PELVE: Probability Equivalent Level of VaR and ES

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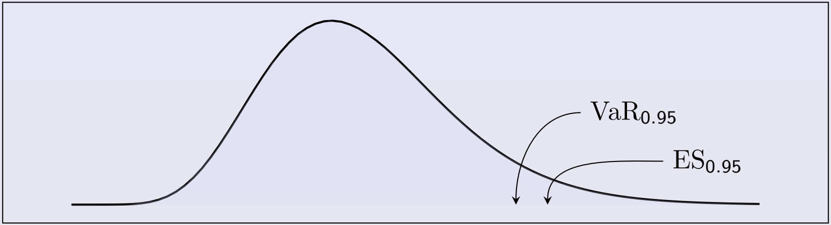
Agenda

1. Background
2. PELVE: A tale of two risk measures
3. Theoretical properties
4. Parametric and heavy tailed distributions
5. Non-parametric estimation
6. Empirical analysis
7. Concluding remarks

Based on joint work with Hengxin Li (UC Berkeley)

VaR and ES



Value-at-Risk (VaR), $p \in (0,1)$

$\mathrm{VaR}_p : L^0 \to \mathbb{R}$,

$$\mathrm{VaR}_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) \ge p\}.$$

(left-quantile)

Expected Shortfall (ES), $p \in (0,1)$

$\mathrm{ES}_p : L^1 \to \mathbb{R}$,

$$\mathrm{ES}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q(X)\mathrm{d}q$$

(also: TVaR/CVaR/AVaR)

FRTB

The Basel Committee on Banking Supervision

Fundamental Review of the Trading Book (FRTB), live Jan 2019

- Widely discussed since 2012
- Planned implementation
  - March 2021 (most Europe)
  - Jan 2022 (North America, some of East Asia)

FRTB

$$\mathrm{VaR}_{0.99} \implies \mathrm{ES}_{0.975}$$

- $\mathrm{VaR}_{0.99}$ is replaced by $\mathrm{ES}_{0.975}$ as the standard risk measure for market risk in the internal model approach
- 10-day portfolio loss forecast
- In a survey in 2015, 2/3 of banks reported higher capital charge under the (back-then) proposed FTRB
- Is there a general relationship between $\mathrm{VaR}_{0.99}$ and $\mathrm{ES}_{0.975}$?

A tiny portion of literature on VaR and ES

ES is coherent and VaR is not

- Artzner-Delbaen-Eber-Heath'99 MF; Acerbi-Tasche'02 JBF

VaR is elicitable and ES is not

- Gneiting'11 JASA

Axiomatic characterizations

- ES: W.-Zitikis'20 MS
- VaR: Chambers'09 MF; Kou-Peng'16 OR; He-Peng'18 OR; Liu-W.'20 MOR

Optimization properties

- Rockafellar-Uryasev'00/02 JR/JBF; Gaivoronski-Pflug'05 JR

Statistical inference and time series

- Scaillet'04 MF; Engle-Manganelli'04 JBES; Chen'08 JFEc

Risk aggregation

- Embrechts-Puccetti-Rüschendorf'13 JBF; Embrechts-Wang-W.'15 FS

A tiny portion of literature on VaR and ES

Investment and portfolio management

- Basak-Shapiro'01 RFS; Krokhmal-Palmquist-Uryasev'02 JR; Natarajan-Pachamanova-Sim'08 MS; Adrian-Shin'14 RFS

Capital allocation

- Dhaene-Goovaerts-Kaas'03 IME; Kalkbrener'05 MF; Dhaene-Tsanakas-Valdez-Vanduffel'12 JRI; Asimit-Peng-W.-Yu'19 MF

Insurance, reinsurance, and risk sharing

- Cai-Tan'07 ASTIN; Cai-Tan-Weng-Zhang'08 IME; Chi-Tan'11 ASTIN; Embrechts-Liu-W.'18 OR; Weber'18 IME

Systemic risk, CoVaR/CoES

- Acharya-Engle-Richardson'12 AER; Adrian-Brunnermeier'16 AER

Forecasting and backtesting

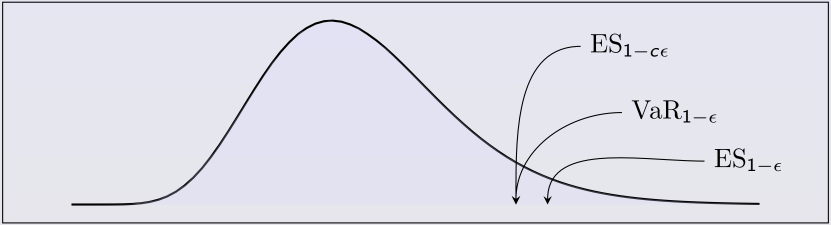
- Fissler-Ziegel'16 AoS; Du-Escanciano'17 MS; Kratz-Lok-McNeil'18 JBF

Progress

Definition of PELVE

Given a random loss X, how do we compare $\mathrm{VaR}_{0.99}(X)$ and $\mathrm{ES}_{0.975}(X)$, or generally $\mathrm{VaR}_p(X)$ and $\mathrm{ES}_q(X)$?

