x + c) = \tau_x(X) + c$ for $c \geqslant 0$.

A special case of ΛVaR is the two-level ΛVaR in Example 7 of Bellini and Peri (2022), which is the simplest form of ΛVaR different from VaR; we give a more explicit formula for this case. Fix $0 < \alpha < \beta < 1$ and $z \in \mathbb{R}$. Define $\Lambda' : x \mapsto \beta\mathbb{1}_{\{x \leqslant z\}} + \alpha\mathbb{1}_{\{x > z\}}$. The corresponding risk measure is given by $\Lambda'\text{VaR}(X) = \min\{\text{VaR}_\beta(X), \text{VaR}_\alpha(X \vee z)\}$, $X \in \mathcal{X}$. Write $\mathcal{C}_{t,x} = \{\tau_x : \tau \in \mathcal{C}_t\}$ for $x \in \mathbb{R}$ and $t \in (0, 1)$. By (4.5),

$$\Lambda'\text{VaR}(X) = \min\{\rho(X) \mid \rho \in \mathcal{C}_\beta \cup \mathcal{C}_{\alpha,z}\}, \quad X \in \mathcal{X}.$$

The representation (4.5) is parallel to the property that VaR_t can be represented as the lower envelope the coherent risk measures dominating VaR_t in (4.4). Since ΛVaR has a similar interpretation to VaR via assessing risks with loss probability, and cash subadditivity and quasi-convexity generalize cash additivity and convexity, the representation (4.5) arises quite naturally.

Next, we look at a more explicit representation of cash-subadditive risk measures. An existing result of Cerreia-Vioglio et al. (2011) states that a quasi-convex cash-subadditive risk measure can be represented by the supremum of a family of functions $(t, Q) \mapsto R(t, Q)$ that are upper semicontinuous, quasi-concave, increasing and 1-Lipschitz in its first argument t. Combining Theorem 4.1 and Theorem 3.1 of Cerreia-Vioglio et al. (2011), we obtain a representation of a general cash-subadditive risk measure based on the above functions R.

Proposition 4.1. *A functional $\rho : \mathcal{X} \to \mathbb{R}$ is a cash-subadditive risk measure if and only if there exists a set $\mathcal{R}$ of upper semi-continuous, quasi-concave, increasing and 1-Lipschitz in the first argument functions $R : \mathbb{R} \times \mathcal{M}_f \to \mathbb{R}$ such that*

$$\rho(X) = \min_{R \in \mathcal{R}} \max_{Q \in \mathcal{M}_f} R\left(\mathbb{E}_Q[X], Q\right), \quad \text{for all } X \in \mathcal{X}.$$

Proposition 4.1 has a similar form to the minimax representation of star-shaped risk measures in Proposition 5 of Castagnoli et al. (2022). We note that the set $\mathcal{R}$ is not unique in this representation, different from R in Cerreia-Vioglio et al. (2011), which is unique. Instead, the largest choice of $\mathcal{R}$ is unique, which is the set of all R satisfying the conditions in Proposition 4.1 such that $\max_{Q \in \mathcal{M}_f} R\left(\mathbb{E}_Q[X], Q\right) \geqslant \rho(X)$ for all X.

5 Cash-subadditive risk measures with further properties

5.1 Normalized and quasi-star-shaped cash-subadditive risk measures

In this section, we give a representation of cash-subadditive risk measures that are normalized and quasi-star-shaped in Theorem 5.1. Some other relevant technical results are also obtained. Before showing Theorem 5.1, we need the representation result below of quasi-normalized, quasi-star-shaped and cash-subadditive risk measures, which is in similar sense with Lemma 4.1 but based on a more sophisticated construction with techniques different from the literature. In what follows, the convention is $\sup \emptyset = -\infty$ so that all quantities are well defined.

Proposition 5.1. *Let $\rho : \mathcal{X} \to \mathbb{R}$ be a quasi-normalized, quasi-star-shaped and cash-subadditive risk measure. For $Z \in \mathcal{X}$ and $t \in \mathbb{R}$, define*

$$m_Z(t) = \sup\{m \in \mathbb{R} \mid \rho(Z + m) = t\} \quad \text{and} \quad \mathcal{A}_Z^t = \bigcup_{\lambda \in [0,1]} \{X \in \mathcal{X} \mid X \leqslant \lambda(Z + m_Z(t)) + (1 - \lambda)t\}.$$

We have $\rho(X) = \min_{Z \in \mathcal{X}} \tilde{\rho}_Z(X)$ for $X \in \mathcal{X}$, where $\tilde{\rho}_Z(X) = \inf\{t \in \mathbb{R} \mid X \in \mathcal{A}_Z^t\}$.

Proof. Since a cash-subadditive risk measure is L^∞-continuous, for each $Z \in \mathcal{X}$, the range of the function $m \mapsto \rho(Z + m)$ on $\mathbb{R}$ is an interval of $\mathbb{R}$. Moreover, recall the definition of $D_\rho = \{\rho(X) \mid X \in \mathcal{X}\}$, since ρ is quasi-normalized, the function $m \mapsto \rho(m)$ on $\mathbb{R}$ takes all possible values in D_ρ, which is an interval on $\mathbb{R}$, and by monotonicity, so does $m \mapsto \rho(Z + m)$. Hence, $\rho(Z + m_Z(t)) = t$ for all $t \in D_\rho$. For $X, Z \in \mathcal{X}$, we can write

$$\begin{aligned}
\tilde{\rho}_Z(X) &= \inf\{t \in \mathbb{R} \mid X \in \mathcal{A}_Z^t\} \\
&= \inf\{t \in \mathbb{R} \mid X \leqslant \lambda(Z + m_Z(t)) + (1 - \lambda)t \text{ for some } \lambda \in [0,1]\} \\
&= \inf_{\lambda \in [0,1]} \inf\{t \in \mathbb{R} \mid X \leqslant \lambda(Z + m_Z(t)) + (1 - \lambda)t\}.
\end{aligned}$$

It is straightforward that $\tilde{\rho}_Z(X) \in D_\rho$. For $X, Z \in \mathcal{X}$ and $t \in D_\rho$, if $\tilde{\rho}_Z(X) < t$, then $X \leqslant \lambda(Z + m_Z(t)) + (1 - \lambda)t$ for some $\lambda \in [0,1]$. By monotonicity, quasi-normalization and

quasi-star-shapedness of ρ, we have

$$\rho(X) \leqslant \rho(\lambda(Z + m_Z(t)) + (1-\lambda)t) \leqslant \max\{\rho(Z + m_Z(t)), t\} = t.$$

Thus we have $\rho(X) \leqslant \inf_{Z \in \mathcal{X}} \tilde{\rho}_Z(X)$. On the other hand,

$$\tilde{\rho}_Z(X) \leqslant \inf\{t \in \mathbb{R} \mid X \leqslant Z + m_Z(t)\}$$
$$= \inf\{t \in \mathbb{R} \mid m_Z(t) = \text{ess-sup}(X - Z)\} = \rho(Z + \text{ess-sup}(X - Z)),$$

which is $\rho_Z(X)$ in Lemma 4.1. Using Lemma 4.1, we have

$$\rho(X) = \min_{Z \in \mathcal{X}} \rho_Z(X) \geqslant \inf_{Z \in \mathcal{X}} \tilde{\rho}_Z(X) \geqslant \rho(X). \tag{5.1}$$

Moreover, attainability of the infimum is guaranteed by $\rho(X) = \rho_X(X) \geqslant \tilde{\rho}_X(X) \geqslant \rho(X)$. Therefore, $\rho(X) = \min_{Z \in \mathcal{X}} \tilde{\rho}_Z(X)$ holds. $\qquad\square$

The representation in Proposition 5.1 is closely linked to that in Lemma 4.1 through (5.1).

