Quantile-based Risk Sharing, Market Equilibria, and Belief Heterogeneity

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Content

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- Embrechts-Liu-W., Quantile-based risk sharing
  SSRN: 2744142, 2018, Operations Research
- Embrechts-Liu-Mao-W., Quantile-based risk sharing with heterogeneous beliefs
  SSRN: 3079998, 2018, Mathematical Programming
- W.-Wei, Characterizing optimal allocations in quantile-based risk sharing
  SSRN: 3173503, 2018

Agenda

Risk Sharing Games

Setup

- n agents sharing a total risk (or asset) $X \in \mathcal{X}$ (a set of rvs)

The set of allocations of X:

$$\mathbb{A}_n(X) = \left\{ (X_1, \dots, X_n) \in \mathcal{X}^n : \sum_{i=1}^{n} X_i = X \right\}.$$

- Collaborative risk sharing: Pareto optimality; an allocation impossible to strictly improve
- Competitive risk sharing: an equilibrium arrived at via each agent optimizing their objectives individually

Risk Sharing Games

What is the "canonical form" of an optimal (sensible) allocation?

If we assume the preferences of the agents are "similar" ...

- $X_i = a_i X +$ side payments for some $\sum_{i=1}^n a_i = 1$?
- $X_i = \mathbb{1}_{A_i} X +$ side payments for some $\bigcup_{i=1}^n A_i = \Omega$?
- other forms?

Risk Measures

A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ maps a risk (via a model) to a number

- regulatory capital calculation $\leftarrow$ our main interpretation
- decision making (management, optimization, ...)
- performance analysis and capital allocation
- pricing

Risks ...

- modelled by random losses in one period in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$

Value-at-Risk and Expected Shortfall

Value-at-Risk (VaR) at level $\alpha \geq 0$

$\mathrm{VaR}_\alpha : L^0 \to [-\infty, \infty]$,

$$\mathrm{VaR}_\alpha(X) = F_X^{-1}(1-\alpha) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq 1-\alpha\}.$$

Note: for $\alpha \geq 1$, $\mathrm{VaR}_\alpha(X) = -\infty$.

Expected Shortfall (ES/TVaR/CVaR/AVaR) at level $\beta \in (0,1)$

$\mathrm{ES}_\beta : L^1 \to (-\infty, \infty)$,

$$\mathrm{ES}_\beta(X) = \frac{1}{\beta}\int_0^\beta \mathrm{VaR}_\alpha(X)\mathrm{d}\alpha \underset{(F_X \text{ cont.})}{=} \mathbb{E}\left[X | X > \mathrm{VaR}_\beta(X)\right].$$

Remarks: small α convention ... relevance of $\mathbb{P}$ ...

Value-at-Risk and Expected Shortfall

The ongoing co-existence of VaR and ES:

- Basel IV - both
- Solvency II - VaR
- Swiss Solvency Test - ES
- US Solvency Framework (NAIC ORSA) - both

Questions from Regulation

Basel Committee on Banking Supervision (BCBS)

Consultative Document, May 2012, page 41. *Question 8*:

"What are the likely constraints with moving from VaR to ES, including any challenges in delivering robust backtesting, and how might these be best overcome?"

Standards, Jan 2016, page 1. *Executive Summary*:

"Use of ES will help to ensure a more prudent capture of "tail risk" and capital adequacy ..."

Questions from Regulation

International Association of Insurance Supervisors (IAIS)

Consultation Document, December 2014, page 43. *Question 42*:

"Which risk measure - VaR, Tail-VaR [ES] or another - is most appropriate for ICS [insurance capital standard] capital requirement purposes? Why?"

