A Theory for Measures of Tail Risk

Ruodu Wang

http://sas.uwaterloo.ca/~wang

Department of Statistics and Actuarial Science
University of Waterloo, Canada

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Outline

Based on joint work with Fangda Liu (CUFE, Beijing) and joint work with Edward Furman (York, Toronto) and Ričardas Zitikis (Western, London Ontario)

Risk measures

A risk measure ...

- quantifies the "riskiness" of a random loss in a fixed period of time (e.g. 1 year, 10 days, 1 day)
- primary examples in practice: the Value-at-Risk and the Expected Shortfall
- main interpretation: the amount of regulatory capital of a financial institution taking a risk (random loss) in a fixed period

Value-at-Risk and Expected Shortfall

For $X \in L^0$, the Value-at-Risk (VaR) at confidence level $p \in (0,1)$ has two versions:

$$\mathrm{VaR}_p^L(X) = \inf\{x \in \mathbb{R} : F_X(x) \geq p\} = F_X^{-1}(p),$$

and

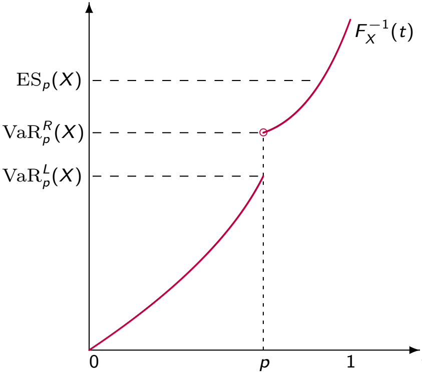
$$\mathrm{VaR}_p^R(X) = \inf\{x \in \mathbb{R} : F_X(x) > p\} = F_X^{-1}(p+).$$

The Expected Shortfall (ES) at confidence level $p \in (0,1)$:

$$\mathrm{ES}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q^L(X)\mathrm{d}q = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q^R(X)\mathrm{d}q$$

- Typical choices of p: 0.975, 0.99, 0.995, 0.999 ...

Value-at-Risk and Expected Shortfall



Value-at-Risk and Expected Shortfall

The ongoing debate on "VaR versus ES":

- Basel III (mixed; in transition from VaR to ES as standard metric for market risk)
- Solvency II (VaR based)
- Swiss Solvency Test (ES based)

Some academic references

- Embrechts-Puccetti-Rüschendorf-W.-Beleraj 2014
- Emmer-Kratz-Tasche 2015

Tail risk

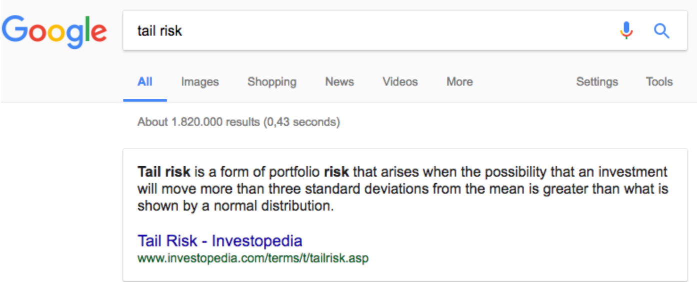
Quoting **Basel Committee on Banking Supervision**: Standards, January 2016, Minimum capital requirements for Market Risk. Page 1. *Executive Summary*:

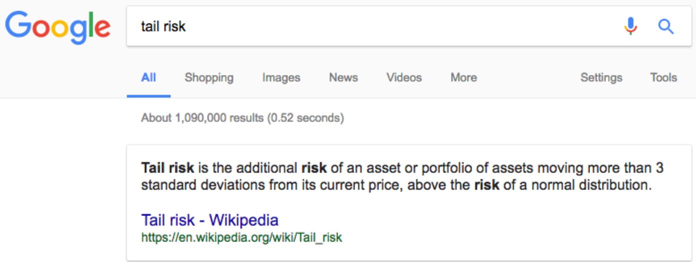
"... A shift from Value-at-Risk (VaR) to an Expected Shortfall (ES) measure of risk under stress. Use of ES will help to ensure a more prudent capture of "tail risk" and capital adequacy during periods of significant financial market stress."