Remark 5.1. Although arising from completely different considerations, the risk measure $\tilde{\rho}_Z$ in Proposition 5.1 has a similar form to an acceptability index of Cherny and Madan (2009), which has the form $\alpha(X) = \sup\{x \in \mathbb{R}_+ \mid X \in \mathcal{A}_x\}$ where $(\mathcal{A}_x)_{x \in \mathbb{R}_+}$ is a decreasing family of subsets of $\mathcal{X}$. For more recent results on acceptability indices, see e.g., Righi (2024).

The following representation result concerns cash-subadditive risk measures that are normalized and quasi-star-shaped. We show that a normalized, quasi-star-shaped and cash-subadditive risk measure can be represented by the lower envelope of a family of ones that are normalized, quasi-convex, and cash subadditive.

Theorem 5.1. *For a functional $\rho : \mathcal{X} \to \mathbb{R}$, the following statements are equivalent.*

(i) ρ is a normalized, quasi-star-shaped and cash-subadditive risk measure.

(ii) There exists a family $\mathcal{C}$ of normalized, quasi-convex and cash-subadditive risk measures such that

$$\rho(X) = \min_{\psi \in \mathcal{C}} \psi(X), \quad \text{for all } X \in \mathcal{X}. \tag{5.2}$$

Proof. "(ii) $\Rightarrow$ (i)": Assume that there exists a family $\mathcal{C}$ of normalized, quasi-convex and cash-subadditive risk measures such that $\rho = \min_{\psi \in \mathcal{C}} \psi$. Monotonicity, normalization and cash subadditivity of ρ are straightforward. Quasi-star-shapedness follows from Lemma 3.1.

"(i) $\Rightarrow$ (ii)": Assume that ρ is a normalized, quasi-star-shaped and cash-subadditive risk measure. Using Proposition 5.1, it suffices to show that $\tilde{\rho}_Z(X)$ defined via

$$\tilde{\rho}_Z(X) = \inf_{\lambda \in [0,1]} \inf\{t \in \mathbb{R} \mid X \leqslant \lambda(Z + m_Z(t)) + (1-\lambda)t\}$$

for each $Z \in \mathcal{X}$ is a normalized, quasi-convex and cash-subadditive risk measure.

We first verify that each $\tilde{\rho}_Z$ is normalized. For all $s \in \mathbb{R}$, by taking $\lambda = 0$, we have $\tilde{\rho}_Z(s) \leqslant$ $\inf\{t \in \mathbb{R} \mid s \leqslant t\} = s$. On the other hand, for all $t \in \mathbb{R}$ and $\lambda \in [0,1]$ such that $s \leqslant \lambda(Z + m_Z(t)) + (1-\lambda)t$, by normalization, monotonicity and quasi-star-shapedness of ρ, we have

$$s = \rho(s) \leqslant \rho(\lambda(Z + m_Z(t)) + (1-\lambda)t) \leqslant \max\{\rho(Z + m_Z(t)), t\} = t.$$

Hence we obtain $\tilde{\rho}_Z(s) \geqslant s$, and further $\tilde{\rho}_Z(s) = s$.

Next, we show that each $\tilde{\rho}_Z$ is quasi-convex. We first note that $\mathcal{A}_Z^t$ is a convex set for each $t \in \mathbb{R}$, which follows from the fact that $\mathcal{A}_Z^t$ is the set of all $X \in \mathcal{X}$ dominated by the segment $\{\lambda(Z + m_Z(t)) + (1-\lambda)t \mid \lambda \in [0,1]\}$. Take $t \in \mathbb{R}$. For X_1, X_2 satisfying $\tilde{\rho}_Z(X_1) \leqslant \tilde{\rho}_Z(X_2) \leqslant t$, for any $s > t$, we have $X_1, X_2 \in \mathcal{A}_Z^s$. Convexity of $\mathcal{A}_Z^s$ implies, for each $\lambda \in [0,1]$, $\lambda X_1 + (1-\lambda)X_2 \in \mathcal{A}_Z^s$, and it further gives $\tilde{\rho}_Z(\lambda X_1 + (1-\lambda)X_2) \leqslant s$. Since $s > t$ is arbitrary, we have $\tilde{\rho}_Z(\lambda X_1 + (1-\lambda)X_2) \leqslant t$. This gives quasi-convexity of $\tilde{\rho}_Z$.

Finally, we prove that $\tilde{\rho}_Z$ is cash subadditive for all $Z \in \mathcal{X}$. Note that continuity in Remark 2.1 implies

$$m_Z(t) = \sup\{m \in \mathbb{R} \mid \rho(Z + m) \leqslant t\}.$$

Since ρ is cash subadditive, for all $Z \in \mathcal{X}$, $t \in \mathbb{R}$ and $c \geqslant 0$,

$$m_Z(t + c) = \sup\{m + c \in \mathbb{R} \mid \rho(Z + m + c) \leqslant t + c\}$$
$$\geqslant \sup\{m + c \in \mathbb{R} \mid \rho(Z + m) + c \leqslant t + c\} = m_Z(t) + c.$$

For all $c \geqslant 0$ and $X \in \mathcal{X}$, we have

$$\tilde{\rho}_Z(X + c) = \inf_{\lambda \in [0,1]} \inf\{t \in \mathbb{R} \mid X + c \leqslant \lambda(Z + m_Z(t)) + (1-\lambda)t\}$$
$$= \inf_{\lambda \in [0,1]} \inf\{t + c \in \mathbb{R} \mid X + c \leqslant \lambda(Z + m_Z(t + c)) + (1-\lambda)(t + c)\}$$
$$= \inf_{\lambda \in [0,1]} \inf\{t \in \mathbb{R} \mid X \leqslant \lambda(Z + m_Z(t + c) - (t + c)) + t\} + c$$
$$\leqslant \inf_{\lambda \in [0,1]} \inf\{t \in \mathbb{R} \mid X \leqslant \lambda(Z + m_Z(t) - t) + t\} + c = \tilde{\rho}_Z(X) + c.$$

In summary, $\{\tilde{\rho}_Z \mid Z \in \mathcal{X}\}$ is a desired family of normalized, quasi-convex and cash-subadditive risk measures. $\qquad\square$

The proof of Theorem 5.1 is based on a delicate construction of the dominating risk measures, different from those used for Theorem 4.1. Normalization in both (i) and (ii) of Theorem 5.1 is important and cannot be removed, but it can be replaced by quasi-normalization. The modified version of Theorem 5.1 using quasi-normalization follows from combining Proposition 5.1 and Lemma 3.1.