Academic Inputs

- ES is generally advocated by academia for desirable properties in the past two decades; in particular,
  - subadditivity or coherence (Artzner-Delbaen-Eber-Heath'99)
  - convex optimization properties (Rockafellar-Uryasev'00)
- Some other examples of impact from academic research
  - Gneiting'11: backtesting ES is unclear, whereas backtesting VaR is straightforward
  - Cont-Deguest-Scandolo'10: ES is not robust, whereas VaR is
  - Embrechts-Wang-W.'15: VaR is sensitive to risk aggregation

VaR versus ES

Features/Risk measure	VaR	Tail-VaR
Frequency captured?	Yes	Yes
Severity captured?	No	Yes
Sub-additive?	Not always	Always
Diversification captured?	Issues	Yes
Back-testing?	Straight-forward	Issues
Estimation?	Feasible	Issues with data limitation
Model uncertainty?	Sensitive to aggregation	Sensitive to tail modelling
Robustness I (with respect to "Lévy metric[33]")?	Almost, only minor issues	No
Robustness II (with respect to "Wasserstein metric[34]")?	Yes	Yes

Table copied from IAIS Dec 2014, page 42

Basel III & IV

Basel III & IV for market risk $(\mathrm{ES}_{0.025})$

- internal model approach
  - subject to approval
  - consistency to risk management and decision making
  - favourable capital calculation
- standard approach

Progress of the Talk

Targets

Via studying risk sharing problems, we aim to understand:

- optimal allocations and competitive equilibria
- capital adequacy of the system
- management of tail risk
- robustness
- internal models
- implications for regulation, VaR or ES?

Risk Sharing Games

Simplistic risk sharing problem

- n agents sharing a total risk $X \in \mathcal{X}$
- risk measures $\rho_1, \dots, \rho_n$ (individual objectives to minimize)

Pareto-optimal allocation

An allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is Pareto-optimal if for any $(Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$, $\rho_i(X_i) \geq \rho_i(Y_i)$, $i = 1, \dots, n$ implies equality.

For finite monetary (monotone and cash-additive) risk measures:

$$\text{Pareto optimality} \quad \Leftrightarrow \quad \text{sum-optimality}$$

- an allocation $(X_1, \dots, X_n)$ is sum-optimal if $\sum_{i=1}^n \rho_i(X_i)$ is minimal among $\mathbb{A}_n(X)$.

Inf-convolution

The inf-convolution of n risk measures is a risk measure $\square_{i=1}^n \rho_i$ mapping $\mathcal{X}$ to $[-\infty, \infty]$:

$$\mathop{\square}_{i=1}^{n} \rho_i(X) = \inf\left\{\sum_{i=1}^{n} \rho_i(X_i) : (X_1, \dots, X_n) \in \mathbb{A}_n(X)\right\}.$$

- $\square_{i=1}^n \rho_i(X)$ is the smallest total capital in the economy
- For finite monetary risk measures,

$$(X_1^*, \dots, X_n^*) \text{ is Pareto-optimal} \quad \Leftrightarrow \quad \sum_{i=1}^{n} \rho_i(X_i^*) = \mathop{\square}_{i=1}^{n} \rho_i(X).$$

Some classic references (mainly on convex objectives): Barrieu-El Karoui'05, Jouini-Schachermayer-Touzi'08, Delbaen'12, Rüschendorf'13

Homogeneous and Heterogeneous Beliefs

Homogeneous beliefs: each agent has the same probability measure $\mathbb{P}$

- central coordination (e.g. fragmentation of a firm)
- public credit rating
- standard approach

Heterogeneous beliefs: agent i has a private probability measure Q_i

- individual management
- information asymmetry
- internal model approach

Range-Value-at-Risk (RVaR)

A two-parameter family of risk measures, for $\alpha, \beta \in \mathbb{R}_+ := [0, \infty)$,

$$\mathrm{RVaR}_{\alpha,\beta}(X) = \begin{cases} \frac{1}{\beta}\int_\alpha^{\alpha+\beta} \mathrm{VaR}_\gamma(X)\mathrm{d}\gamma & \beta > 0, \\ \mathrm{VaR}_\alpha(X) & \beta = 0, \end{cases} \quad X \in \mathcal{X},$$

where and from now on $\mathcal{X} = L^1$ ($\mathbb{P}$-integrable random variables).

