Recent Advances in Risk Aggregation and Dependence Uncertainty

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Mainly based on some joint work with

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- Paul Embrechts (Zurich)
- Andreas Tsanakas (London)
- Bin Wang (Beijing)

Outline

1. The Question
2. Mixability
3. Risk Aggregation under Uncertainty
4. Challenges
5. References

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Risk aggregation

Two aspects of modeling and inference of a multivariate model:

marginal distribution and **dependence structure**.

"*copula thinking*"

- Assumption: certain margins, uncertain dependence.
- A common setup in operational risk

For example,

$$S_n = X_1 + \cdots + X_n.$$

X_i: individual risks; S_n: risk aggregation

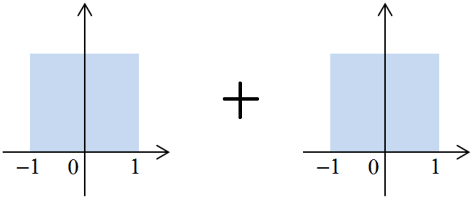
Key question

What are possible distributions of S_n?

Simple example

The simplest case: $n = 2$, $F_1 = F_2 = \mathrm{U}[-1, 1]$.

What is a possible distribution of $S_2 = X_1 + X_2$?



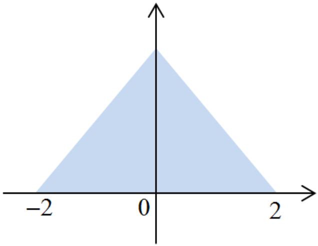
Obvious constraints

- $\mathbb{E}[S_2] = 0$
- range of S_2 in $[-2, 2]$
- $\mathrm{Var}(S_2) \le 4/3$

- In fact, $S_2 \prec_{\mathrm{cx}} 2X_1$ i.e. $S_2 \overset{d}{=} 2\mathbb{E}[X_1|\mathcal{G}]$ (sufficient?)

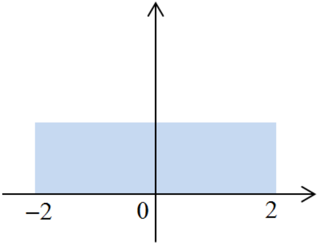
Uniform example I

Is the following distribution possible for S_2?



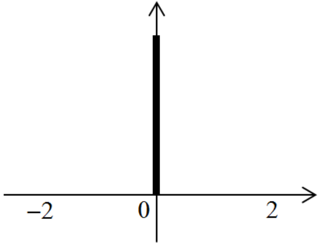
Uniform example II

Is the following distribution possible for S_2?



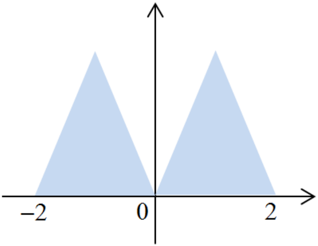
Uniform example III

Is the following distribution possible for S_2?



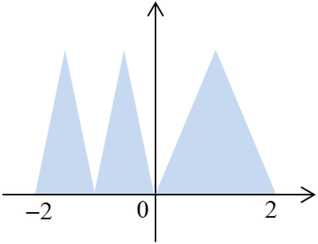
Uniform example IV

Is the following distribution possible for S_2?



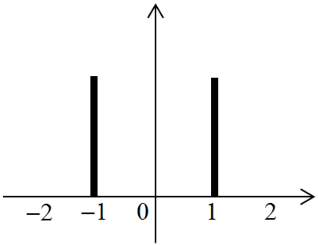
Uniform example V

Is the following distribution possible for S_2?



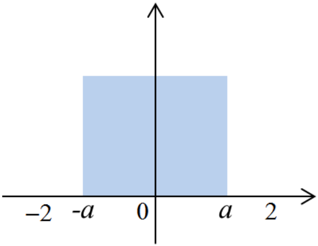
Uniform example VI

Is the following distribution possible for S_2?



Uniform example VII

Is the following distribution possible for S_2?

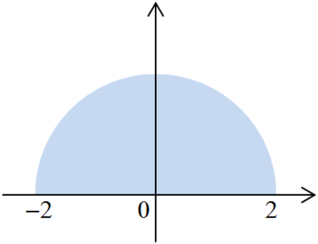


This it not trivial any more[1].