Theorem 5.1 can be seen as a parallel result, although obtained via different techniques, to the representation result of Castagnoli et al. (2022), which uses star-shapedness, convexity, and cash additivity instead of quasi-star-shapedness, quasi-convexity and cash subadditivity. It is clear that

(ii) of Theorem 5.1 is equivalent to the following alternative formulation

$$\rho(X) = \min\left\{ \psi(X) \;\middle|\; \begin{array}{l} \psi \text{ is a normalized, quasi-convex and} \\ \text{cash-subadditive risk measure, } \psi \geqslant \rho \end{array} \right\}, \quad X \in \mathcal{X}. \tag{5.3}$$

5.2 SSD-consistent cash-subadditive risk measures

Let $(\Omega, \mathcal{F}, P)$ be a nonatomic probability space in this section. We present below the representation result of SSD-consistent cash-subadditive risk measures. For this, we define the *Expected Shortfall* (ES) at level $t \in [0, 1]$ as

$$\mathrm{ES}_t(X) = \frac{1}{1-t} \int_t^1 \mathrm{VaR}_\alpha(X) \, \mathrm{d}\alpha, \ t \in [0, 1) \ \text{ and } \ \mathrm{ES}_1(X) = \text{ess-sup}(X), \ \ X \in \mathcal{X}.$$

As a coherent alternative to VaR, ES is the most important risk measure in current banking regulation; see Wang and Zitikis (2021) for its role in the Basel Accords and an axiomatization. It is well known that the class of ES characterizes SSD via

$$X \succeq_2 Y \iff \mathrm{ES}_t(X) \geqslant \mathrm{ES}_t(Y) \text{ for all } t \in [0, 1].$$

Mao and Wang (2020) investigated SSD-consistent monetary risk measures and provided four equivalent conditions of SSD-consistency; see their Theorem 2.1. The result can also be extended to L^∞-continuous risk measures, which is shown in the following lemma.

Lemma 5.1. *Let ρ be an L^∞-continuous risk measure on $\mathcal{X}$. The following are equivalent.*

(i) ρ is SSD-consistent.

(ii) $\rho(X) \geqslant \rho(Y)$ for all $X, Y \in \mathcal{X}$ with $X \succeq_2 Y$ and $\mathbb{E}[X] = \mathbb{E}[Y]$.

(iii) $\rho(X) \geqslant \rho(Y)$ for all $X, Y \in \mathcal{X}$ with $\mathbb{E}\left[(X - K)_+\right] \geqslant \mathbb{E}\left[(Y - K)_+\right]$ for all $K \in \mathbb{R}$.

(iv) $\rho(X) \geqslant \rho(Y)$ for all $X, Y \in \mathcal{X}$ with $Y = \mathbb{E}[X \mid Y]$.

(v) $\rho(X^c + Y^c) \geqslant \rho(X + Y)$ for all $X, Y, X^c, Y^c \in \mathcal{X}$ such that (X^c, Y^c) is comonotonic, $X \stackrel{\mathrm{d}}{=} X^c$, and $Y \stackrel{\mathrm{d}}{=} Y^c$.

Moreover, any of these properties imply that ρ is law invariant.

Proof. Since $(\Omega, \mathcal{F}, P)$ is nonatomic, the equivalence among (i)-(iv) is easy to verify from classic properties of SSD by the same logic of the proof of Theorem 2.1 in Mao and Wang (2020). The equivalence between (i) and (v) for L^∞-continuous functions follows from Theorem 2 of Wang and Wu (2020). $\square$

The following lemma is needed in the proof of Theorem 5.2, which was obtained by Cerreia-Vioglio et al. (2011) with the additional assumption of continuity from above. We include a self-contained proof of Lemma 5.2.

Lemma 5.2. *If $\rho : \mathcal{X} \to \mathbb{R}$ is a quasi-convex cash-subadditive risk measure, then ρ is law invariant if and only if ρ is SSD-consistent.*

Proof. It is obvious that SSD-consistency implies law invariance. We will only show the "only if" statement. By Lemma 5.1, it suffices to show that $\rho(X) \leqslant \rho(Y)$ for $X \preceq_2 Y$ satisfying $\mathbb{E}[X] = \mathbb{E}[Y]$. By Proposition 3.6 of Mao and Wang (2015), there exists a sequence of $\mathbf{Y}^k = (Y_1^k, \ldots, Y_{n_k}^k)$, $k \in \mathbb{N}$, such that each $Y_j^k \stackrel{\mathrm{d}}{=} Y$, $n_k \to \infty$ as $k \to \infty$, and

$$\frac{1}{n_k} \sum_{j=1}^{n_k} Y_j^k \to X \quad \text{in } L^\infty.$$

Note that a cash-subadditive risk measure is L^∞-continuous. Quasi-convexity and L^∞-continuity lead to

$$\rho(Y) \geqslant \rho\left(\frac{1}{n_k} \sum_{j=1}^{n_k} Y_j^k \right) \to \rho(X),$$

and thus ρ is SSD-consistent. $\qquad\square$

In the following theorem, we establish a representation for an SSD-consistent cash-subadditive risk measure as the lower envelope of some family of law-invariant, quasi-convex and cash-subadditive risk measures.

Theorem 5.2. *For a functional $\rho : \mathcal{X} \to \mathbb{R}$, the following statements are equivalent.*

(i) *ρ is an SSD-consistent cash-subadditive risk measure.*

(ii) *There exists a family $\mathcal{C}$ of law-invariant, quasi-convex and cash-subadditive risk measures such that*
$$\rho(X) = \min_{\psi \in \mathcal{C}} \psi(X), \quad \text{for all } X \in \mathcal{X}.$$

Proof. "(ii) $\Rightarrow$ (i)" is implied by Lemma 5.2 (i) and the fact that cash subadditivity and SSD-consistency are preserved under the infimum operation. We will show "(i) $\Rightarrow$ (ii)".

Suppose that ρ is an SSD-consistent cash-subadditive risk measure. For all $X \in \mathcal{X}$ and $Z \in \mathcal{X}$, define the risk measure

$$\psi_Z(X) = \inf\{\rho(Z + m) \mid m \in \mathbb{R}, \ Z + m \succeq_2 X\}.$$

It is straightforward to check that $\rho(X) = \min_{Z \in \mathcal{X}} \psi_Z(X)$ and

$$\psi_Z(X) = \inf\{\rho(Z + m) \mid m \in \mathbb{R}, \ \mathrm{ES}_t(Z) + m \geqslant \mathrm{ES}_t(X), \text{ for all } t \in [0,1]\}$$
$$= \rho\left(Z + \sup_{t \in [0,1]} (\mathrm{ES}_t(X) - \mathrm{ES}_t(Z)) \right).$$

It is clear that ψ_Z is monotone, cash subadditive and law invariant. We prove that ψ_Z is quasi-convex with similar manner to the proof of Theorem 4.1. Assume that $\psi_Z(X_i) \leqslant \alpha$ for $i = 1, 2$. For all $\varepsilon > 0$