Some interpretation:

- "tail risk" is a crucial concern for prudent risk management
- "tail risk" is associated with financial market stress

So ... what is “tail risk”?





Tail risk

> *That is wrong on so many levels.*
>
> – Apple's Siri, as quoted by Sidney Resnick, April 2017, Zurich

- Probability of moving downside more than 3 standard deviations:
  - normal risk: 0.135%
  - Pareto(5) risk: 1.86%
  - Pareto(3) risk: 1.45%
  - Pareto(2.01) risk: 0.05%
  - Cantalli's inequality: $\leq$ 10% (10% $\Rightarrow$ Bernoulli)
- No tail risk for very heavy-tailed distributions?
- A Bernoulli distribution has the most severe tail risk?

Tail risk

Our motivation

- establish a framework for regulatory concerns of tail risks
- complementary risk metrics to VaR and ES
  - VaR and ES are similar to "median" and "mean"
- alternative risk measures (internal risk management)
- better understand the roles of VaR and ES

Progress of the talk

Notation

Some notation.

- $(\Omega, \mathcal{F}, \mathbb{P})$ is an atomless probability space
- $\mathcal{X}$ is a convex cone of random variables containing L^∞
  - e.g. $\mathcal{X} = L^\infty$
- A risk measure is a functional $\rho : \mathcal{X} \to (-\infty, \infty]$ such that $\rho(X) \in \mathbb{R}$ for $X \in L^\infty$.
- For $X \in \mathcal{X}$,
  - F_X: cdf of X
  - $F_X^{-1}(p) = \inf\{x \in \mathbb{R} : F_X(x) \geq p\}$, $p \in (0,1]$
  - U_X: a uniform random variable satisfying $F_X^{-1}(U_X) = X$ a.s.[1]

[1] such U_X always exists; see Lemma A.32 of Föllmer-Schied 2016

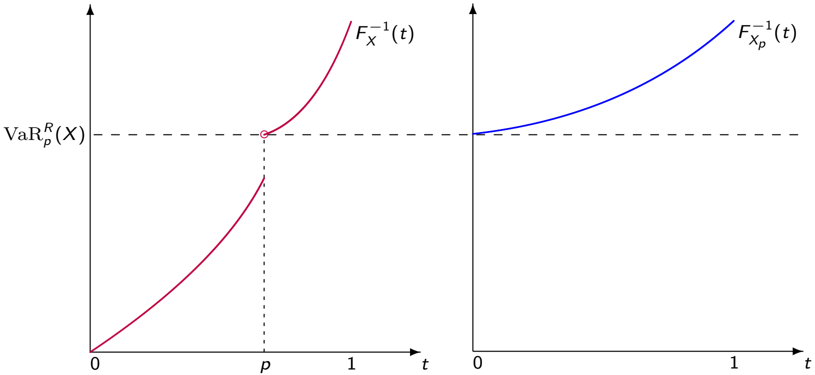
Tail risk

For any random variable $X \in \mathcal{X}$ and $p \in (0,1)$, let X_p be the tail risk of X beyond its p-quantile

$$X_p = F_X^{-1}(p + (1-p)U_X).$$

- $p + (1-p)U_X$ is uniform on $[p, 1]$.

Tail risk



Tail risk measures

Definition 1

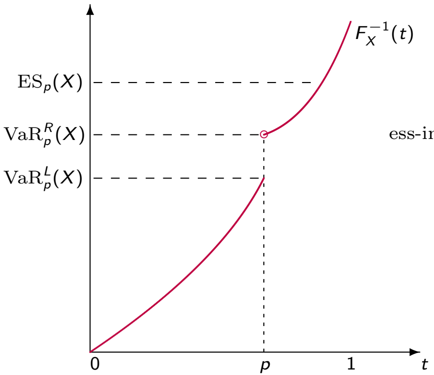
For $p \in (0,1)$, a risk measure ρ is a p-tail risk measure if $\rho(X) = \rho(Y)$ for all $X, Y \in \mathcal{X}$ satisfying $X_p \stackrel{\mathrm{d}}{=} Y_p$.