[1] the case $[-1,1]$ obtained in Rüschendorf (1982); general case $[-a,a]$ obtained in Wang-W. (2015+ MOR)

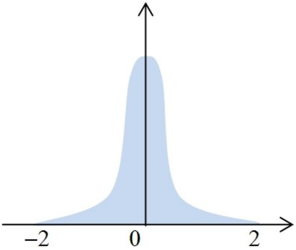
Uniform example VIII

Is the following distribution possible for S_2?



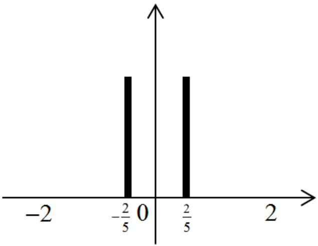
Uniform example IX

Is the following distribution possible for S_2?



Uniform example X

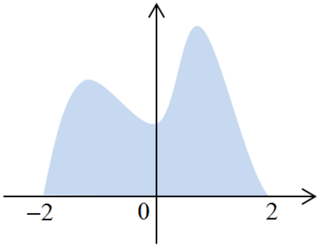
Is the following distribution possible for S_2?



Example 3.3 of Mao-W. (2015 JMVA)

Uniform example XI

Is the following distribution possible for S_2?



Aggregation set

Denote the aggregation set

$$\mathcal{D}_n = \mathcal{D}_n(F_1, \ldots, F_n) = \{\text{cdf of } S_n | X_i \sim F_i,\ i = 1, \ldots, n\}.$$

- $\mathcal{D}_n$ is a convex set, and closed with respect to weak convergence.

Aggregation set

Some questions to ask:

- **(Compatibility)** For a given F, is $F \in \mathcal{D}_n$?
- **(Mimicking)** What is the best approximation in $\mathcal{D}_n$ to F? That is, find $G \in \mathcal{D}_n$ such that $d(F, G)$ is minimized for some metric d.
- **(Extreme values)** What is $\sup_{F \in \mathcal{D}_n} \rho(F)$ for some functional ρ? $\leftarrow$ risk aggregation with dependence uncertainty

Other applications

Many applications and related problems

- Simulation: variance reduction
- Model-independent option pricing
- (Multi-dimensional) Monge-Kantorovich optimal transportation
- Change of measure
- Decision making
- Assembly and scheduling[2]

Many natural questions are not related to statistical uncertainty

[2] traditional problem in OR: e.g. Coffman-Yannakakis (1984 MOR)

Aggregation of risk measures

Attention coming from **Quantitative Risk Management**

- Most research looks at extreme values of some quantities (e.g. risk measures, pricing function) on the aggregate position S_n:

$$\sup\{\rho(S_n) : F_{S_n} \in \mathcal{D}\} \quad \text{and} \quad \inf\{\rho(S_n) : F_{S_n} \in \mathcal{D}\}$$

  where $\mathcal{D}$ is typically a subset of $\mathcal{D}_n$.

Earlier research:

- VaR: Embrechts-Puccetti (2006 F&S)
  Distribution functions: Makarov (1981 TPA), Rüschendorf (1982 JAP)

Risk aggregation and dependence uncertainty

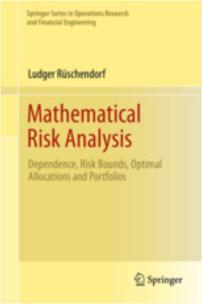
An active field for the past few years:

- Some recent papers (many more not listed)
  - W.-Peng-Yang (2013 F&S)
  - Embrechts-Puccetti-Rüschendorf (2013 JBF)
  - Bernard-Jiang-W. (2014 IME)
  - Aas-Puccetti (2014 Extremes)
  - Embrechts-Wang-W. (2015 F&S)
  - W.-Bignozzi-Tsanakas (2015 SIFIN)
  - Bignozzi-Puccetti-Rüschendorf (2015 IME)
  - Bernard-Vanduffel (2015 JBF)
  - Bernard-Vanduffel-Rüschendorf (2015+ JRI)
  - Wang-W. (2015+ MOR)

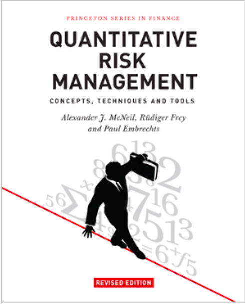
Risk aggregation and dependence uncertainty

Books covering topics in this field:

Rüschendorf (2013)



McNeil-Frey-Embrechts (2015)



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Mixability

Observe that

$$S = X_1 + \cdots + X_n \;\Leftrightarrow\; X_1 + \cdots + X_n - S = 0$$

Hence,

$$F_S \in \mathcal{D}_n(F_1, \ldots, F_n) \;\Leftrightarrow\; \delta_0 \in \mathcal{D}_{n+1}(F_1, \ldots, F_n, F_{-S}).$$

To answer

is a distribution in $\mathcal{D}_n$, $n \geq 2$?