24

and $i = 1, 2$, there exists some $m_i \in \mathbb{R}$ such that $Z + m_i \succeq_2 X_i$ and $\rho(Z + m_i) \leqslant \psi_Z(X_i) + \varepsilon \leqslant \alpha + \varepsilon$. By Theorem 3.5 of Rüschendorf (2013), we have $Z + \lambda m_1 + (1 - \lambda)m_2 \succeq_2 \lambda X_1 + (1 - \lambda)X_2$ for all $\lambda \in [0, 1]$. It follows that

$$\psi_Z(\lambda X_1 + (1 - \lambda)X_2) \leqslant \rho(Z + \lambda m_1 + (1 - \lambda)m_2) \leqslant \rho(Z + \max\{m_1, m_2\}) \leqslant \alpha + \varepsilon.$$

The arbitrariness of ε implies that $\psi_Z(\lambda X_1 + (1 - \lambda)X_2) \leqslant \alpha$. Therefore, ψ_Z is quasi-convex. We conclude that $\{\psi_Z \mid Z \in \mathcal{X}\}$ is a desired family of law-invariant, quasi-convex and cash-subadditive risk measures. $\qquad\square$

Theorem 5.2 can be seen as a parallel result to Theorem 3.3 of Mao and Wang (2020) which showed that any SSD-consistent monetary risk measure is the lower envelope of law-invariant and convex monetary risk measures. Similarly to (5.3), we can reformulate (ii) of Theorem 5.2 as

$$\rho(X) = \min \left\{ \psi(X) \ \middle| \ \begin{array}{l} \psi \text{ is a law-invariant, quasi-convex and} \\ \text{cash-subadditive risk measure, } \psi \geqslant \rho \end{array} \right\}, \quad X \in \mathcal{X}. \tag{5.4}$$

A representation result in a similar spirit to Proposition 4.1 for SSD-consistent cash-subadditive risk measures follows directly from Theorem 5.1 of Cerreia-Vioglio et al. (2011) and Theorem 5.2.

Proposition 5.2. *Let $\rho : \mathcal{X} \to \mathbb{R}$ be a functional that is continuous from above. We have ρ is an SSD-consistent cash-subadditive risk measure if and only if there exists a set $\mathcal{R}$ of upper semi-continuous, quasi-concave, increasing and 1-Lipschitz in the first component functions $R : \mathbb{R} \times \mathcal{M} \to \mathbb{R}$ such that*

$$\rho(X) = \min_{R \in \mathcal{R}} \max_{Q \in \mathcal{M}} R \left(\int_0^1 \mathrm{VaR}_t(X)\mathrm{VaR}_t \left(\frac{\mathrm{d}Q}{\mathrm{d}P} \right) \mathrm{d}t, \ Q \right), \quad \text{for all } X \in \mathcal{X}.$$

6 Conclusion

We provide a systemic study of cash-subadditive risk measures, which were traditionally studied together with convexity (El Karoui and Ravanelli, 2009) or quasi-convexity (Cerreia-Vioglio et al., 2011). Different from the literature, our study focuses on cash-subadditive risk measures without quasi-convexity, which include many natural examples as discussed in the paper. As our major technical contributions, a general cash-subadditive risk measure is shown to be representable by the lower envelope of a family of quasi-convex cash-subadditive risk measures (Theorem 4.1). The notions of quasi-star-shapedness and quasi-normalization were introduced as analogues of star-shapedness and normalization studied by Castagnoli et al. (2022). It turns out that quasi-star-shapedness and quasi-normalization fit naturally in the setting of cash subadditivity, leading to a new representation result (Theorem 5.1). A representation result of SSD-consistent cash-subadditive risk measures was also obtained (Theorem 5.2). We summarize some related results in the literature and compare them with our results in Table 1.

	a (...) risk measure	is an infimum of (...) risk measures
Mao and Wang (2020)	CA, SSD-consistent	CA, convex, law-invariant
Jia et al. (2020)	CA	CA, convex
Castagnoli et al. (2022)	CA, star-shaped, normalized	CA, convex, normalized
Theorem 5.2	CS, SSD-consistent	CS, quasi-convex, law-invariant
Theorem 4.1	CS	CS, quasi-convex
Theorem 5.1	CS, quasi-star-shaped, normalized	CS, quasi-convex, normalized

Table 1: Representation results related to this paper, where monotonicity is always assumed; definitions of the properties are in Sections 2 and 3. CA stands for cash additivity and CS stands for cash subadditivity.

Furthermore, we obtain several results on the risk measure ΛVaR proposed by Frittelli et al. (2014), including a new representation result (Theorem 3.1). In particular, the class of ΛVaR serves as a natural example of quasi-star-shaped, quasi-normalized and cash-subadditive risk measures, which are not star-shaped, normalized, or cash additive.

Risk measures without cash additivity have received increasing attention in the recent literature due to their technical generality and intimate connection to decision analysis, risk transforms, portfolio optimization, and stochastic interest rates; many references and examples were mentioned in the introduction and throughout the paper. Results in this paper serve as a building block for future studies on cash subadditivity and the new properties of quasi-star-shapedness and quasi-normalization, for which many questions and applications remain to be explored.

Acknowledgments

References

Anscombe, F. J. and Aumann, R. J. (1963). A definition of subjective probability. *Annals of Mathematical Statistics*, **34**(1), 199–205.

Arduca, M. and Munari, C. (2024). Risk measures beyond frictionless markets. *SIAM Journal on Financial Mathematics*, **15**(2), 537–570

Artzner, P., Delbaen, F., Eber, J.-M. and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, **9**(3), 203–228.

Barrieu, P. and El Karoui, N. (2005). Inf-convolution of risk measures and optimal risk transfer. *Finance and Stochastics*, **9**(2), 269–298.

Basak, S. and Shapiro, A. (2001). Value-at-Risk-based risk management: Optimal policies and asset prices. *Review of Financial Studies*, **14**(2), 371–405.

Basak, S., Shapiro, A. and Teplá, L. (2006). Risk management with benchmarking. *Management Science*, **52**, 542–557.

Bellini, F. and Peri, I. (2022). An axiomatization of Λ-quantiles. *SIAM Journal on Financial Mathematics*, **13**(1), SC26–SC38.

Burzoni, M., Peri, I. and Ruffo, C.M. (2017). On the properties of the Lambda Value at Risk: Robustness, elicitability and consistency. *Quantitative Finance*, **17**(11), 1735–1743.

Carlier, G. and Dana, R.-A. (2003). Core of convex distortions of a probability. *Journal of Economic Theory*, **113**, 199–222.

Castagnoli, E., Cattelan, G., Maccheroni, F., Tebaldi, C. and Wang, R. (2022). Star-shaped risk measures. *Operations Research*, **70**(5), 2637–2654.

Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M. and Montrucchio, L. (2011). Risk measures: Rationality and diversification. *Mathematical Finance*, **21**(4), 743–774.

Cherny, A. S. and Madan, D. (2009). New measures for performance evaluation. *Review of Financial Studies*, **22**(7), 2571–2606.

Choquet, G. (1954). Theory of capacities. *Annales de l'institut Fourier*, **5**, 131–295.

Cont, R., Deguest, R. and He, X. D. (2013). Loss-based risk measures. *Statistics and Risk Modeling*, **30**, 133–167.

Daníelsson, J., Embrechts, P., Goodhart, C., Keating, C., Muennich, F., Renault, O. and Shin, H. S. (2001). An academic response to Basel II. *LSE Special Paper Series May 2001*.