- A number $c \in [1, 1/\epsilon]$ such that $\mathrm{ES}_{1-c\epsilon}(X) = \mathrm{VaR}_{1-\epsilon}(X)$



- For $\epsilon = 0.01 \longleftrightarrow \mathrm{VaR}_{0.99}$ in FRTB:
  - $c > 2.5 \Rightarrow \mathrm{ES}_{0.975} > \mathrm{VaR}_{0.99} \Rightarrow$ capital increases
  - $c \approx 2.5 \Rightarrow \mathrm{ES}_{0.975} \approx \mathrm{VaR}_{0.99} \Rightarrow$ little or no change in capital
  - $c < 2.5 \Rightarrow \mathrm{ES}_{0.975} < \mathrm{VaR}_{0.99} \Rightarrow$ capital decreases

Definition of PELVE

Definition 1

For $\epsilon \in (0,1)$, the Probability Equivalent Level of VaR-ES (PELVE) is defined as $\Pi_\epsilon : L^1 \to \mathbb{R}$

$$\Pi_\epsilon(X) = \inf\left\{c \in [1, 1/\epsilon] : \mathrm{ES}_{1-c\epsilon}(X) \leq \mathrm{VaR}_{1-\epsilon}(X)\right\}$$

with the convention $\inf(\varnothing) = \infty$.

- Always well defined
- Almost always $\mathrm{ES}_{1-c\epsilon}(X) = \mathrm{VaR}_{1-\epsilon}(X)$
- $\Pi_\epsilon(X) < \infty$ if ϵ is small

Typical values of PELVE

ϵ	Dirac	U	N	Exp	LN(σ^2)		
					0.04	0.25	1
0.100	1.00	2.00	2.46	2.72	2.56	2.76	3.23
0.050			2.51		2.61	2.79	3.19
0.010			2.58		2.66	2.81	3.13
0.005			2.59		2.67	2.81	3.10

ϵ	t(ν)			Pareto(α)		
	2	10	30	2	4	10
0.100	3.60	2.58	2.49	4.00	3.16	2.87
0.050	3.80	2.65	2.55			
0.010	3.96	2.74	2.63			
0.005	3.98	2.77	2.65			

Typical values of PELVE

Quick observations on Π_ϵ:

- Common range $[2, 4]$
- $\Pi_\epsilon(X) = 1 \Leftarrow$ point-mass
- $\Pi_\epsilon(X) = 4 \Leftarrow$ Pareto(2), infinite variance
- $\Pi_\epsilon(X) \approx 2.5 \Leftarrow$ normal
  - $\mathrm{VaR}_{0.99} \approx \mathrm{ES}_{0.975}$ in FRTB for normal X
- Relatively stable across different ϵ for the same distribution
  - Constant in ϵ for degenerate, uniform, exponential and Pareto
- higher value $\Leftarrow$ heavy tails
- lower value $\Leftarrow$ light tails
- Can be used as a measure of variability

Progress

Existence and uniqueness

Assumption 1 (Existence & uniqueness condition)

$\mathbb{E}[X] < \mathrm{VaR}_{1-\epsilon}(X)$ and $\mathrm{VaR}_p(X)$ is not a constant for $p \in [1-\epsilon, 1)$.

Proposition 1 (Existence & uniqueness of PELVE)

Under Assumption 1, there exists a unique $c \in [1, 1/\epsilon]$ *such that*

$$\mathrm{ES}_{1-c\epsilon}(X) = \mathrm{VaR}_{1-\epsilon}(X).$$

Theoretical features of PELVE

Features

- Location-scale invariance
  - location-scale free risk assessment (e.g., Sharp ratio)
- Monotone in convex transformation
  - $X \sim \mathrm{N}(0,1)$ vs $e^X \sim \mathrm{LN}(0,1)$
  - $X \sim \mathrm{Pareto}(4)$ vs $X^2 \sim \mathrm{Pareto}(2)$
- Betweenness
  - quasi-convexity and quasi-concavity wrt quantile-mixture
  - combining two comonotonic losses does not give a PELVE value beyond the worse one or below the better one
  - quasi-convex programming

Theoretical features of PELVE

Theorem 1

Suppose that $X \in L^1$, $\epsilon \in (0,1)$ *and* $\mathbb{E}[X] \le \mathrm{VaR}_{1-\epsilon}(X)$.