We study

is a point-mass in $\mathcal{D}_{n+1}$, $n \geq 2$?

Joint mixability

Joint mix

A random vector $(X_1, \ldots, X_n)$ is a joint mix if $X_1 + \cdots + X_n$ is a constant.

- Example: a multinomial random vector

Joint mixability

Joint mixability (W.-Peng-Yang, 2013 F&S)

An n-tuple of univariate distributions $(F_1, \ldots, F_n)$ is jointly mixable (JM) if there exists a joint mix with marginal distributions $(F_1, \ldots, F_n)$.

- Equivalently, $\mathcal{D}_n(F_1, \ldots, F_n)$ contains a point-mass.
- This concerned point-mass can be chosen at the sum of the means of $F_1, \ldots, F_n$ whenever it is finite.
- We say a univariate distribution F is n-completely mixabe (n-CM) if exists an n-dimensional joint mix with identical marginal distributions F.

Mixability

An open research area:

what distributions are CM/JM?

The research in this area is very much marginal-dependent - copula techniques do not help much!

- Recent summary paper: Puccetti-W. (2015 StS)

Mean condition

Let $\mu_i, a_i, b_i \in \mathbb{R}$ be respectively the mean, essential infimum, and essential supremum of the support of F_i;

$l = \max\{b_i - a_i : i = 1, \dots, n\}$.

Mean condition

If $(F_1, \dots, F_n)$ is JM, then

$$\sum_{i=1}^n a_i + l \le \sum_{i=1}^n \mu_i \le \sum_{i=1}^n b_i - l \quad (1)$$

Sufficiency of mean condition

Sufficiency:

Theorem 1 (Wang-W., 2015+ MOR)

The mean condition (1) is sufficient for a tuple of distributions with increasing (decreasing) densities to be JM.

- The homogeneous case is shown in Wang-W. (2011 JMVA).
- Corollary: $(\mathrm{U}[0, a_1], \ldots, \mathrm{U}[0, a_n])$ is JM if and only if

$$\max_{i=1,\ldots,n} a_i \leq \frac{1}{2} \sum_{i=1}^{n} a_i.$$

- In particular[3]: $\mathrm{U}[0, 1]$ is n-CM for $n \geq 2$.

[3] known in Rüschendorf (1982 JAP)

Variance condition

Another necessary condition:

Variance condition

If $(F_1, \dots, F_n)$ is JM with finite variance $\sigma_1^2, \dots, \sigma_n^2$, then

$$\max_{i=1,\dots,n} \sigma_i \le \frac{1}{2} \sum_{i=1}^{n} \sigma_i. \quad (2)$$

(A polygon inequality[4].)

[4] the standard deviation can be replaced by any law-based central norm

Sufficiency of variance condition

Theorem 2 (Wang-W., 2015+ MOR)

The variance condition (2) is sufficient for the joint mixability of

(i) *a tuple of uniform distributions,*

(ii) *a tuple of marginal distributions of a multivariate elliptical distribution,*

(iii) *a tuple of distributions with unimodal-symmetric densities in the same location-scale family.*

Joint mixability

Theorem 3 (Wang-W., 2015+ MOR)

Suppose that F has a unimodal-symmetric density. For $a > 0$, $(\mathrm{U}[0,a], \mathrm{U}[0,a], F)$ is JM if and only if F is supported in an interval of length at most $2a$.

Joint mixability

Some remarks:

- Determination of JM is still open
- 12 open questions on mixability: W. (2015 PS)
- Determination of JM in discrete setting is NP-complete[5].

