Risk Aversion in Regulatory Capital Principles

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Outline

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Regulatory Capital Principles

Risk measures as regulatory capital principles

A (regulatory) risk measure is a functional $\rho : \mathcal{X} \to (-\infty, \infty]$ which calculates the amount of regulatory capital of a financial institution taking a risk (random loss) X in a fixed period.

- $(\Omega, \mathcal{F}, \mathbb{P})$ is an atomless probability space
- $\mathcal{X}$ is a convex cone of random variables
  - e.g. $\mathcal{X} = L^q(\Omega, \mathcal{F}, \mathbb{P})$, $q \in [1, \infty]$
- $X \in \mathcal{X}$ represent loss/profit (discounted to present)

Very general question

What is a good risk measure to use?

Regulatory Capital Principles

	regulator	firm manager
usage	external regulation	internal management performance analysis capital allocation
interest	social wellfare	shareholders
risk	systemic risk	risk of a single firm
role	designs a principle	reacts to a principle
goal	maintain enough capital	reduce regulatory capital
risk-averse	yes	not necessarily

Value-at-Risk and Expected Shortfall

Value-at-Risk (VaR) at level $p \in (0,1)$

$\mathrm{VaR}_p : L^0 \to \mathbb{R}$,

$$\mathrm{VaR}_p(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) \ge p\}.$$

Expected Shortfall (ES/TVaR/CVaR/AVaR) at level $p \in (0,1)$

$\mathrm{ES}_\beta : L^1 \to \mathbb{R}$,

$$\mathrm{ES}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q(X)\mathrm{d}q, \quad p \in (0,1).$$

Value-at-Risk and Expected Shortfall

The ongoing debate on "VaR versus ES":

- Basel III (mixed; in transition from VaR to ES as standard metric for market risk[1])
- Solvency II (VaR based)
- Swiss Solvency Test (ES based)

[1] e.g. **Basel Committee on Banking Supervision**: Standards, January 2016, Minimum capital requirements for Market Risk.

Value-at-Risk and Expected Shortfall

Many perspectives

- regulator's versus firms' standpoints
- economic interpretation
- statistical issues: estimation, robustness, backtesting, model uncertainty
- computation, simulation and optimization
- systemic risk
- There is no single "perfect" risk measure

Some academic references

- Embrechts et al. (2014)
- Emmer-Kratz-Tasche (2015)

Value-at-Risk and Expected Shortfall

We provide a new perspective: incorporating risk aversion to the above issue on risk measures.

Standard Properties of Risk Measures

Some standard properties of risk measures

(M) Monotonicity: $\rho(X) \le \rho(Y)$ for $X, Y \in \mathcal{X}$, $X \le Y$ almost surely;

(TI) Translation-invariance: $\rho(X - m) = \rho(X) - m$ for all $m \in \mathbb{R}$ and $X \in \mathcal{X}$.

(LI) Law-invariance: $\rho(X) = \rho(Y)$ if $X, Y \in \mathcal{X}$ and $X \overset{\mathrm{d}}{=} Y$.

Definition 1

A monetary risk measure is a functional on $\mathcal{X}$ satisfying (M) and (TI).

- VaR and ES are monetary and law-invariant.

Progress of the Talk

1. Regulatory capital principles
2. Risk measures in financial decisions: an example
3. Consistent risk measures
4. Mathematical Characterization
5. Risk sharing
6. Discussions
7. References

Simple Example

A simplified example:

- $\Omega = \{\omega_1, \omega_2, \omega_3\}$: future (e.g. one-year) economic states
  - ω_1: a normal economic state
  - ω_2: an adverse economic state
  - ω_3: an extreme scenario
- $\mathbb{P}(\{\omega_1\}) = 0.99$, $\mathbb{P}(\{\omega_2\}) = 0.0099$ and $\mathbb{P}(\{\omega_3\}) = 0.0001$
- A financial institution has to choose between two risks (decisions)

Simple Example

Risks X and Y (in millions of USD):

$$X = \begin{cases} -1 & \omega = \omega_1, \\ 10 & \omega = \omega_2, \\ 20 & \omega = \omega_3, \end{cases} \qquad Y = \begin{cases} -1.1 & \omega = \omega_1, \\ 9.9 & \omega = \omega_2, \\ 2,000 & \omega = \omega_3. \end{cases}$$

- Possible interpretations:
  - X is benchmark - Y is X plus an bet against event ω_3 (e.g. AAA bond with high leverage)
  - Y is benchmark - X is Y plus a hedge against event ω_3 (e.g. insurance contract)
- $\mathbb{P}(Y < X) = 99.99\%$

Simple Example

- Assume that the financial institution has 10M (economic) capital
    - $\mathrm{VaR}_{0.999}(X) = 10$, $\mathrm{VaR}_{0.999}(Y) = 9.9$
- Which risk would the financial institution prefer?
    - The manager of the financial institution is not necessarily risk averse
    - Limited liability
    - $\mathbb{P}(\omega_3)$ is too small to notice or accurately model
- Which risk would a regulator prefer?
    - A regulator cares about loss to the society
    - What if all firms in the system are doing this? ... Aggregation!