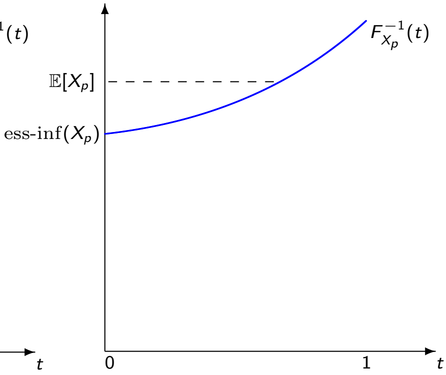
For $p \in (0,1)$,

- ES_p is a p-tail risk measure
- VaR_p^R is a p-tail risk measure
- VaR_p^L is not a p-tail risk measure, but a $(p-\epsilon)$-tail risk measure for all $\epsilon \in (0,p)$
- a p-tail risk measure is law-invariant

(A0) Law-invariance: $\rho(X) = \rho(Y)$, if $X \stackrel{\mathrm{d}}{=} Y$, $X, Y \in \mathcal{X}$.

Tail risk


$F_X^{-1}(t)$
$\mathrm{ES}_p(X)$
$\mathrm{VaR}_p^R(X)$
$\mathrm{VaR}_p^L(X)$
0
p
1
t


$F_{X_p}^{-1}(t)$
$\mathbb{E}[X_p]$
ess-inf(X_p)
0
1
t

Generators of tail risk measures

Observe simple relations

$$\mathrm{VaR}_p^R(X) = \text{ess-inf}(X_p) \ \text{ and } \ \mathrm{ES}_p(X) = \mathbb{E}[X_p], \quad X \in \mathcal{X}.$$

Generally, for any law-invariant risk measure ρ^* on $\mathcal{X}$, define

$$\rho(X) = \rho^*(X_p), \quad X \in \mathcal{X}.$$

then ρ is a p-tail risk measure.

- ρ is generated by ρ^* and ρ^* is a p-generator of ρ.
- There is a one-to-one relationship between ρ and ρ^*

Tail pair of risk measures

A pair of risk measures (ρ, ρ^*) is called a *p*-tail pair if ρ^* is law-invariant and is a *p*-generator of ρ.

Examples.

- $(\mathrm{VaR}_p^R, \text{ess-inf})$
- $(\mathrm{VaR}_{(p+1)/2}^R, \text{right-median})$
- $(\mathrm{ES}_p, \mathbb{E})$
- $(\mathrm{VaR}_q^L, \mathrm{VaR}_{(q-p)/(1-p)}^L)$, $q > p$

Properties of tail risk measures

Estimation.

To estimate a tail risk measure

- estimate the tail distribution (distribution of X_p; maybe through Extreme Value Theory if p is close to 1)
- apply a standard procedure to estimate the generator $\rho^*(X_p)$

Question.

How do we generate a tail risk measure with desirable properties?

- what properties may be passed from ρ^* to ρ?

For instance, if ρ^* is convex, is ρ also convex?

Risk measure properties

Classic properties. For $X, Y \in \mathcal{X}$,

Monetary risk measures

(A1) Monotonicity: $\rho(X) \le \rho(Y)$ if $X \le Y$.

(A2) Translation invariance: $\rho(X + m) = \rho(X) + m$ if $m \in \mathbb{R}$.

(Artzner-Delbaen-Eber-Heath 1999, Föllmer-Schied 2002, Frittelli-Rosazza Gianin 2002)

Risk measure properties

Classic properties. For $X, Y \in \mathcal{X}$,

Convex risk measures

(A1) Monotonicity: $\rho(X) \leq \rho(Y)$ if $X \leq Y$.

(A2) Translation invariance: $\rho(X+m) = \rho(X) + m$ if $m \in \mathbb{R}$.

(A3) Convexity: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1+\lambda)\rho(Y)$ for $\lambda \in [0,1]$.

(Artzner-Delbaen-Eber-Heath 1999, Föllmer-Schied 2002, Frittelli-Rosazza Gianin 2002)

Risk measure properties

Classic properties. For $X, Y \in \mathcal{X}$,

Coherent risk measures

(A1) Monotonicity: $\rho(X) \le \rho(Y)$ if $X \le Y$.

(A2) Translation invariance: $\rho(X+m) = \rho(X) + m$ if $m \in \mathbb{R}$.