(i) *For all* $\lambda > 0$ *and* $a \in \mathbb{R}$, $\Pi_\epsilon(\lambda X + a) = \Pi_\epsilon(X)$.

(ii) $\Pi_\epsilon(f(X)) \le \Pi_\epsilon(X)$ *for all increasing concave functions* $f : \mathbb{R} \to \mathbb{R}$ *with* $f(X) \in L^1$.

(iii) $\Pi_\epsilon(g(X)) \ge \Pi_\epsilon(X)$ *for all strictly increasing convex functions* $g : \mathbb{R} \to \mathbb{R}$ *with* $g(X) \in L^1$.

(iv) *For all comonotonic* $Y, Z \in L^1$, $\Pi_\epsilon(Y + Z)$ *is between* $\Pi_\epsilon(Y)$ *and* $\Pi_\epsilon(Z)$.

(i/iv) $\Rightarrow \Pi_\epsilon(Y) \wedge \Pi_\epsilon(Z) \le \Pi_\epsilon(\lambda Y + (1-\lambda)Z) \le \Pi_\epsilon(Y) \vee \Pi_\epsilon(Z), \quad \forall \lambda \in [0,1].$

Examples

Example 1 (PELVE for time-series models)

Risk measure forecast is usually done via conditional models. Consider an AR-GARCH type of time-series for risk factors

$$X_t = \mu_t + \sigma_t Z_t, \quad t \in \mathbb{Z},$$

where μ_t and σ_t are the conditional mean and standard deviation given a σ-field $\mathcal{F}_{t-1}$, and Z_t is independent of $\mathcal{F}_{t-1}$. Then

$$\Pi_\epsilon(X_t|\mathcal{F}_{t-1}) = \Pi_\epsilon(Z_t).$$

Examples

Example 2 (Reducing PELVE with options)

- Losses from an asset and a European call or put (same maturity)

$$X_A\text{: asset;}\quad X_C\text{: call;}\quad X_{AC}\text{: asset+call;}\quad X_{AP}\text{: asset+put}$$

- Put-call parity: $X_C + c = X_{AP}$ for some $c \in \mathbb{R}$
- All of X_C, X_{AC}, and X_{AP} are increasing concave functions of X_A
- X_{AP} is an increasing concave function of X_{AC}

Theorem 1 (ii) implies:

$$\Pi_\epsilon(X_C) = \Pi_\epsilon(X_{AP}) \le \Pi_\epsilon(X_{AC}) \le \Pi_\epsilon(X_A)$$

Consistent with intuition & no model assumption

Examples

Example 3 (Comparison of PELVE using quasi-convexity/concavity)

Following the previous example:

- X_A and X_C are comonotonic
- Theorem 1 (iv) betweenness $\Rightarrow$

$$\Pi_\epsilon(X_C) \wedge \Pi_\epsilon(X_A) \le \Pi_\epsilon(X_{AC}) \le \Pi_\epsilon(X_C) \vee \Pi_\epsilon(X_A)$$

- Remains true if the call option in X_C and X_{AC} is replaced by any other payoffs increasing in the asset price
- Use $\Pi_\epsilon(X_C) \le \Pi_\epsilon(X_A) \Rightarrow$ the result in the previous example

Examples

Example 4 (Linear loss and log-loss)

For an asset with price X_t at time $t = 0, 1, \dots$ (e.g. daily prices)

- linear return: $R_t = X_t/X_{t-1} - 1$
- log-return: $r_t = \log(X_t/X_{t-1})$
- linear loss (negative return): $-R_t = 1 - X_t/X_{t-1}$
- log-loss (negative log-return): $-r_t = -\log(X_t/X_{t-1})$
- $y \mapsto -\log(1-y)$ is strictly increasing and convex
- Theorem 1 (iii) $\Rightarrow \Pi_\epsilon(-r_t) \geq \Pi_\epsilon(-R_t)$
- Using log-loss $\Rightarrow$ a (slightly) higher PELVE than linear loss

Consistency

Assumption 2 (Continuous quantile)

$F_X^{-1}(q)$ is continuous at $q = 1 - \epsilon$.

Assumption 3 (Uniform integrability)

$\{X_n\}_{n\in\mathbb{N}}$ is uniformly integrable.