Financial Decisions and Risk Preference

Question

How can the regulator leads/encourages the financial institution to choose X over Y?

Idea:

(1) A firm has incentives to reduce its regulatory capital

- Firms are "effectively risk averse" because holding capital is costly
- Froot-Stein (1998), Zanjani (2002), Bauer-Zanjani (2016)

(2) View a regulatory risk measure ρ as a decision principle for the firm

(3) Choose a properly designed ρ

Financial Decisions and Risk Preference

A regulator uses ρ to calculate regulatory capital

- Formally, assume that for two decisions X and Y, if $\rho(X) \ll \rho(Y)$, then a firm has the incentive to choose X (smaller capital) over Y (larger capital).
- If the regulator prefers X to Y, then she should design ρ such that $\rho(X) < \rho(Y)$.
- In the previous example

	X	Y
$\mathrm{VaR}_{0.999}$	10	9.9
$\mathrm{ES}_{0.999}$	11	208.91
StDev	1.109	20.039

Financial Decisions and Risk Preference

What is a suitable preference for the regulator?

- very complicated question
- for the interest of the society
- decision theory $\longleftrightarrow$ regulatory risk measures

Progress of the Talk

Expected Loss to the Society

A company has capital K and decides between two risks $X, Y \in \mathcal{X} \subset L^1$.

- If $\mathbb{E}[(X-K)_+] \le \mathbb{E}[(Y-K)_+]$ then taking X has less expected loss to the society.
- If $\mathbb{E}[(X-K)_+] \le \mathbb{E}[(Y-K)_+]$ holds for all K, then it is reasonable that X requires a smaller capital.

Formally, define the property

(EL) Consistency with expected loss to the society: for $X, Y \in \mathcal{X}$, $\rho(X) \le \rho(Y)$ if $\mathbb{E}[(X-K)_+] \le \mathbb{E}[(Y-K)_+]$ for all $K \in \mathbb{R}$.

(EL) is equivalent to the consistency with respect to second-order stochastic dominance (SSD).

Risk Aversion

Definition 2 (Second-order stochastic dominance)

For $X, Y \in L^1$, X has second-order stochastic dominance (SSD) over Y, denoted as $X \prec_{\mathrm{sd}} Y$, if $\mathbb{E}[f(X)] \le \mathbb{E}[f(Y)]$ for all increasing convex functions f such that the expectations exist.

- Also known as increasing convex order or stop-loss order
- $X \prec_{\mathrm{sd}} Y$ in the previous three-state example

(SC) SSD consistentcy: $\rho(X) \le \rho(Y)$ if $X \prec_{\mathrm{sd}} Y$, $X, Y \in \mathcal{X}$.

- (SC) is called strong risk aversion in decision theory
- (SC) $\Leftrightarrow$ (EL)

Consistent Risk Measures

Assume $\mathcal{X} \subset L^1$ in the following.

Definition 3 (Consistent risk measures)

A risk measure is a consistent risk measure if it satisfies (SC) and (TI).

- Consistent risk measures are monetary
- Interpretation: the regulator penalizes more risky financial decisions (ones that have higher expected social impact)

Consistent Risk Measures

Some examples

- An Expected Shortfall ES_p, $p \in (0,1)$ is consistent
- The mean $\mathbb{E}[\cdot]$ on L^1 is consistent
- Any law-invariant convex risk measure on L^∞ is consistent
- Any finite law-invariant convex risk measure on L^q, $q \geq 1$ is consistent
- Any Value-at-Risk VaR_p, $p \in (0,1)$ is not consistent

Is a consistent risk measure necessarily convex?

Properties

Similar properties for risk measures

(CC) Convex order consistency: $\rho(X) \le \rho(Y)$ if $X \prec_{\mathrm{cx}} Y$, $X, Y \in \mathcal{X}$.

(DM) Dilatation monotonicity: $\rho(X) \le \rho(Y)$ if $(X, Y) \in \mathcal{X}^2$ is a martingale.

(DC) Diversification consistency: $\rho(X + Y) \le \rho(X^c + Y^c)$ if $X, Y, X^c, Y^c \in \mathcal{X}$, $X \overset{\mathrm{d}}{=} X^c$, $Y \overset{\mathrm{d}}{=} Y^c$, and (X^c, Y^c) is comonotonic.