(A3) Convexity: $\rho(\lambda X + (1-\lambda)Y) \le \lambda\rho(X) + (1+\lambda)\rho(Y)$ for $\lambda \in [0,1]$.

(A4) Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ for $\lambda > 0$.

(A5) Subadditivity: $\rho(X+Y) \le \rho(X) + \rho(Y)$.

(Artzner-Delbaen-Eber-Heath 1999, Föllmer-Schied 2002, Frittelli-Rosazza Gianin 2002)

More properties

(A6) Comonotonic additivity: $\rho(X+Y)=\rho(X)+\rho(Y)$ if $X, Y \in \mathcal{X}$ are comonotonic.

(A7) $\prec_{\mathrm{cx}}$-monotonicity: $\rho(X) \le \rho(Y)$ if $X \prec_{\mathrm{cx}} Y$.

Notes. For $X, Y \in \mathcal{X}$,

- X, Y are comonotonic if $(X(\omega)-X(\omega'))(Y(\omega)-Y(\omega')) \ge 0$ for all $(\omega, \omega') \in \Omega \times \Omega$, $\mathbb{P} \times \mathbb{P}$-almost surely.
- Y is smaller than X in convex order, denoted as $X \prec_{\mathrm{cx}} Y$, if $X \overset{\mathrm{d}}{=} \mathbb{E}[Y|\mathcal{G}]$ for some σ-algebra $\mathcal{G} \subset \mathcal{F}$; equivalently, $\mathbb{E}[f(X)] \le \mathbb{E}[f(Y)]$ (if both exist) for all convex f.

VaR and ES

Take $p \in (0,1)$.

- For $(\rho, \rho^*) = (\mathrm{VaR}_p^R, \text{ess-inf})$, both ρ and ρ^* are monotone, translation-invariant, positively homogeneous, and comonotonically additive.
- For $(\rho, \rho^*) = (\mathrm{ES}_p, \mathbb{E})$, both ρ and ρ^* are, in addition to the above, subadditive, convex, and $\prec_{\mathrm{cx}}$-monotone.

Tail standard deviation risk measure

Take $p \in (0,1)$, $\mathcal{X} = L^2$ and let ρ^* be the standard deviation risk measure for some $\beta > 0$

$$\rho^*(X) = \mathbb{E}[X] + \beta\sqrt{\mathrm{var}(X)}, \quad X \in L^2. \tag{1}$$

Let

$$\rho(X) = \rho^*(X_p) = \mathbb{E}[X_p] + \beta\sqrt{\mathrm{var}(X_p)}, \quad X \in L^2.$$

- ρ^* is convex, subadditive, and $\prec_{\mathrm{cx}}$-monotone
- but ρ is NOT convex, subadditive, or $\prec_{\mathrm{cx}}$-monotone!

Properties of tail risk measures

Theorem 2

Suppose that $p \in (0,1)$ and (ρ, ρ^*) is a p-tail pair of risk measures on $\mathcal{X}$ and $\mathcal{X}^*$, respectively. The following statements hold.

(i) *ρ is monotone (translation-invariant, positively homogeneous, comonotonically additive) if and only if so is ρ^*.*

(ii) *If ρ is subadditive (convex, $\prec_{\mathrm{cx}}$-monotone) then so is ρ^*.*

Remark.

- The converse of (ii) is not true because of the previous example.

Properties of tail risk measures

Theorem 3

Suppose that $p \in (0,1)$ and (ρ, ρ^*) is a p-tail pair of risk measures on $\mathcal{X}$ and $\mathcal{X}^*$, respectively. Then ρ is a coherent (convex, monetary) risk measure if and only if so is ρ^*.

Remark.

- Monotonicity is essential for the other properties to pass through.
- To construct coherent tail risk measures: apply an existing coherent risk measure to the tail risk X_p.

Smallest tail risk measures

Theorem 4

For $p \in (0,1)$, if ρ is a monetary p-tail risk measure with $\rho(0) = 0$, then $\rho \geq \mathrm{VaR}_p^R$ on $\mathcal{X}$, and if ρ is a coherent p-tail risk measure, then $\rho \geq \mathrm{ES}_p$ on $\mathcal{X}$.