Theorem 2

Suppose that $\{X_n\}_{n\in\mathbb{N}} \subset L^1$, $X \in L^1$ and $\epsilon \in (0,1)$ satisfy Assumptions 1-3 and $X_n \to X$ in distribution as $n \to \infty$. Then $\Pi_\epsilon(X_n) \to \Pi_\epsilon(X)$ as $n \to \infty$.

Consistency

Corollary 1

Suppose that $X, X_1, X_2, \dots \in L^1$ *are iid,* $\epsilon \in (0,1)$*, and Assumptions 1-2 hold. Then* $\widehat{\Pi}_\epsilon(n) \to \Pi_\epsilon(X)$ *as* $n \to \infty$*, where* $\widehat{\Pi}_\epsilon(n)$ *is the* ϵ*-PELVE of the empirical distribution based on sample* $X_1, \dots, X_n$*.*

Diversification effect

VaR and ES: which of them rewards diversification more?

- Elliptical family: the same
- Generally: unclear

Consider a general setting

- $X_1, \ldots, X_n$: individual losses with finite variances
- $S_n = \sum_{i=1}^n w_i X_i$ for some weights $w_1, \ldots, w_n \geq 0$
- Allow $n \to \infty$

Diversification effect

- Assume CLT: $(S_n - a_n)/b_n \to N \sim \mathrm{N}(0,1)$ for some $a_n \in \mathbb{R}$ and $b_n > 0$
- Theorems 1-2:

$$\Pi_\epsilon(S_n) = \Pi_\epsilon\left(\frac{S_n - a_n}{b_n}\right) \to \Pi_\epsilon(N)$$

- most asset PELVE $> \Pi_\epsilon(N)$, e.g., $\Pi_{0.01}(N) = 2.58$
- Aggregate PELVE is likely smaller than individual PELVE
- True even without CLT

Diversification effect

Example 5 (Diversification ratio)

- Diversification ratio: for a risk measure ρ,

$$\Delta(\rho) = \frac{\rho(S_n)}{\sum_{i=1}^n w_i \rho(X_i)} = \frac{\text{with diversification}}{\text{without diversification}}$$

- $c \geq 1$: $\sum_{i=1}^n w_i \mathrm{VaR}_{1-\epsilon}(X_i) = \sum_{i=1}^n w_i \mathrm{ES}_{1-c\epsilon}(X_i)$
- $c' = \Pi_\epsilon(S_n)$: $\mathrm{VaR}_{1-\epsilon}(S_n) = \mathrm{ES}_{1-c'\epsilon}(S_n)$
- Assume $c > \Pi_\epsilon(N)$ (commonly observed)
- CLT: $c' < c$ for n large $\Rightarrow \mathrm{ES}_{1-c\epsilon}(S_n) < \mathrm{VaR}_{1-\epsilon}(S_n)$
- $\Delta(\mathrm{ES}_{1-c\epsilon}) < \Delta(\mathrm{VaR}_{1-\epsilon})$ and $\Delta(\mathrm{ES}_{1-c'\epsilon}) < \Delta(\mathrm{VaR}_{1-\epsilon})$

Remark. The result will be flipped if we assume $c < \Pi_\epsilon(N)$, e.g., uniform individual losses (nothing to do with the coherence of ES).

Progress

PELVE of parametric models

Example 6 (Uniform/Exponential/Pareto distributions)

- $X \sim$ uniform: $\Pi_\epsilon(X) = 2$ for $\epsilon \le 1/2$.
- $X \sim$ exponential: $\Pi_\epsilon(X) = e$ for $\epsilon \le 1/e$.
- $X \sim$ Pareto(α), $\alpha > 1$ (i.e., $\mathbb{P}(X > x) = x^{-\alpha},\ x \ge 1$):

$$\Pi_\epsilon(X) = \left(\frac{\alpha}{\alpha - 1}\right)^{\alpha} \quad \text{for} \quad \epsilon \le \left(\frac{\alpha}{\alpha - 1}\right)^{-\alpha}.$$

- For Pareto(α), $\alpha \mapsto (\frac{\alpha}{\alpha-1})^\alpha$ is decreasing in α and

$$\Pi_\epsilon(X) \ge \lim_{\alpha\to\infty} \left(\frac{\alpha}{\alpha - 1}\right)^{\alpha} = e \approx 2.718.$$

- e is a threshold for heavy and light tails

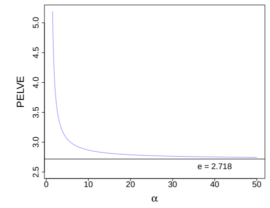
PELVE of parametric models

Example 7 (Normal/t/log-normal distributions)

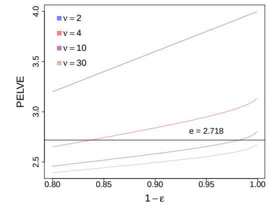
Π_ϵ has no explicit formula. VaR and ES have explicit formulas.