[5] see Haus (2015 ORL)

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Risk Aggregation under Uncertainty

To study aggregation sets $\mathcal{D}_n$:

- To measure model uncertainty for quantities (e.g. risk measures, moments, etc) of S_n.
- Targets:

$$\sup_{F_S \in \mathcal{D}_n} \rho(S) \quad \text{and} \quad \inf_{F_S \in \mathcal{D}_n} \rho(S) \tag{3}$$

where $\rho : \mathcal{X} \to \mathbb{R}$ is a risk measure[6].

[6] ρ is law-determined; $\mathcal{X}$ is a set of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$

VaR and ES

Two regulatory risk measures

Value-at-Risk VaR_p

For $p \in (0,1)$,

$$\mathrm{VaR}_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbb{R} : F_X(x) \geq p\}$$

Expected Shortfall ES_p

For $p \in (0,1)$,

$$\mathrm{ES}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q(X)\mathrm{d}q \underset{(F \text{ cont.})}{=} \mathbb{E}\left[X|X > \mathrm{VaR}_p(X)\right]$$

Worst- and best-values of VaR and ES

The Fréchet (unconstrained) problems for VaR_p: For given $F_1, \ldots, F_n$ with finite means, and $p \in (0,1)$, let

$$\overline{\mathrm{VaR}}_p(n) = \sup\{\mathrm{VaR}_p(S) : F_S \in \mathcal{D}_n(F_1, \ldots, F_d)\},$$

$$\underline{\mathrm{VaR}}_p(n) = \inf\{\mathrm{VaR}_p(S) : F_S \in \mathcal{D}_n(F_1, \ldots, F_d)\}.$$

Same notation for ES_p.

Worst- and best-values of VaR and ES

Uncertainty intervals

$$[\underline{\mathrm{VaR}}_p(n), \overline{\mathrm{VaR}}_p(n)], \quad [\underline{\mathrm{ES}}_p(n), \overline{\mathrm{ES}}_p(n)]$$

- ES is subadditive: $\overline{\mathrm{ES}}_p(n) = \sum_{i=1}^{n} \mathrm{ES}_p(X_i)$.
- $\overline{\mathrm{VaR}}_p(n)$, $\underline{\mathrm{VaR}}_p(n)$ and $\underline{\mathrm{ES}}_p(n)$: generally open questions

Challenge for $\underline{\mathrm{ES}}_p(n)$

To calculate $\underline{\mathrm{ES}}_p(n)$ one naturally seeks a safest risk in $\mathcal{D}_n$.

Mathematical difficulty

Common understanding of the most dangerous scenario:

- Comonotonicity - well accepted notion; even for a collection of random vectors

Understanding concerning the safest scenario:

- $n = 2$: counter-monotonicity
- $n \geq 3$: unclear
  - Calls for notions of extremal negative dependence.

Summary of existing results

$n = 2$: (based on counter-comonotonicity)

- fully solved analytically[7]

$n \geq 3$: (based on joint mixability)

- $\underline{\mathrm{ES}}_p(n)$ solved analytically for decreasing densities, e.g. Pareto, Exponential
- $\overline{\mathrm{VaR}}_p(n)$ solved analytically for tail-decreasing densities, e.g. Pareto, Gamma, Log-normal[8]
- $\underline{\mathrm{VaR}}_p(n)$ similar to $\overline{\mathrm{VaR}}_p(n)$

[7] Makarov (1981 TPA) and Rüschendorf (1982 JAP)

[8] homogeneous model: W.-Yang-Peng (2013 F&S); inhomogeneous model: Jakobsons-Han-W. (2015+ SAJ)

Remarks

Remarks:

- For general marginal distributions the problem is still open
- Numerical methods: Rearrangement Algorithm[9]

[9] Puccetti-Rüschendorf (2012 JCAM); Embrechts-Puccetti-Rüschendorf (2013 JBF), Hofert-Memartoluie-Saunders-Wirjanto (2015+ arXiv)

Aggregation of risk measures

Let $\mathcal{D}_n(F) = \mathcal{D}_n(F, \dots, F)$ (homogeneous model).

For a law-determined risk measure ρ, define

$$\Gamma_\rho(X) = \lim_{n\to\infty} \frac{1}{n} \sup\left\{\rho(S) : F_S \in \mathcal{D}_n(F_X)\right\}.$$

Γ_ρ is also a law-determined risk measure which inherits some properties of ρ.