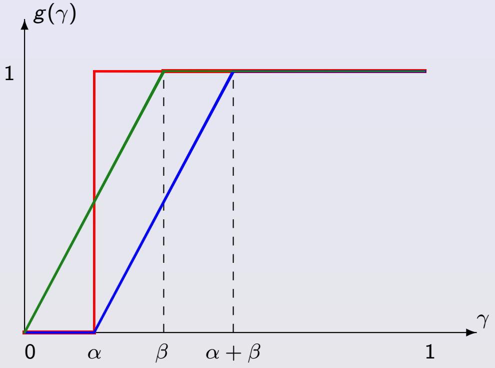
RVaR bridges the gap between VaR and ES:

- $\mathrm{VaR}_\alpha(X) = \mathrm{RVaR}_{\alpha,0}(X) = \lim_{\beta \to 0^+} \mathrm{RVaR}_{\alpha,\beta}(X), \ \alpha \in \mathbb{R}_+.$
- $\mathrm{ES}_\beta(X) = \mathrm{RVaR}_{0,\beta}(X) = \lim_{\alpha \to 0^+} \mathrm{RVaR}_{\alpha,\beta}(X), \ \beta \in (0,1).$

Practically:

$$\mathrm{RVaR}_{\alpha,\beta}(X) \underset{(F_X \text{ cont.})}{=} \mathbb{E}[X | \mathrm{VaR}_{\alpha+\beta}(X) < X \leq \mathrm{VaR}_\alpha(X)].$$

Range-Value-at-Risk (RVaR)



Distortion functions of VaR_α (red), ES_β (green) and $\mathrm{RVaR}_{\alpha,\beta}$ (blue)

in the form of $\int_0^1 \mathrm{VaR}_\gamma(X)\mathrm{d}g(\gamma)$

Range-Value-at-Risk (RVaR)

For $\alpha, \beta > 0$ and $\alpha + \beta < 1$,

- $\mathrm{RVaR}_{\alpha,\beta}$ is a distortion risk measure (Yaari's dual utility): monetary, comonotonic additive, positive homogeneous, ...
  non-convex
- $\mathrm{RVaR}_{\alpha,\beta}$ is robust (continuous wrt weak convergence)
  - VaR_α and ES_β are not continuous wrt weak convergence (VaR_α is "almost continuous")

On risk measures robustness issues: e.g. Cont-Deguest-Scandolo'10, Kou-Peng-Heyde'13, Krätschmer-Schied-Zähle'14,'17, Embrechts-Wang-W.'15

Range-Value-at-Risk (RVaR)

Von Neumann-Morgenstein expected utility

$$F_X \mapsto \mathbb{E}[u(X)] = \int_{\mathbb{R}} u(x) \mathrm{d} F_X(x)$$

- linear in the distribution function F_X

Yaari's dual utility (Yaari'87)

$$F_X \mapsto \int_{\mathbb{R}} x \mathrm{d} h(F_X(x)) = \int_0^1 F_X^{-1}(t) \mathrm{d} h(t)$$

- linear in the quantile function F_X^{-1}

Quantile Inequalities

Theorem 1

For any $X_1, \dots, X_n \in \mathcal{X}$ *and* $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{R}_+$, *we have*

$$\mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}\left(\sum_{i=1}^n X_i\right) \le \sum_{i=1}^n \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i).$$

- $\bigvee_{i=1}^n \beta_i = \max\{\beta_1, \dots, \beta_n\}$
- RVaR enjoys a special type of subadditivity
  - $+$ and $\vee$ are both popular additive operations on $\mathbb{R}$

Quantile Inequalities

Corollary: Taking $\beta_1 = \cdots = \beta_n = 0$,

$$\mathrm{VaR}_{\sum_{i=1}^n \alpha_i}\left(\sum_{i=1}^n X_i\right) \le \sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(X_i).$$

(Also valid for $\mathcal{X} = L^0$)

"Corollary": Taking $\alpha_1 = \cdots = \alpha_n = 0$ and $\beta_1 = \cdots = \beta_n = \beta$,

$$\mathrm{ES}_{\beta}\left(\sum_{i=1}^n X_i\right) \le \sum_{i=1}^n \mathrm{ES}_{\beta}(X_i).$$

(Classic subadditivity of ES)