Remark.

- VaRs and ES serve as benchmarks for tail risk measures.
- The converse statements are not true in general. For instance, take $\rho(X) = \max\{\mathbb{E}[X], \mathrm{VaR}_p^R(X)\}$.
- For distortion risk measures, the converse statements are true.

Notes.

- A distortion risk measure is a law-invariant, comonotonic-additive and monetary risk measure.

Other properties

Other mathematical or statistical properties:

- Superadditivity and concavity. Counter-example: $(\mathrm{VaR}_p^R, \text{ess-inf})$
- Linearity. Counter-example: $(\mathrm{ES}_p, \mathbb{E})$
- Common robustness (continuity) properties such as continuity with respect to Wasserstein L^q-norm ($q \geq 1$) or convergence in distribution can be naturally passed on from ρ^* to ρ.
- Elicitability cannot be passed from ρ^* to ρ (will be discussed later)

Progress of the talk

Gini mean deviation

- C. Gini noticed the center-free version of the variance

$$\mathrm{Var}(X) = \frac{1}{2}\mathbb{E}[(X^* - X^{**})^2], \quad X \in L^2,$$

where X^* and X^{**} are two independent copies of X.

- Consequently, he introduced

$$\mathrm{Gini}(X) = \mathbb{E}[|X^* - X^{**}|], \quad X \in L^1,$$

which is nowadays known as the Gini mean difference.

- The Gini functional is comonotonic-addtive and satisfies

$$\mathrm{Gini}(X) = 2\int_0^1 F_X^{-1}(u)(2u-1)\mathrm{d}u = 4\mathrm{Cov}(X, U_X).$$

Gini principle

- D. Denneberg, insisting comonotonic-additivity, introduced the Gini principle to replace the standard deviation risk measure, for $\lambda > 0$.

$$\mathrm{GP}^{\lambda}(X) = \mathbb{E}[X] + \lambda\,\mathrm{Gini}(X), \quad X \in L^1.$$

- Using the language of risk measures, the Gini principle is convex, subadditive, $\prec_{\mathrm{cx}}$-monotone, positively homogeneous, comonotonic-additive and translation invariant, but not necessarily monotone.

(Denneberg 1990)

Gini Shortfall

By applying the Gini principle to the tail risk, define the Gini Shortfall for $p \in [0,1)$ and $\lambda > 0$,

$$\mathrm{GS}_p^{\lambda}(X) = \mathbb{E}[X_p] + \lambda\,\mathrm{Gini}(X_p), \quad X \in L^1,$$

and equivalently,

$$\mathrm{GS}_p^{\lambda}(X) = \mathrm{ES}_p(X) + \lambda \mathbb{E}[|X_p^* - X_p^{**}|], \quad X \in L^1,$$

where X_p^* and X_p^{**} are two independent copies of X_p.

- A Gini Shortfall combines magnitude (captured by the ES part) and variability (captured by the Gini part) of tail risk.

Gini Shortfall

Theorem 5

Let $p \in (0,1)$ and $\lambda \in [0,\infty)$.

1. *The functional GS_p^λ is translation invariant, positively homogeneous, and comonotonic-additive.*
2. *The following statements are equivalent:*
   (i) GS_p^λ *is monotone;*
   (ii) GS_p^λ *is convex;*
   (iii) GS_p^λ *is subadditive;*
   (iv) GS_p^λ *is $\prec_{\mathrm{cx}}$-monotone;*
   (v) GS_p^λ *is a coherent risk measure;*
   (vi) $\lambda \in [0, 1/2]$.

Gini Shortfall

- Unlike other distortion risk measures, a Gini Shortfall has a simple non-parametric estimator

$$\widehat{\mathrm{GS}}_p^\lambda = \frac{1}{m}\sum_{i=1}^m X_i + \frac{\lambda}{m(m-1)}\sum_{i,j=1}^m |X_i - X_j|,$$

where $X_1, \ldots, X_m$ are the largest $m = \lfloor np \rfloor$ observations in an iid sample of size n.

- A Gini shortfall is well defined on L^1 and is continuous with respect to L^1 convergence.