Aggregation of risk measures

Distortion risk measures:

$$\rho_h(X) = \int_0^1 F_X^{-1}(t)\mathrm{d}h(t), \quad X \in \mathcal{X} = L^\infty$$

h is the distortion function: a probability measure on $(0,1)$.

- ES and VaR are special cases

Theorem 4 (W.-Bignozzi-Tsanakas, 2015 SIFIN)

We have

$$\Gamma_{\rho_h}(X) = \rho_{h^*}(X), \quad X \in \mathcal{X},$$

where h^* is the largest convex distortion function dominated by h.

Aggregation of risk measures

For distortion risk measures

- Example: $\Gamma_{\mathrm{VaR}_p} = \mathrm{ES}_p$
- ρ_h is coherent if and only if $h^* = h$
- Application: when arbitrary dependence is allowed, the worst-case VaR_p of a portfolio behaves like the worst-case ES_p

For law-determined convex risk measures.

- Γ_ρ is the smallest coherent risk measure dominating ρ
- If ρ is a convex shortfall risk measure, then Γ_ρ is a coherent expectile

Dependence-uncertainty spread

Theorem 5 (Embrechts-Wang-W., 2015 F&S)

Take $1 > q \geq p > 0$. Under weak regularity conditions, for inhomogeneous models,

$$\liminf_{n\to\infty} \frac{\overline{\mathrm{VaR}}_q(n) - \underline{\mathrm{VaR}}_q(n)}{\overline{\mathrm{ES}}_p(n) - \underline{\mathrm{ES}}_p(n)} \geq 1.$$

- The uncertainty-spread of VaR is generally bigger than that of ES.
- In recent Consultative Documents of the Basel Committee, $\mathrm{VaR}_{0.99}$ is compared with $\mathrm{ES}_{0.975}$: $p = 0.975$ and $q = 0.99$.

Dependence-uncertainty spread

ES and VaR of $S_n = X_1 + \cdots + X_n$, where

- $X_i \sim \mathrm{Pareto}(2 + 0.1i)$, $i = 1, \ldots, 5$;
- $X_i \sim \mathrm{Exp}(i - 5)$, $i = 6, \ldots, 10$;
- $X_i \sim \mathrm{Log\text{–}Normal}(0, (0.1(i - 10))^2)$, $i = 11, \ldots, 20$.

	$n = 5$			$n = 20$		
	best	worst	spread	best	worst	spread
$\mathrm{ES}_{0.975}$	22.48	44.88	22.40	29.15	102.35	73.20
$\mathrm{VaR}_{0.975}$	9.79	41.46	31.67	21.44	100.65	79.21
$\mathrm{VaR}_{0.99}$	12.96	62.01	49.05	22.29	136.30	114.01
$\frac{\overline{\mathrm{ES}}_{0.975}}{\overline{\mathrm{VaR}}_{0.975}}$		1.08			1.02	

Dependence-uncertainty spread

Features/Risk measure	VaR	Tail-VaR
Frequency captured?	Yes	Yes
Severity captured?	No	Yes
Sub-additive?	Not always	Always
Diversification captured?	Issues	Yes
Back-testing?	Straight-forward	Issues
Estimation?	Feasible	Issues with data limitation
Model uncertainty?	Sensitive to aggregation	Sensitive to tail modelling
Robustness I (with respect to "Lévy metric[33]")?	Almost, only minor issues	No
Robustness II (with respect to "Wasserstein metric[34]")?	Yes	Yes

From the International Association of Insurance Supervisors Consultation Document (December 2014).

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Open questions

Concrete mathematical questions:

- Full characterization of $\mathcal{D}_n$ and mixability
- Existence and determination of smallest $\prec_{\mathrm{cx}}$-element in $\mathcal{D}_n$
- General analytical formulas for $\overline{\mathrm{VaR}}_p$ ($\underline{\mathrm{VaR}}_p$) and $\underline{\mathrm{ES}}_p$
- Aggregation of random vectors

Practical questions:

- Capital calculation under uncertainty
- Robust decision making under uncertainty
- Regulation with uncertainty

Other directions

Some on-going directions on RADU

- Partial information on dependence[10]
- Connection with Extreme Value Theory
- Connection with martingale optimal transportation
- Both marginal and dependence uncertainty
- Computational solutions
- Other aggregation functionals

[10] Bignozzi-Puccetti-Rüschendorf (2015 IME), Bernard-Rüschendorf-Vanduffel (2015+ JRI), Bernard-Denuit-Vanduffel (2014 SSRN), Bernard-Vanduffel (2015 JBF), many more

References I

Aas, K. and G. Puccetti (2014). Bounds for total economic capital: the DNB case study. *Extremes*, **17**(4), 693–715.