Seven proofs of subadditivity of ES: Embrechts-W.'15

Progress of the Talk

Inf-convolution of RVaR

Theorem 2

For $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{R}_+$ *and* $X \in \mathcal{X}$*, we have*

$$\mathop{\square}_{i=1}^{n} \mathrm{RVaR}_{\alpha_i,\beta_i}(X) = \mathrm{RVaR}_{\sum_{i=1}^n \alpha_i, \bigvee_{i=1}^n \beta_i}(X).$$

Proof of Theorem:

- "$\leq$": by construction; "$\geq$": by the previous RVaR inequality

Remark:

- $\left(\{\mathrm{RVaR}_{\alpha,\beta}\}_{(\alpha,\beta)\in\mathbb{R}_+^2}, \square\right)$ is an Abelian semigroup, isomorphic to the monoid $(\mathbb{R}_+^2, (+, \vee))$.

Inf-convolution of RVaR

Corollary: For $\alpha_1, \ldots, \alpha_n \geq 0$,

$$\mathop{\square}_{i=1}^{n} \mathrm{VaR}_{\alpha_i} = \mathrm{VaR}_{\sum_{i=1}^{n} \alpha_i}.$$

Corollary: For $\alpha, \beta \geq 0$,

$$\mathrm{VaR}_{\alpha} \square \mathrm{ES}_{\beta} = \mathrm{RVaR}_{\alpha,\beta}.$$

Risk Sharing with Homogeneous Beliefs

Setup

- The objective of agent i is $\rho_i = \mathrm{RVaR}_{\alpha_i,\beta_i}$
- Assume $\alpha_i < 1$ and $\alpha_i + \beta_i \leq 1$
- Total risk is $X \in \mathcal{X}$
- Notation: $\alpha = \sum_{i=1}^n \alpha_i$ and $\beta = \bigvee_{i=1}^n \beta_i$

Risk Sharing with Homogeneous Beliefs

Theorem 3

A Pareto-optimal allocation of X exists if and only if one of the following holds:

(A1) $\alpha = \beta = 0$ *and X is bounded from above;*

(A2) $0 < \alpha + \beta < 1$; $\leftarrow$ most relevant

(A3) $\alpha + \beta = 1$, $\beta > 0$ *and X is bounded from below;*

(A4) $\alpha + \beta = 1$ *and there exists $i \in \{1, \ldots, n\}$ such that $\alpha_i = \alpha$ and $\beta_i = \beta$.*

Optimal Allocations

An optimal allocation

Assume (A2) and let j be such that $\beta_j = \beta$. A Pareto-optimal allocation $(X_1^*, \dots, X_n^*)$ of X is

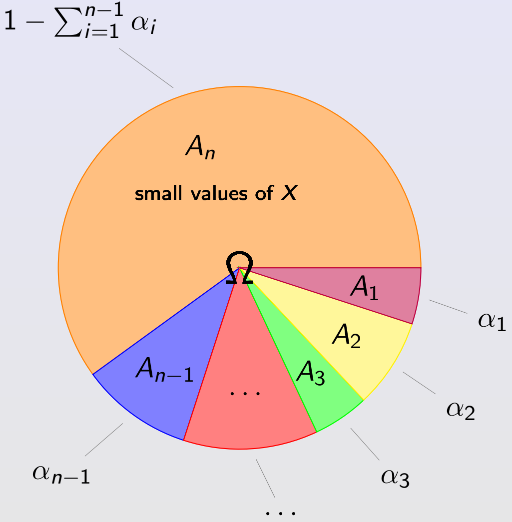
$$X_i^* = (X - z)\mathbb{1}_{A_i} + \frac{z}{n}, \ i = 1, \dots, n.$$

where $z \in (-\infty, \mathrm{VaR}_\alpha(X)]$ and $(A_1, \dots, A_n)$ is a partition of Ω with $\mathbb{P}(A_i) = \alpha_i$ for $i \neq j$ such that $X(\omega) \geq X(\omega')$ for $\omega \in \bigcup_{i \neq j} A_i$ and $\omega' \in A_j$.