Delbaen, F. (2002). Coherent risk measures on general probability spaces. In *Advances in Finance and Stochastics* (pp. 1–37). Springer, Berlin, Heidelberg.

Drapeau, S. and Kupper, M. (2013). Risk preferences and their robust representation. *Mathematics of Operations Research*, **38**(1), 28–62.

El Karoui, N. and Ravanelli, C. (2009). Cash subadditive risk measures and interest rate ambiguity. *Mathematical Finance*, **19**(4), 562–590.

Farkas, W., Koch-Medina, P. and Munari, C. (2014). Beyond cash-additive risk measures: When changing the numeraire fails. *Finance and Stochastics*, **18**, 145–173.

Filipović, D. and Svindland, G. (2008). Optimal capital and risk allocations for law- and cash-invariant convex functions. *Finance and Stochastics*, **12**, 423–439.

Föllmer, H. and Schied, A. (2002). Convex measures of risk and trading constraints. *Finance and Stochastics*, **6**(4), 429–447.

Föllmer, H. and Schied, A. (2016). *Stochastic Finance. An Introduction in Discrete Time*. Fourth Edition. Walter de Gruyter, Berlin.

Frittelli, M. and Maggis, M. (2011). Dual representation of quasiconvex conditional maps. *SIAM Journal on Financial Mathematics*, **2**, 357–382.

Frittelli, M., Maggis, M. and Peri, I. (2014). Risk measures on $\mathcal{P}(\mathbb{R})$ and value at risk with probability/loss function. *Mathematical Finance*, **24**(3), 442–463.

Frittelli, M. and Rosazza Gianin, E. (2002). Putting order in risk measures. *Journal of Banking and Finance*,

26, 1473–1486.

Ghirardato, P., Maccheroni, F. and Marinacci, M. (2004). Differentiating ambiguity and ambiguity attitude. *Journal of Economic Theory*, **118**(2), 133–173.

Jarrow, R. (2002). Put option premiums and coherent risk measures. *Mathematical Finance*, **12**(2), 135–142.

Jia, G., Xia, J. and Zhao, R. (2020). Monetary risk measures. *arXiv*: 2012.06751.

Li, H. and Wang, R. (2023). PELVE: Probability equivalent level of VaR and ES. *Journal of Econometrics*, **234**(1), 353–370.

Liebrich, F. B. and Svindland, G. (2019). Efficient allocations under law-invariance: A unifying approach. *Journal of Mathematical Economics*, **84**, 28–45.

Maccheroni, F., Marinacci, M. and Rustichini, A. (2006). Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica*, **74**(6), 1447–1498.

Mao, T. and Wang, R. (2015). On aggregation sets and lower-convex sets. *Journal of Multivariate Analysis*, **136**, 12–25.

Mao T. and Wang, R. (2020). Risk aversion in regulatory capital calculation. *SIAM Journal on Financial Mathematics,* **11**(1), 169–200.

Marinacci, M. (2002). Probabilistic sophistication and multiple priors. *Econometrica*, **70**(2), 755–764.

Mastrogiacomo, E. and Rosazza Gianin, E. (2015). Portfolio optimization with quasiconvex risk measures. *Mathematics of Operations Research*, **40**(4), 1042–1059.

McNeil, A. J., Frey, R. and Embrechts, P. (2015). *Quantitative Risk Management: Concepts, Techniques and Tools*. Revised Edition. Princeton, NJ: Princeton University Press.

Moresco, M. R. and Righi, M. B. (2022). On the link between monetary and star-shaped risk measures. *Statistics and Probability Letters*, **184**, 109345.

Müller, A., Scarsini, M., Tsetlin, I. and Winkler. R. L. (2017). Between first and second-order stochastic dominance. *Management Science*, **63**(9), 2933–2947.

Munari, C. A. (2015). Measuring risk beyond the cash-additive paradigm. *Doctoral Dissertation*, ETH Zurich.

Principi, G., Wakker, P. and Wang, R. (2024). Anticomonotonicity for preference axioms: The natural counterpart to comonotonicity. *Theoretical Economics*, forthcoming.

Quiggin, J. (1982). A theory of anticipated utility. *Journal of Economic Behavior & Organization*, **3**(4), 323–343.

Righi, M. B. (2024). Star-shaped acceptability indexes. *Insurance: Mathematics and Economics*, **117**, 170–181.

Rüschendorf, L. (2013). *Mathematical Risk Analysis. Dependence, Risk Bounds, Optimal Allocations and Portfolios*. Springer, Heidelberg.

Schmeidler, D. (1986). Integral representation without additivity. *Proceedings of the American Mathematical Society*, **97**(2), 255–261.

Schmeidler, D. (1989). Subjective probability and expected utility without additivity. *Econometrica*, **57**(3), 571–587.

Song, Y. and Yan, J. (2006). The representations of two types of functions on $L^\infty(\Omega, \mathcal{F})$ and $L^\infty(\Omega, \mathcal{F}, P)$. *Science in China Series A: Mathematics*, **49**, 1376–1382.

Song, Y. and Yan, J. (2009). Risk measures with comonotonic subadditivity or convexity and respecting stochastic orders. *Insurance: Mathematics and Economics*, **45**(3), 459–465.

Tian, D. and Long, J. (2015). Quasiconvex risk statistics with scenario analysis. *Mathematics and Financial Economics*, **9**(2), 111–121.

Trautmann, S. T. and Van De Kuilen, G. (2015). Ambiguity attitudes. *The Wiley Blackwell Handbook of Judgment and Decision Making*, **2**, 89–116.

Wang, R. and Wu, Q. (2020). Dependence and risk attitudes: An equivalence. *SSRN*: 3707709.

Wang, R. and Zitikis, R. (2021). An axiomatic foundation for the Expected Shortfall. *Management Science*, **67**(3), 1413–1429.

Xia, J. (2013). Comonotonic convex preferences. *SSRN*: 2298884.

Yaari, M. E. (1987). The dual theory of choice under risk. *Econometrica*, **55**(1), 95–115.

A   Additional results and technical discussions

This appendix includes a few additional technical results, examples and discussions of the representation results of cash-subadditive risk measures, which are not used in the main text of the paper. Some of them may be of independent interest.

A.1   Connection to a representation of monetary risk measures

Theorem 4.1 is more general than the result of Jia et al. (2020) for monetary risk measures, which says that any monetary risk measure can be written as the infimum of some convex monetary risk measures. Indeed, the proof of Theorem 4.1 does not depend on, but leads to Theorem 3.1 of Jia et al. (2020) as a special case.

The proposition below illustrates how Lemma 4.1 (the key lemma proving Theorem 4.1) can be used to show that any monetary risk measure is the minimum of some convex risk measures.

Proposition A.1. *A functional $\rho : \mathcal{X} \to \mathbb{R}$ is a monetary risk measure if and only if*

$$\rho(X) = \min_{Z \in \mathcal{A}_\rho} \text{ess-sup}(X - Z), \quad \text{for all } X \in \mathcal{X},$$

where $\mathcal{A}_\rho$ is the acceptance set of ρ given by $\mathcal{A}_\rho = \{Z \in \mathcal{X} \mid \rho(Z) \leqslant 0\}$.