Gini Shortfall

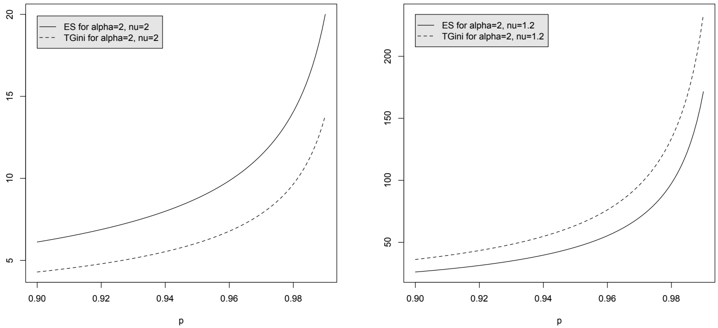


Figure: $\mathrm{ES}_p(X)$ and $\mathrm{TGini}_p(X) = \mathbb{E}[|X_p^* - X_p^{**}|]$, $p \in [0.9, 0.99]$ for skew-t risks with $\alpha = 2$ and $\nu = 2$ (left) and $\alpha = 2$ and $\nu = 1.2$ (right)

Progress of the talk

Motivation

Measures of tail risk

Gini Shortfall

Elicitable tail risk measures

Risk aggregation and dual representations

Conclusion

References

Elicitability

Quoting Acerbi-Szekely 2014:

"Eliciwhat?"
Risk professionals had never heard of elicitability until 2011, when Gneiting proved that ES is not elicitable as opposed to VaR. This result sparked a confusing debate.

- In 2011, a notion is proposed for comparing risk measure forecasts: elicitability (Gneiting 2011)
- Issues related to elicitability soon raised the attention of regulators in the Bank for International Settlements (mentioned in their official consultative document in May 2012)

(earlier study in statistical literature: Osband 1985, Lambert-Pennock-Shoham 2008)

Elicitability

Modified definition of elicitability for law-invariant risk measures

Definition 6

A law-invariant risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is elicitable if there exists a function $S : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\rho(X) = \min \left\{ \arg\min_{x \in \mathbb{R}} \mathbb{E}[S(x, X)] \right\}, \quad X \in \mathcal{X}.$$

- Typically one may further require $S(x, y) \geq 0$ and $S(x, x) = 0$ for $x, y \in \mathbb{R}$

Elicitability

Assuming all integrals are finite:

- the mean is elicitable with

$$S(x,y) = (x-y)^2.$$

- the median is elicitable with

$$S(x,y) = |x-y|.$$

- VaR_p^L is elicitable with

$$S(x,y) = (1-p)(x-y)_+ + p(y-x)_+.$$

- an expectile[2] e_p is elicitable with

$$S(x,y) = (1-p)(x-y)_+^2 + p(y-x)_+^2.$$

[2] introduced by Newey-Powell 1987

Comparative forecasting

- for simplicity, suppose that observations are iid
- for a risk measure ρ, different forecasting procedures $\rho^{(1)}, \dots, \rho^{(k)}$
- at time $t-1$, the estimated/modeled value of $\rho(X_t)$ is $\rho_t^{(i)}$
- collect the statistics $S(\rho_t^{(i)}, X_t)$; a summary statistic can typically be chosen as $T_n(\rho^{(i)}) = \frac{1}{n}\sum_{t=1}^{n} S(\rho_t^{(i)}, X_t)$
- the above procedure is model-independent
- forecasting comparison: compare $T_n(\rho^{(1)}), \dots, T_n(\rho^{(k)})$
  - risk analyst: compare forecasting procedures/models
  - regulator: compare internal model forecasts with a standard model

Estimation procedures of an elicitable risk measure are straightforward to compare.

Elicitable risk measures

Some characterizations (under suitable conditions)

- (Gneiting 2011)
  VaR is elicitable whereas ES is not.
- (Ziegel 2016)
  Among all coherent risk measures, only expectiles (including the mean) are elicitable.
- (Bellini-Bignozzi 2015, Delbaen-Bellini-Bignozzi-Ziegel 2016)
  Among all convex risk measures, only shortfall risk measures are elicitable.
- (Kou-Peng 2016, W.-Ziegel 2015)
  Among all distortion risk measures, only the mean and the quantiles are elicitable.
- (Acerbi-Székely 2014, Fissler-Ziegel 2016)
  (VaR,ES) is co-elicitable

Elicitable risk measures

Remarks.