If $\mathrm{VaR}_\alpha(X) \geq 0$, an optimal allocation can be chosen as

$$X_i^* = X\mathbb{1}_{A_i}, \ i = 1, \dots, n. \qquad (\star)$$

Optimal Allocations



- $X_i^* = X\mathbb{1}_{A_i}$, $\beta_n = \beta$
- For $i = 1, \dots, n-1$, $\mathbb{P}(X_i^* > 0) = \alpha_i \Rightarrow \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i^*) = 0$; "Agent i walks away thinking the risk is free"
- All the remaining risk is taken by agent n (most tolerant)
- "Neglecting the tail risk" ($\alpha_i > 0$) vs "capturing the tail risk" ($\alpha_i = 0$)

Sharp Contrasts to Classic Framework

Sharp contrast I

- For classic utility-based agents, a Pareto-optimal allocation is to divide the risk X "proportionally"
- For quantile-based agents, a Pareto-optimal allocation is to divide the space Ω "proportionally"

"when von Neumann-Morgenstern meets Yaari"

Uniqueness of the Form

Pareto-optimal allocations

- generally not unique
- large degree of freedom because RVaR ignores part of the risk
- all optimal allocations can be characterized; all involve dividing Ω among agents with $\alpha_i > 0$

Competitive Equilibria

Question

Can the optimal allocation $(\star)$ be achieved in a competitive market?

- Agent i has an initial risk $\xi_i \in \mathcal{X}$. Assume $X = \sum_{i=1}^n \xi_i \geq 0$.
- $\psi \geq 0$: the pricing rule (pricing density) $\leftarrow$ market output
- One is allowed to make side-payments s_i
- No short selling or over-taking: $0 \leq X_i \leq X$ $\leftarrow$ non-trivial
- For a given ψ, agent i aims to

$$\begin{aligned} \text{minimize} \quad & \mathrm{RVaR}_{\alpha_i,\beta_i}(X_i) + s_i \quad \text{over } X_i \in \mathcal{X} \\ \text{subject to} \quad & s_i + \mathbb{E}[\psi X_i] \geq \mathbb{E}[\psi \xi_i], \\ & 0 \leq X_i \leq X, \ s_i \in \mathbb{R}. \end{aligned} \tag{E}$$

Competitive Equilibria

Competitive equilibrium

A pair $(\psi^*, (X_1^*, \ldots, X_n^*))$ is a competitive equilibrium if X_i^* solves (E) and $X_1^* + \cdots + X_n^* = X$.

- ψ^*: equilibrium price
- $(X_1^*, \ldots, X_n^*)$: equilibrium allocation
- An equilibrium allocation is necessarily Pareto-optimal; thus First Fundamental Theorem of Welfare Economics ("the Invisible Hand") holds

e.g. Arrow-Debreu'54, Föllmer-Schied'16

Competitive Equilibria

Theorem 4

Assume (A2) and $X \geq 0$ *with* $\mathbb{P}(X > 0) \leq \max\{\bigwedge_{i=1}^n \alpha_i + \beta, \alpha\}$. *Let* $(X_1^*, \ldots, X_n^*)$ *be given by* $(\star)$*, and*

$$\psi^* = \min\left\{\frac{x}{X\beta}, \frac{1}{\beta}\right\} \mathbb{1}_{\{X\beta > 0\}} \quad \textit{where } x = \mathrm{VaR}_\alpha(X).$$

Then $(\psi^*, (X_1^*, \ldots, X_n^*))$ *is a competitive equilibrium.*

- We assumed $\mathbb{P}(X > 0)$ is not too large - e.g. credit portfolio
- Second Fundamental Theorem of Welfare Economics
- The pricing rule ψ^* is a reciprocal function of X pasted to a constant

Sharp Contrasts to Classic Framework

Sharp contrast II

- For classic utility-based agents, Pareto-optimal allocations are generally equilibrium allocations
- For quantile-based agents, Pareto-optimal allocations are not necessarily equilibrium allocations

Sharp Contrasts to Classic Framework

Sharp contrast III

- For classic utility-based agents, Pareto-optimal and equilibrium allocations are generally comonotonic (positive dependence)
- For quantile-based agents, Pareto-optimal and equilibrium allocations are generally mutually exclusive (negative dependence)