Proof. The "if" part is straightforward. We prove the "only if" part. For all $X \in \mathcal{X}$, by Lemma 4.1 and cash additivity of ρ, we have

$$\rho(X) = \min_{Z \in \mathcal{X}} \rho(Z + \text{ess-sup}(X - Z)) = \min_{Z \in \mathcal{X}} \left\{ \rho(Z) + \text{ess-sup}(X - Z) \right\}.$$

By taking $Z_0 = X - \rho(X)$, we have $\rho(Z_0) + \text{ess-sup}(X - Z_0) = \rho(X)$, where the minimum is obtained. Define $\mathcal{A}_\rho^0 = \{Z \in \mathcal{X} \mid \rho(Z) = 0\}$. We have $Z_0 \in \mathcal{A}_\rho^0$ and thus

$$\rho(X) = \min_{Z \in \mathcal{A}_\rho^0} (\rho(Z) + \text{ess-sup}(X - Z)) = \min_{Z \in \mathcal{A}_\rho^0} \text{ess-sup}(X - Z) \geqslant \min_{Z \in \mathcal{A}_\rho} \text{ess-sup}(X - Z).$$

On the other hand, since $\rho(Z) \leqslant 0$ for all $Z \in \mathcal{A}_\rho$, we have

$$\rho(X) = \min_{Z \in \mathcal{A}_\rho} \left\{ \rho(Z) + \text{ess-sup}(X - Z) \right\} \leqslant \min_{Z \in \mathcal{A}_\rho} \text{ess-sup}(X - Z).$$

Therefore, we have $\rho(X) = \min_{Z \in \mathcal{A}_\rho} \text{ess-sup}(X - Z)$. □

A.2 Comonotonic quasi-convexity

Let $(\Omega, \mathcal{F}, P)$ be a nonatomic probability space. Since law-invariant, quasi-convex and cash-subadditive risk measures are SSD-consistent (Lemma 5.2), a general law-invariant cash-subadditive risk measure (such as VaR in Section 2) does not admit a representation via the lower envelope of a family of law-invariant, quasi-convex and cash-subadditive risk measures. One remaining question is whether a law-invariant cash-subadditive risk measure can be represented as the infimum of a set of law-invariant cash-subadditive risk measures with some other properties. For such a representation, we need comonotonic quasi-convexity.

> *Comonotonic quasi-convexity*: $\rho(\lambda X + (1 - \lambda)Y) \leqslant \max\{\rho(X), \rho(Y)\}$ for all comonotonic $(X, Y) \in \mathcal{X}^2$ and $\lambda \in [0, 1]$.

The property of comonotonic quasi-convexity appeared in various contexts; e.g., Xia (2013), Tian and Long (2015) and Li and Wang (2023). Before showing the representation result, we first give the following equivalence result demonstrating the relations among several properties of ρ, similarly to Lemma 5.2.

Lemma A.1. *If $\rho : \mathcal{X} \to \mathbb{R}$ is a cash-subadditive risk measure, then ρ is law invariant and quasi-convex if and only if ρ is SSD-consistent and comonotonic quasi-convex.*

Proof. The "only if" part follows directly from Lemma 5.2. We prove the "if" part. Suppose that ρ is SSD-consistent and comonotonic quasi-convex. It is clear that ρ is law invariant by taking $X \stackrel{\mathrm{d}}{=} Y$ and observing $X \succeq_2 Y$ and $Y \succeq_2 X$. For all $X, Y \in \mathcal{X}$, take $X^c, Y^c \in \mathcal{X}$ such that (X^c, Y^c) is comonotonic, $X^c \stackrel{\mathrm{d}}{=} X$, and $Y^c \stackrel{\mathrm{d}}{=} Y$. Again, by Theorem 3.5 of Rüschendorf (2013), we have $\lambda X^c + (1 - \lambda)Y^c \succeq_2 \lambda X + (1 - \lambda)Y$ for all $\lambda \in [0, 1]$. Hence, we have

$$\rho(\lambda X + (1 - \lambda)Y) \leqslant \rho(\lambda X^c + (1 - \lambda)Y^c) \leqslant \max\{\rho(X^c), \rho(Y^c)\} = \max\{\rho(X), \rho(Y)\},$$

which indicates that ρ is quasi-convex. $\qquad\square$

With the extra requirement of comonotonic quasi-convexity, we obtain a unifying umbrella for the representation of cash-subadditive risk measures with various properties. This result is parallel to the result of Jia et al. (2020) on monetary risk measures, where comonotonic convexity (Song and Yan, 2006, 2009) is in place of our comonotonic quasi-convexity.

Proposition A.2. *For a functional $\rho : \mathcal{X} \to \mathbb{R}$, we have the following statements.*

(i) *ρ is a cash-subadditive risk measure if and only if it is the lower envelope of a family of comonotonic quasi-convex and cash-subadditive risk measures.*

(ii) *ρ is a law-invariant cash-subadditive risk measure if and only if it is the lower envelope of a family of law-invariant, comonotonic quasi-convex and cash-subadditive risk measures.*

The equivalence (ii) holds true if "law-invariant" is replaced by "normalized and quasi-star-shaped" or "SSD-consistent".

Proof. Note that each of law invariance, normalization, quasi-star-shapedness, SSD-consistency and cash subadditivity is preserved under taking an infimum, and hence the "if" parts in all statements are obvious. Since comonotonic quasi-convexity is weaker than quasi-convexity, the representations ("only if") in Theorems 4.1, 5.1 and 5.2 hold true by replacing quasi-convexity with comonotonic quasi-convexity. This, together with Lemma 5.2, gives the "only if" parts except for the case of law-invariant cash-subadditive risk measures in (ii). Below we show this part.

Assume ρ is a law-invariant cash-subadditive risk measure. According to Proposition A.3 below, for all $X \in \mathcal{X}$, we have $\rho(X) = \min_{Z \in \mathcal{X}} \phi_Z(X)$ in which

$$\phi_Z(X) = \rho \left(Z + \sup_{t \in (0,1)} (\mathrm{VaR}_t(X) - \mathrm{VaR}_t(Z)) \right).$$

It is clear that ϕ_Z is monotone, cash subadditive and law invariant. We prove that ϕ_Z is comonotonic quasi-convex by the similar way to Theorems 4.1 and 5.2. Assume that $(X_1, X_2) \in \mathcal{X}^2$ is comonotonic and $\phi_Z(X_i) \leqslant \alpha$ for $i = 1, 2$. For all $\varepsilon > 0$ and $i = 1, 2$, there exists some $m_i \in \mathbb{R}$ such that $Z + m_i \succeq_1 X_i$ and $\rho(Z + m_i) \leqslant \phi_Z(X_i) + \varepsilon \leqslant \alpha + \varepsilon$. For all $\lambda \in [0,1]$, comonotonic additivity of VaR_t yields that

$$\mathrm{VaR}_t(\lambda X_1 + (1-\lambda)X_2) = \lambda \mathrm{VaR}_t(X_1) + (1-\lambda)\mathrm{VaR}_t(X_2) \leqslant \mathrm{VaR}_t(Z + \lambda m_1 + (1-\lambda)m_2),$$

for all $t \in (0,1)$. We thus have $Z + m \succeq_1 \lambda X_1 + (1-\lambda)X_2$. It follows that

$$\phi_Z(\lambda X_1 + (1-\lambda)X_2) \leqslant \rho(Z + \lambda m_1 + (1-\lambda)m_2) \leqslant \rho(Z + \max\{m_1, m_2\}) \leqslant \alpha + \varepsilon.$$