- For a *p*-tail pair (ρ, ρ^*): elicitability cannot be pass from ρ^* to ρ: take $(\mathrm{ES}_p, \mathbb{E})$. $\mathbb{E}$ is elicitable, whereas ES_p is not.
- A necessary condition for elicitability is closely related to the class of shortfall risk measure (a result of Weber 2006).

Elicitable tail risk measures

Implications.

For monetary risk measures:

$$\begin{array}{ccc} \text{Shortfall risk measure} & \Rightarrow & \text{Elicitability} \\ \Uparrow \text{(Weber 2006)} & & \Downarrow \text{(Osband 1985)} \\ \text{CxLS} + \text{DLC} + \text{AP} & & \text{CxLS} \end{array}$$

Additional assumptions.

(A8) *Distribution-wise lower-semi-continuous* (DLC). $\liminf_{n\to\infty} \rho(X_n) \geq \rho(X)$ for $X_n \to X$ in distribution as $n \to \infty$.

(A9) *Absorbing property* (AP). There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $Z_\lambda \in \mathcal{A}_\rho$ where $Z_\lambda \sim (1-\lambda)\delta_x + \lambda\delta_y \in \mathcal{N}_\rho$ for some $\lambda > 0$.

Elicitable tail risk measures

Theorem 7

1. *For any $p \in (0,1)$, the only elicitable p-tail convex risk measure is VaR_1^L (the essential supremum).*
2. *For $p \in (0,1)$, a monetary and positively homogeneous p-tail risk measure ρ satisfying DLC is elicitable if and only if $\rho = \mathrm{VaR}_q^L$ for some $q \in (p, 1]$.*

Remark.

- A new axiomatic characterization of VaRs
- To arrive at VaR_q^R: "lower-continuity" → "upper-continuity", and "min" → "max" in the definition of elicitable risk measures
- Full characterization of elicitable *p*-tail risk measures is available

Elicitable tail risk measures

Question.

Suppose that (ρ, ρ^*) is a p-tail pair of risk measures.

- ρ^* is elicitable $\Rightarrow (\mathrm{VaR}_p^R, \rho)$ is co-elicitable?
- (VaR_p^R, ρ) is co-elicitable $\Rightarrow \rho^*$ is elicitable?

Note that $(\mathrm{VaR}_p^R, \mathrm{ES}_p)$ is co-elicitable under suitable continuity conditions.

Progress of the talk

Risk aggregation

Target: for given univariate distributions $F_1, \dots, F_n$, calculate

$$\sup\{\rho(S) : S \in \mathcal{S}_n(F_1, \dots, F_n)\}$$

where $\mathcal{S}_n(F_1, \dots, F_n)$ is the aggregation set defined as

$$\mathcal{S}_n(F_1, \dots, F_n) = \{X_1 + \cdots + X_n : X_i \in \mathcal{X},\ X_i \sim F_i,\ i = 1, \dots, n\}.$$

- This setting is called risk aggregation with dependence uncertainty
- A particularly relevant case is $\rho = \mathrm{VaR}_p^L$ or $\rho = \mathrm{VaR}_p^R$ for some $p \in (0,1)$.

(e.g. Embrechts-Puccetti-Rüschendorf 2013, W.-Peng-Yang 2013, Embrechts-Wang-W. 2015)

Risk aggregation

For $X \in L^0$, $F_X^{[p]}$ is the distribution of X_p.

Theorem 8

Let $p \in (0,1)$, (ρ, ρ^*) be a p-tail pair of monotone risk measures. For any univariate distributions $F_1, \dots, F_n$, we have

$$\sup\{\rho(S) : S \in \mathcal{S}_n(F_1, \dots, F_n)\} = \sup\{\rho^*(T) : T \in \mathcal{S}_n(F_1^{[p]}, \dots, F_n^{[p]})\}.$$

Remark.