Sharp Contrasts to Classic Framework

Sharp contrast IV

- For classic utility-based agents, given initial risks, an equilibrium allocation is often unique
- For quantile-based agents, given initial risks, equilibrium allocations are not unique

Progress of the Talk

Heterogeneous Beliefs

- Agent i has a belief Q_i about future randomness
- Each agent uses an ES, namely $\rho_i = \mathrm{ES}_{\alpha_i}^{Q_i}$, $\alpha_i \in (0,1)$.
- Take $\mathcal{X}$ as the set of bounded random variables

Proposition

For $X \in \mathcal{X}$, $\square_{i=1}^{n} \mathrm{ES}_{\alpha_i}^{Q_i}(X) = \sup\{\mathbb{E}^Q[X] : Q \in \overline{\mathcal{Q}}\}$, where

$$\overline{\mathcal{Q}} = \left\{ Q \in \mathcal{P} : \frac{\mathrm{d}Q}{\mathrm{d}Q_i} \le \frac{1}{\alpha_i},\ i = 1, \dots, n \right\}.$$

Moreover, a Pareto-optimal allocation exists iff $\overline{\mathcal{Q}}$ is non-empty.

Classic convex analysis; e.g. Barrieu-El Karoui'05

ES Agents with Heterogeneous Beliefs

Let $Q = \frac{1}{n}\sum_{i=1}^n Q_i$,

$$B_j = \left\{ \frac{1}{\alpha_j}\frac{\mathrm{d}Q_j}{\mathrm{d}Q} = \bigwedge_{i=1}^n \frac{1}{\alpha_i}\frac{\mathrm{d}Q_i}{\mathrm{d}Q} \right\},\ j = 1, \dots, n,$$

and

$$y^* = \inf\left\{ x \in \mathbb{R} : \sum_{i=1}^n \frac{1}{\alpha_i} Q_i(X > x, B_i) < 1 \right\}.$$

- B_j is the set of random outcomes which is the least likely (weighted by α_j) according to agent j relative to other agents.
- If $B_j = \Omega$, then y^* is the Q_j-right-quantile of X at level α_j.

ES Agents with Heterogeneous Beliefs

Theorem 5

Assume $\overline{\mathcal{Q}}$ is non-empty and $B_1, \dots, B_n$ are disjoint. A Pareto-optimal allocation $(X_1^*, \dots, X_n^*)$ of $X \in \mathcal{X}$ is given by

$$X_i^* = (X - y^*)\mathbb{1}_{B_i} + \frac{y^*}{n}, \quad i = 1, \dots, n.$$

- the Pareto-optimal allocation is unique on the set $\{X > y^*\}$ up to constant shifts
- Pareto-optimal allocation $\Leftrightarrow$ equilibrium allocation
- the equilibrium price is unique for a fixed X
- generalizes the classic optimization property of ES

Sharp Contrasts to Classic Framework

Sharp contrast V

- For classic utility-based agents, if their beliefs are not equivalent, then no Pareto-optimal allocations or equilibria exist
- For quantile-based agents, Pareto-optimal allocations and equilibria exist even if beliefs are not equivalent

Mixed VaR, ES and RVaR Agents

For VaR agents, mixed VaR/ES agents, or RVaR agents, a Pareto-optimal allocation $(X_1^*, \dots, X_n^*)$ of $X \in \mathcal{X}$ is given by

$$X_i^* = (X - x^*)\mathbb{1}_{A_i} + \frac{x^*}{n}, \quad i = 1, \dots, n.$$

for some partition $(A_1, \dots, A_n)$ and $x^* \in \mathbb{R}$.

- Similar forms to the case of ES agents/homogeneous beliefs
- Analytical determination of $(A_1, \dots, A_n)$ and x^* is unavailable
- Competitive equilibrium without trading constraints often does not exist, unless all agents are ES agents

Progress of the Talk

Robustness

Simplistic setup: Assume homogeneous beliefs

- Robustness: small model misspecification does not ruin the optimality of an allocation
- Treat allocations as functions of X (assumed to have finitely many discontinuity points)

Robust allocations

For n risk measures $\rho_1, \dots, \rho_n$, a (pseudo-)metric π on $\mathcal{X}$ and $X \in \mathcal{X}$, an allocation $(f_1(X), \dots, f_n(X)) \in \mathbb{A}_n(X)$ is π-robust if $\sum_{i=1}^n (\rho_i \circ f_i)$ is continuous at X with respect to π.