Properties

Proposition 4

For a monetary risk measure on L^∞, (SC), (EL), (CC), (DM), (DC) are equivalent. Moreover, each of them implies (LI).

Progress of the Talk

Characterization of Consistent Risk Measures

The next question is a characterization of all consistent risk measures.

- We assume $\mathcal{X} = L^\infty$ for simplicity
- All results hold for $\mathcal{X} = L^q$, $q \geq 1$

Characterization Theorem

Theorem 5

A risk measure ρ on L^∞ is consistent if and only if there exists a set $\mathcal{G}$ of functions mapping $(0,1)$ to $(-\infty,\infty]$ such that

$$\rho(X) = \inf_{g \in \mathcal{G}} \sup_{p \in (0,1)} \left\{ \mathrm{ES}_p(X) - g(p) \right\}, \quad X \in L^\infty. \tag{1}$$

- Example: If ρ is ES_p ($p \in (0,1)$), then one can take $\mathcal{G} = \{g_p\}$ where $g_p(p) = 0$ and $g_p(x) = \infty$ for $x \in (0,1) \setminus p$.
- $\mathcal{G}$ in (1) is not unique. It may be chosen as the adjustment set of ρ

$$\mathcal{G} = \left\{ g_Y : \ Y \in \mathcal{X}, \ \rho(Y) \le 0 \right\},$$

where $g_Y : (0,1) \to \mathbb{R}, \ p \mapsto \mathrm{ES}_p(Y)$.

Characterization Theorem

On the representation:

$$\rho(X) = \inf_{g \in \mathcal{G}} \sup_{p \in (0,1)} \{\mathrm{ES}_p(X) - g(p)\}, \quad X \in L^\infty.$$

- $g \in \mathcal{G}$ are benchmarks: if for some $g \in \mathcal{G}$, $\mathrm{ES}_\cdot(X) \le g(\cdot)$, then $\rho(X) \le 0$ (an accepted risk without extra capital); otherwise $\rho(X) > 0$ (or ≥ 0).
- Any risk-averse regulator or risk manager is essentially using a collection of Expected Shortfalls up to some adjustments.

Relation to Classic Risk Measures

Classic properties in the theory of monetary risk measures

(PH) Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ for all $\lambda \in (0, \infty)$ and $X \in \mathcal{X}$;

(CX) Convexity: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ for all $\lambda \in [0, 1]$ and $X, Y \in \mathcal{X}$;

(CA) Comonotonic additivity: $\rho(X + Y) = \rho(X) + \rho(Y)$ if $(X, Y) \in \mathcal{X}^2$ is comonotonic.

Definition 6

A risk measure is called a convex risk measure if it satisfies (M), (TI) and (CX). A risk measure is called a coherent risk measure if it satisfies (M), (TI), (PH) and (CX).

(Artzner-Delbaen-Eber-Heath 1999, Föllmer-Schied 2002, Kusuoka 2001)

Relation to Classic Risk Measures

Consistent risk measures are closely related to law-invariant convex risk measures.

Theorem 7

A risk measure ρ on L^∞ is consistent if and only if there exists a set $\mathcal{C}$ of law-invariant convex risk measures such that

$$\rho(X) = \inf_{\tau \in \mathcal{C}} \tau(X), \quad X \in L^\infty.$$

Relation to Classic Risk Measures

Yet we obtain a new characterization of convex (coherent) risk measures.

Proposition 8

A law-invariant risk measure ρ on L^∞ is a convex (resp. coherent) risk measure if and only if there exists a convex set (resp. convex cone) $\mathcal{G}$ of functions mapping $(0,1)$ to $(-\infty,\infty]$ such that

$$\rho(X) = \inf_{g\in\mathcal{G}} \sup_{p\in(0,1)} \{\mathrm{ES}_p(X) - g(p)\}, \quad X \in L^\infty.$$

Consistency vs Convexity

Consistency versus convexity:

(SC) Consistentcy: $\rho(X) \le \rho(Y)$ if $X \prec_{\mathrm{sd}} Y$, $X, Y \in \mathcal{X}$.

(CX) Convexity: $\rho(\lambda X + (1-\lambda)Y) \le \lambda\rho(X) + (1-\lambda)\rho(Y)$ for all $\lambda \in [0,1]$ and $X, Y \in \mathcal{X}$.