- monotonicity is essential for the above equation to hold

Risk aggregation

For the cases of VaR and ES:

(i) Take $\rho = \mathrm{VaR}_p^R$.

$$\begin{aligned}&\sup\{\mathrm{VaR}_p^R(S) : S \in \mathcal{S}_n(F_1, \dots, F_n)\}\\&= \sup\{\text{ess-inf}(T) : T \in \mathcal{S}_n(F_1^{[p]}, \dots, F_n^{[p]})\}.\end{aligned}$$

(ii) Take $\rho = \mathrm{ES}_p$. For any $X_1, \dots, X_n \in L^1$,

$$\begin{aligned}\mathrm{ES}_p(X_1 + \cdots + X_n) &\le \sup\{\mathbb{E}[T] : T \in \mathcal{S}_n(F_{X_1}^{[p]}, \dots, F_{X_n}^{[p]})\}\\&= \sum_{i=1}^{n} \mathbb{E}[(X_i)_p] = \sum_{i=1}^{n} \mathrm{ES}_p(X_i),\end{aligned}$$

which is (yet another proof of) the classic subadditivity of ES_p.

(cf. Bernard-Jiang-W. 2014, Embrechts-W. 2015)

Dual representations

Suppose $p \in (0,1)$, $\mathcal{D}_p$ is the set of distributions functions on $[p,1]$ and ρ is a functional mapping $\mathcal{X} = L^\infty$ to $\mathbb{R}$.

(i) ρ is a comonotonically additive and coherent p-tail risk measure if and only if there exists $g \in \mathcal{D}_p$ such that

$$\rho(X) = \int_p^1 \mathrm{ES}_q(X)\mathrm{d}g(q),\ X \in \mathcal{X}.$$

(ii) ρ is a coherent p-tail risk measure if and only if there exists a set $\mathcal{G} \subset \mathcal{D}_p$ such that

$$\rho(X) = \sup_{g\in\mathcal{G}} \int_p^1 \mathrm{ES}_q(X)\mathrm{d}g(q),\ X \in \mathcal{X}.$$

(i) and (ii) on non-tail risk measures: Kusuoka 2001, Jouini-Schachermayer-Touzi 2006

Dual representations

(iii) ρ is a convex and monetary p-tail risk measure if and only if there exists a function $v : \mathcal{D}_p \to \mathbb{R}$ such that

$$\rho(X) = \sup_{g \in \mathcal{P}} \left\{ \int_p^1 \mathrm{ES}_q(X) \mathrm{d}g(q) - v(g) \right\}, \ X \in \mathcal{X}.$$

(iv) ρ is a $\prec_{\mathrm{cx}}$-monotone and monetary p-tail risk measure if and only if there exists a set $\mathcal{H}$ of functions mapping $[p, 1]$ to $\mathbb{R}$ such that

$$\rho(X) = \inf_{\alpha \in \mathcal{H}} \sup_{q \in [p,1]} \{\mathrm{ES}_q(X) - \alpha(q)\}, \ X \in \mathcal{X}.$$

(iii) and (iv) on non-tail risk measures: Frittelli-Rosazza Gianin 2005, Mao-W. 2016

Progress of the talk

Concluding remarks

- replacing a generic risk measure by its tail counterpart is philosophically analogous to replacing the expectation by an ES
- Gini shortfall seems to be a promising risk measure to consider
- potential applications and future research in portfolio selection, decision analysis, risk sharing, etc.
- better position VaR and ES among all tail risk measures
- generalization of the tail distributional transform $X \mapsto X_p$
  - choose a general distributional transform $T : \mathcal{X} \to \mathcal{X}$ and a law-invariant risk measure ρ^*
  - define a risk measure $\rho_T(X) = \rho^*(T(X))$, $X \in \mathcal{X}$.
  - study properties of the triplet (ρ_T, ρ^*, T)

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Thank you

Thank you

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- Liu, F. and Wang, R. (2016)
  A theory for measures of tail risk.
  Manuscript available at `https://ssrn.com/abstract=2841909`
- Furman, E., Wang, R. and Zitikis, R. (2017)
  Gini-type measures of risk and variability: Gini shortfall, capital allocations, and heavy-tailed risks.
  Journal of Banking and Finance, forthcoming.
  Manuscript available at `https://ssrn.com/abstract=2836281`