- metrics: e.g. L^1, L^∞, Wasserstein, $\pi_W =$ Lévy, ...

Robustness

RVaR can be arranged into three categories:

- ES: $\alpha = 0$
- true VaR: $\beta = 0$, $\alpha > 0$
- true RVaR: $\beta > 0$, $\alpha > 0$

Robustness

Theorem 6

Assume (A2) and $X \in L^\infty$ has continuous cdf and inverse cdf.

(i) *There exists an L^1- or L^∞-robust optimal allocation of X if and only if $\beta_1, \dots, \beta_n > 0$ (all ES or true RVaR).*

(ii) *There exists a π_W-robust optimal allocation of X if and only if $\beta_1, \dots, \beta_n > 0$ and $\alpha_i > 0$ for some $i = 1, \dots, n$ (all ES or true RVaR, and at least one true RVaR).*

- No true VaR is allowed for robust optimal allocations
- True RVaR is the most robust

Progress of the Talk

Implications

Is risk positions of type $(\star)$ realistic?

"*Starting in 2006, the CDO group at UBS noticed that their risk-management systems treated AAA securities as essentially riskless even though they yielded a premium (the proverbial free lunch). So they decided to hold onto them rather than sell them.*"

- From Feb 06 to Sep 07, UBS increased investment in AAA-rated CDOs by more than 10 times; many large banks did the same.
  - Take a risk of big loss with small probability, $X_i = X\mathbb{1}_{A_i}$
  - Treat it as free money - profit
  - Financial crisis?

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quoted from Acharya-Cooley-Richardson-Walter'10

Implications

Some issues with VaR as the regulatory risk measure:

- Recall that $\square_{i=1}^{n} \mathrm{VaR}_{\alpha}(X) = \mathrm{VaR}_{n\alpha}(X)$ (= total capital).

$$\mathrm{VaR}_{n\alpha}(X) \ll \mathrm{VaR}_{\alpha}(X) \quad \text{typically}$$

(i) A firm has incentives to split its risk: regulatory arbitrage

(ii) Sharing is not robust: insolvency under model uncertainty

(iii) Total capital at optimum is much smaller than $\mathrm{VaR}_{\alpha}(X)$: insufficient capital for the whole economy

(iv) Firms treat big losses with small probability as risk-free: problematic risk management

Implications

Implications

- The implementation of ES generally solves (i)-(iv)
- In case of non-equivalent heterogeneous beliefs, (iv) might still be problematic. This suggests
  - internal models need to be carefully monitored
  - for some risks the standard approach might be necessary
- It is the regulator's responsibility to prevent something like $(\star)$ to happen in a systemic scale

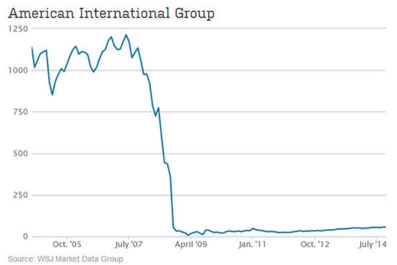
Basel III

BCBS Standards - Minimum capital requirements for Market Risk Jan 2016, page 1. *Executive Summary*:

"... A shift from Value-at-Risk (VaR) to an Expected Shortfall (ES) measure of risk under stress. Use of ES will help to ensure a more prudent capture of "tail risk" and capital adequacy during periods of significant financial market stress."

"... A revised internal models-approach (IMA). The new approach introduces a more rigorous model approval process that enables supervisors to remove internal modelling permission for individual trading desks, ..."

Thank You



CEO of AIG Financial Products, August 2007:

"*It is hard for us, without being flippant, to even see a scenario within any kind of realm of reason that would see us losing one dollar in any of those transactions.*"

- AIGFP sold protection on super-senior tranches of CDOs
- $180 billion bailout from the federal government in September 2008

Thank you

Thank you for your kind attention

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