(i) Consistency compares between risks (decisions) while convexity does not

(ii) For risk-types other than market risk, portfolio diversification is not appropriate

(iii) There is no direct reason why a regulator would favour diversification in a single company, unless some social benefit could be expected (cf. Ibragimov-Jaffee-Walden 2011)

Kusuoka Representations

Kusuoka Representations

- Let $\mathcal{P}$ be the set of all probability measures on $[0,1]$ and $\mathcal{U}$ be the set of all functions mapping $\mathcal{P}$ to $\mathbb{R}$.
- A law-invariant coherent risk measure ρ on L^∞ has the following representation

$$\rho = \sup_{h\in\mathcal{R}} \left\{ \int_0^1 \mathrm{ES}_p \mathrm{d}h(p) \right\} \quad \text{for some } \mathcal{R} \subset \mathcal{P}.$$

- A law-invariant convex risk measure ρ on L^∞ has the following representation

$$\rho = \sup_{h\in\mathcal{P}} \left\{ \int_0^1 \mathrm{ES}_p \mathrm{d}h(p) - \alpha(h) \right\} \quad \text{for some } \alpha \in \mathcal{U}.$$

(Kusuoka 2001, Frittelli-Rosazza Gianin 2005)

Kusuoka Representations

Grand summary: for a risk measure on L^∞,

$$\text{(TI)+(SC)} = \inf_{\alpha\in\mathcal{V}} \sup_{h\in\mathcal{P}} \left\{ \int_0^1 \mathrm{ES}_p \mathrm{d}h(p) - \alpha(h) \right\} \qquad \text{for some } \mathcal{V}\subset\mathcal{U}$$

$$\xrightarrow{+\text{(CX)}} \sup_{h\in\mathcal{P}} \left\{ \int_0^1 \mathrm{ES}_p \mathrm{d}h(p) - \alpha(h) \right\} \qquad \text{for some } \alpha\in\mathcal{U}$$

$$\xrightarrow{+\text{(PH)}} \sup_{h\in\mathcal{R}} \left\{ \int_0^1 \mathrm{ES}_p \mathrm{d}h(p) \right\} \qquad \text{for some } \mathcal{R}\subset\mathcal{P}$$

$$\xrightarrow{+\text{(CA)}} \int_0^1 \mathrm{ES}_p \mathrm{d}h(p) \qquad \text{for some } h\in\mathcal{P}.$$

Remark: (TI)+(SC)+(CA) is sufficient for the last representation

Progress of the Talk

Risk Sharing

General setup

- n agents sharing a total risk $X \in \mathcal{X}$
- $\rho_1, \ldots, \rho_n$: underlying risk measures

Target: for $X \in \mathcal{X}$, find an Pareto-optimal solution of X to minimize

$$\rho_1(X_1), \ldots, \rho_n(X_n) \tag{2}$$

over the set of all allocations:

$$\mathbb{A}_n(X) = \left\{ (X_1, \ldots, X_n) \in \mathcal{X}^n : \sum_{i=1}^{n} X_i = X \right\}.$$

Risk Sharing

Theorem 9

Suppose that $\rho_1, \dots, \rho_n$ are consistent risk measures on $\mathcal{X} = L^q$, $q \in [1, \infty]$ with adjustment sets $\mathcal{G}_1, \dots, \mathcal{G}_n$, respectively. An allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is Pareto-optimal if and only if

$$\sum_{i=1}^{n} \rho_i(X_i) = \rho^*(X),$$

where ρ^* is a consistent risk measure with adjustment set $\sum_{i=1}^{n} \mathcal{G}_i$.

In particular,

$$\rho^*(X) = \inf_{g \in \mathcal{G}_1 + \dots + \mathcal{G}_n} \sup_{\alpha \in [0,1]} \{\mathrm{ES}_\alpha(X) - g(\alpha)\}, \quad X \in \mathcal{X}.$$

Progress of the Talk

Suitable risk measures for regulation

On the current debates regarding the desirability of VaR and ES:

- A suitable risk measure applied in regulatory practice should encourage prudent and socially responsible financial decisions
  - Financial institutions are not necessarily risk-averse or socially responsible for their own interest; a regulator should push them towards risk-aversion
- ES is the basis for any consistent risk measure - supporting the transition from VaR to ES in the recent Basel documents
- ES is the only candidate which preserves consistency and also has simple form and clear economic interpretation

Suitable risk measures for regulation

Further remarks:

- Consistency is more natural than convexity for a regulator
- One can construct non-convex consistent risk measures
  - As far as we are aware of, there are no non-convex consistent risk measures in simple analytical forms other than a minimum
- Criteria for a desirable risk measure used in banking and insurance regulation may vary
- Bring more in decision theory to risk measures and regulation

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Thank You

Thanks you for your kind attendance

The manuscript can be downloaded at

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