Bernard C., Denuit, M. and Vanduffel S.. (2013). Measuring portfolio risk under partial dependence information. *Preprint*, Free University of Brussels.

Bernard, C., Jiang, X. and Wang, R. (2014). Risk aggregation with dependence uncertainty. *Insurance: Mathematics and Economics*, **54**, 93–108.

Bernard C., Rüschendorf L., and Vanduffel S.. (2015+). Value-at-risk bounds with variance constraints. *Journal of Risk and Insurance*, forthcoming.

Bernard, C. and Vanduffel, S. (2015). A new approach to assessing model risk in high dimensions. *Journal of Banking and Finance*, **58**, 166–178.

Bignozzi, V., Puccetti, G. and Rüschendorf, L. (2015). Reducing model risk via positive and negative dependence assumptions. *Insurance: Mathematics and Economics*, **61**, 17–26.

References II

- Coffman, E. G. and Yannakakis, M. (1984). Permuting elements within columns of a matrix in order to minimize maximum row sum. *Mathematics of Operations Research*, **9**(3), 384–390.
- Embrechts, P., Puccetti, G. and Rüschendorf, L. (2013). Model uncertainty and VaR aggregation. *Journal of Banking and Finance*, **37**(8), 2750-2764.
- Embrechts, P., Puccetti, G., Rüschendorf, L., Wang, R. and Beleraj, A. (2014). An academic response to Basel 3.5. *Risks*, **2**(1), 25–48.
- Embrechts, P., Wang, B. and Wang, R. (2015). Aggregation-robustness and model uncertainty of regulatory risk measures. *Finance and Stochastics*, forthcoming.
- Haus, U. (2015). Bounding stochastic dependence, complete mixability of matrices, and multidimensional bottleneck assignment problems. *Operations Research Letters*, **43**(1), 74–79.

References III

- Hofert, M., Memartoluie, A., Saunders, D, and Wirjanto, T. (2015) Improved algorithms for computing worst Value-at-Risk: numerical challenges and the Adaptive Rearrangement Algorithm. *Preprint on arXiv*:1505.02281.
- Jakobsons, E., Han, X. and Wang, R. (2015+). General convex order on risk aggregation. *Scandinavian Actuarial Journal*, forthcoming.
- McNeil, A. J., Frey, R. and Embrechts, P. (2015). *Quantitative Risk Management: Concepts, Techniques and Tools*. Revised Edition. Princeton, NJ: Princeton University Press.
- Puccetti, G. and Rüschendorf, L. (2012). Computation of sharp bounds on the distribution of a function of dependent risks. *Journal of Computational and Applied Mathematics*, **236**(7), 1833–1840.
- Puccetti, G. and Wang, R. (2015). Extremal dependence concepts. *Statistical Science*, **30**(4), 485–517.
- Rüschendorf, L. (2013). *Mathematical risk analysis. Dependence, risk bounds, optimal allocations and portfolios*. Springer, Heidelberg.

References IV

- Wang, B. and Wang, R. (2011). The complete mixability and convex minimization problems with monotone marginal densities. *Journal of Multivariate Analysis*, **102**, 1344–1360.
- Wang, B. and Wang, R. (2015). Extreme negative dependence and risk aggregation. *Journal of Multivariate Analysis*, **136**, 12–25.
- Wang, B. and Wang, R. (2015+). Joint mixability. *Mathematics of Operations Research*, forthcoming.
- Wang, R. (2015). Current open questions in complete mixability. *Probability Surveys*, **12**, 13–32.
- Wang, R., Bignozzi, V. and Tsakanas, A. (2015). How superadditive can a risk measure be? *SIAM Journal on Financial Mathematics*, **6**(1), 776–803.
- Wang, R., Peng, L. and Yang, J. (2013). Bounds for the sum of dependent risks and worst Value-at-Risk with monotone marginal densities. *Finance and Stochastics*, **17**(2), 395–417.

Danke

Thank you for your kind attention