Since ε is arbitrary, $\phi_Z(\lambda X_1 + (1-\lambda)X_2) \leqslant \alpha$. Therefore, ϕ_Z is comonotonic quasi-convex. $\square$

A.3 Law-invariant cash-subadditive risk measures and VaR

Let $(\Omega, \mathcal{F}, P)$ be a nonatomic probability space in this section. We first connect law-invariant cash-subadditive risk measures to VaR defined in Section 2. It is well known that the class of VaR characterizes FSD via

$$X \succeq_1 Y \iff \mathrm{VaR}_t(X) \geqslant \mathrm{VaR}_t(Y) \text{ for all } t \in (0,1).$$

Proposition A.3. *If $\rho : \mathcal{X} \to \mathbb{R}$ is a risk measure, then ρ is law invariant and cash subadditive if and only if it satisfies*

$$\rho(X) = \min_{g \in \mathcal{G}_X} \sup_{t \in (0,1)} \{\mathrm{VaR}_t(X) - g(t)\}, \quad \text{for all } X \in \mathcal{X},$$

where $\mathcal{G}_X$ is a set of measurable functions from $(0,1)$ to $(-\infty, \infty]$ for all $X \in \mathcal{X}$, with $\mathcal{G}_{X_1} \subseteq \mathcal{G}_{X_2}$ for all $X_1, X_2 \in \mathcal{X}$ such that $X_2 \succeq_1 X_1$. Moreover, the set $\mathcal{G}_X$ can be chosen as

$$\mathcal{G}_X = \{g : (0,1) \to (-\infty, \infty], \ t \mapsto \mathrm{VaR}_t(Z) - \rho(Z) \mid Z \in \mathcal{X}, \ X \succeq_1 Z\}, \quad \text{for all } X \in \mathcal{X}.$$

Proof. "⇒": Suppose that ρ is a law-invariant cash-subadditive risk measure. For all $X \in \mathcal{X}$ and $Z \in \mathcal{X}$, define the risk measure

$$\phi_Z(X) = \inf\{\rho(Z + m) \mid m \in \mathbb{R}, \ Z + m \succeq_1 X\}.$$

For all $m \in \mathbb{R}$ such that $Z + m \succeq_1 X$, since any law-invariant risk measure is FSD-consistent (e.g., Föllmer and Schied, 2016, Remark 4.58), we have $\rho(Z + m) \geqslant \rho(X)$. It follows that $\phi_Z(X) \geqslant \rho(X)$ for all $Z \in \mathcal{X}$. Noting that $\phi_X(X) = \rho(X)$, we have $\rho(X) = \min_{Z \in \mathcal{X}} \phi_Z(X)$. By definition of ϕ_Z, we have

$$\phi_Z(X) = \inf\left\{\rho(Z + m) \mid m \in \mathbb{R}, \ \mathrm{VaR}_t(Z) + m \geqslant \mathrm{VaR}_t(X) \text{ for all } t \in (0,1)\right\}$$

$$= \rho\left(Z + \sup_{t \in (0,1)} (\mathrm{VaR}_t(X) - \mathrm{VaR}_t(Z))\right).$$

Further, we have

$$\rho(X) = \min_{Z \in \mathcal{X}, \ X \succeq_1 Z} \sup_{t \in (0,1)} \rho(Z + \mathrm{VaR}_t(X) - \mathrm{VaR}_t(Z))$$

$$\leqslant \min_{Z \in \mathcal{X}, \ X \succeq_1 Z} \sup_{t \in (0,1)} \{\mathrm{VaR}_t(X) - \mathrm{VaR}_t(Z) + \rho(Z)\} \leqslant \rho(X).$$

It follows that

$$\rho(X) = \min_{Z \in \mathcal{X}, \ X \succeq_1 Z} \sup_{t \in (0,1)} \{\mathrm{VaR}_t(X) - g_Z(t)\},$$

where $g_Z(t) = \mathrm{VaR}_t(Z) - \rho(Z)$ for all $t \in (0,1)$. Therefore, $\{g_Z \mid Z \in \mathcal{X}, \ X \succeq_1 Z\}$ is a desired family of measurable functions on $(0,1)$.

"⇐": We first show that ρ is cash subadditive. Indeed, for all $X \in \mathcal{X}$ and $m \geqslant 0$, we have $X + m \succeq_1 X$. Hence, $\mathcal{G}_X \subseteq \mathcal{G}_{X+m}$ and

$$\rho(X + m) = \min_{g \in \mathcal{G}_{X+m}} \sup_{t \in (0,1)} \{\mathrm{VaR}_t(X) - g(t)\} + m$$

$$\leqslant \min_{g \in \mathcal{G}_X} \sup_{t \in (0,1)} \{\mathrm{VaR}_t(X) - g(t)\} + m = \rho(X) + m.$$

To show law invariance of ρ, for all $X, Y \in \mathcal{X}$ such that $X \stackrel{\mathrm{d}}{=} Y$, we have $X \succeq_1 Y$ and $Y \succeq_1 X$. It follows that $\mathcal{G}_X = \mathcal{G}_Y$ and thus ρ is law invariant. $\square$

Remark A.1. Although the functional

$$\phi_Z(X) = \inf\{\rho(Z + m) \mid m \in \mathbb{R}, \ Z + m \succeq_1 X\}, \quad X \in \mathcal{X},$$

defined in the proof of Proposition A.3 is monotone, cash subadditive and law invariant, ϕ_Z is not quasi-convex. This is because VaR does not satisfy quasi-convexity.

A.4 Certainty equivalents of α-maxmin expected utility

We assume $(\Omega, \mathcal{F}, P)$ be a nonatomic probability space. The rank-dependent expected utility (RDEU) of Quiggin (1982) is a popular behavioral decision model specified by the preference functional

$$\int_\Omega \ell(X)\, \mathrm{d}T \circ P, \quad X \in \mathcal{X},$$

where $\ell : \mathbb{R} \to \mathbb{R}$ is a strictly increasing and convex loss function (positive random variables represent losses), $T : [0,1] \to [0,1]$ is a probability distortion function, and the integral with respect to $T \circ P$ is a Choquet integral (Choquet (1954), Schmeidler (1986, 1989)). We consider the choice of T given by $T = \alpha T_1 + (1-\alpha)T_2$ where T_1 (resp. T_2) are increasing, differentiable and convex (resp. concave) probability distortion functions with $T_1(0) = T_2(0) = 0$ and $T_1(1) = T_2(1) = 1$. This corresponds to the well known α-maximin model of Marinacci (2002) and Ghirardato et al. (2004) (see Example 2.3), with the interpretation of balancing between optimistic and pessimistic views on ambiguity. Following Carlier and Dana (2003), for an increasing, differentiable and convex distortion function $h : [0,1] \to [0,1]$ with $h(0) = 0$ and $h(1) = 1$, define the core of $h \circ P$ by

$$\mathrm{core}(h \circ P) = \{Q \in \mathcal{M} \mid Q(A) \geqslant h(P(A)) \text{ for all } A \in \mathcal{F}\}.$$

Continuity of h guarantees that any element in the core of $h \circ P$ is a probability measure absolutely continuous with respect to P, which may be identified with its density with respect to P. Thus, we have

$$\int_\Omega \ell(X)\, \mathrm{d}T \circ P = \alpha \min_{Q_1 \in \mathrm{core}(T_1 \circ P)} \mathbb{E}_{Q_1}[\ell(X)] + (1-\alpha) \max_{Q_2 \in \mathrm{core}(\widehat{T}_2 \circ P)} \mathbb{E}_{Q_2}[\ell(X)], \tag{A.1}$$

where $\widehat{T}_2 : x \mapsto 1 - T_2(1-x)$. The certainty equivalent of the RDEU with an ambiguous discount factor λ is given by

$$\begin{aligned}
\rho(X) &= \sup_{\lambda \in I}\ \ell^{-1}\left(\int_\Omega \ell(\lambda X)\, \mathrm{d}T \circ P\right) \\
&= \min_{Q_1 \in \mathrm{core}(T_1 \circ P)} \max_{Q_2 \in \mathrm{core}(\widehat{T}_2 \circ P)} \sup_{\lambda \in I}\ \ell^{-1}\left(\alpha \mathbb{E}_{Q_1}[\ell(\lambda X)] + (1-\alpha)\mathbb{E}_{Q_2}[\ell(\lambda X)]\right),
\end{aligned} \tag{A.2}$$

where $I \subseteq [0,1]$ is the ambiguity set. For technical tractability, we assume that the discount factor is deterministic here. It is clear that if we take the loss function to be $\ell : x \mapsto \mathrm{e}^{\gamma X}$ for $\gamma > 0$, then ρ is a cash-subadditive risk measure, while ρ becomes a monetary risk measure without ambiguity of the discount factor λ.

Note that for all $\lambda \in I$, $Q_1 \in \mathrm{core}(T_1 \circ P)$ and $Q_2 \in \mathrm{core}(\widehat{T}_2 \circ P)$, the mapping

$$X \mapsto \ell^{-1}\left(\alpha \mathbb{E}_{Q_1}[\ell(\lambda X)] + (1-\alpha)\mathbb{E}_{Q_2}[\ell(\lambda X)]\right)$$

is quasi-convex and upper semi-continuous. Proposition 5.3 of Cerreia-Vioglio et al. (2011) showed an explicit representation of the certainty equivalent of the expected loss given by $\ell^{-1}(\mathbb{E}_P[\ell(\cdot)])$. In the proposition below, we show the representation result of a more general ρ in a similar sense.

Define $\bar{\ell} : [-\infty, \infty] \to [-\infty, \infty]$ as the extended-valued function with inverse function given by

$$\bar{\ell}^{-1}(x) = \begin{cases} \ell^{-1}(x), & x \in (\inf_{t \in \mathbb{R}} \ell(t), \infty), \\ -\infty, & x \in [-\infty, \inf_{t \in \mathbb{R}} \ell(t)], \\ \infty, & x = \infty. \end{cases}$$

Let $\ell^* : [-\infty, \infty] \to [-\infty, \infty]$ be the conjugate function of $\bar{\ell}$ given by

$$\ell^*(x) = \sup_{y \in [-\infty, \infty]} \{xy - \bar{\ell}(y)\}, \quad x \in [-\infty, \infty].$$

Proposition A.4. *Let $\widetilde{Q} = \alpha Q_1 + (1 - \alpha) Q_2$ for $Q_1 \in \mathrm{core}(T_1 \circ P)$ and $Q_2 \in \mathrm{core}(\widehat{T}_2 \circ P)$. For $X \in \mathcal{X}$, the risk measure ρ in (A.2) adopts the following representation:*

$$\rho(X) = \min_{Q_1 \in \mathrm{core}(T_1 \circ P)} \max_{Q_2 \in \mathrm{core}(\widehat{T}_2 \circ P)} \sup_{\lambda \in I} \max_{Q \in \mathcal{M}} R\left(\mathbb{E}_Q[X], \lambda, Q, \widetilde{Q}\right),$$

where

$$R\left(t, \lambda, Q, \widetilde{Q}\right) = \ell^{-1}\left(\max_{x \geqslant 0} \left[\lambda x t - \mathbb{E}_{\widetilde{Q}}\left(\ell^*\left(x \frac{\mathrm{d}Q}{\mathrm{d}\widetilde{Q}}\right)\right)\right]\right), \quad (t, \lambda, Q, \widetilde{Q}) \in \mathbb{R} \times I \times \mathcal{M} \times \mathcal{M}. \quad \text{(A.3)}$$

Proof. For $X \in \mathcal{X}$, $\lambda \in I$, $Q_1 \in \mathrm{core}(T_1 \circ P)$ and $Q_2 \in \mathrm{core}(\widehat{T}_2 \circ P)$, we have

$$\ell^{-1}\left(\alpha \mathbb{E}_{Q_1}[\ell(\lambda X)] + (1 - \alpha)\mathbb{E}_{Q_2}[\ell(\lambda X)]\right) = \ell^{-1}\left(\mathbb{E}_{\widetilde{Q}}[\ell(\lambda X)]\right).$$

Proposition 5.3 of Cerreia-Vioglio et al. (2011) gives

$$\ell^{-1}\left(\mathbb{E}_{\widetilde{Q}}[\ell(\lambda X)]\right) = \max_{Q \in \mathcal{M}} R(\mathbb{E}_Q[X], \lambda, Q, \widetilde{Q}),$$

where $R(t, \lambda, Q, \widetilde{Q})$ is given by (A.3). This and (A.2) yield the desired representation. $\square$

Proposition A.4 gives, for the α-maxmin risk measure in this section, the explicit representation in the form of Theorem 4.1.

A.5   A counter-example

Example A.1 ($\Lambda\mathrm{VaR}$ is not quasi-star-shaped)**.** For $0 < \alpha < 1/2 < \beta < 1$, consider the increasing function $\Lambda(x) = \alpha \mathbb{1}_{\{x \leqslant 1/2\}} + \beta \mathbb{1}_{\{x > 1/2\}}$, $x \in \mathbb{R}$. For $t = 7/4$, $\lambda = 1/8$, and a Bernoulli random loss X given by $P(X = 2) = P(X = 0) = 1/2$, we have $P(\lambda X + (1 - \lambda)t = 57/32) = P(\lambda X + (1 - \lambda)t = 49/32) = 1/2$. Hence, $\Lambda\mathrm{VaR}(X) = \mathrm{VaR}_\alpha(X) = 0$, $\Lambda\mathrm{VaR}(\lambda X + (1 - \lambda)t) = \mathrm{VaR}_\beta(\lambda X + (1 - \lambda)t) = 57/32$, and $\Lambda\mathrm{VaR}(t) = 7/4$. It follows that $\Lambda\mathrm{VaR}(\lambda X + (1 - \lambda)t) > \max\{\Lambda\mathrm{VaR}(X), \Lambda\mathrm{VaR}(t)\}$ and $\Lambda\mathrm{VaR}$ is not quasi-star-shaped.