- Normal: ≈ 2.5
- t(ν): > 2.72 for $\epsilon \approx 0$ and ν not too large
- log-normal: various possibilities

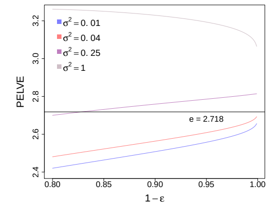
PELVE of parametric models ($\epsilon = 0.01$ if unspecified)



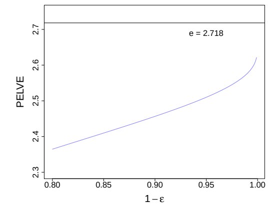
(a) $\mathrm{Pareto}(\alpha)$, $\alpha \in [1.5, 50]$



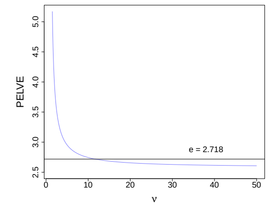
(b) $\mathrm{t}(\nu)$, $\epsilon \in (0, 0.2]$



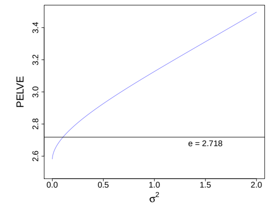
(c) $\mathrm{LN}(\sigma^2)$, $\epsilon \in (0, 0.2]$



(d) $\mathrm{N}(\mu, \sigma^2)$, $\epsilon \in (0, 0.2]$



(e) $\mathrm{t}(\nu)$, $\nu \in [1.5, 50]$



(f) $\mathrm{LN}(\sigma^2)$, $\sigma^2 \in (0, 2]$

PELVE of regularly varying distributions

A survival function $\overline{F}$ is regularly varying (RV) with a tail index $\alpha > 0$, denoted by $f \in \mathrm{RV}_{-\alpha}$, if

$$\lim_{x\to\infty} \frac{\overline{F}(tx)}{\overline{F}(x)} = t^{-\alpha}, \quad \text{for all } t > 0.$$

- e.g., Pareto(α), t(α).

Theorem 3

Suppose that the function $\overline{F}(x) = \mathbb{P}(X > x)$ is $\mathrm{RV}_{-\alpha}$, $\alpha > 1$. Then

$$\lim_{\epsilon \downarrow 0} \Pi_\epsilon(X) = \left(\frac{\alpha}{\alpha - 1}\right)^{\alpha}.$$

PELVE vs tail index

Comparing PELVE and the tail index:

- Both location-scale invariant
- Assumptions
  - The tail index requires regular variation (difficult to check!)
  - PELVE only requires finite mean, well defined for bounded rvs
- Estimation
  - The tail index needs an ad-hoc threshold (e.g., Hill estimator)
  - PELVE needs only ϵ which has a physical meaning
- Interpretation
  - The tail index remains the same for the average of iid risks
  - PELVE can reflect diversification (e.g., CLT)
  - $\Pi_{0.01}$ has a meaning for banking regulation (VaR $\Rightarrow$ ES)

PELVE vs tail index

Proposition 2

Suppose that the function $\overline{F}(x) = \mathbb{P}(X > x)$ *is* $\mathrm{RV}_{-\alpha}$, $\alpha > 1$. *Then, for* $c > 1$,

$$\lim_{\epsilon \downarrow 0} \frac{\mathrm{ES}_{1-c\epsilon}(X)}{\mathrm{VaR}_{1-\epsilon}(X)} = \frac{\alpha}{\alpha - 1} c^{-1/\alpha} = \lim_{\epsilon \downarrow 0} \left(\frac{\Pi_\epsilon(X)}{c} \right)^{1/\alpha}.$$

- In FRTB, $c = 2.5$ leads to

$$R(\alpha) := \frac{\alpha}{\alpha - 1} 2.5^{-1/\alpha} = \lim_{\epsilon \downarrow 0} \left(\frac{\Pi_\epsilon(X)}{2.5} \right)^{1/\alpha} > 1$$

- $R(\alpha) \approx \mathrm{ES}_{0.975} / \mathrm{VaR}_{0.99}$
- $R(8) = 1.02$; $R(4) = 1.06$; $R(2) = 1.26$

Progress

Empirical PELVE estimators

Empirical PELVE estimator $\widehat{\Pi}_\epsilon(n)$: solve

$$\widehat{\mathrm{ES}}_{1-c\epsilon} = \widehat{\mathrm{VaR}}_{1-\epsilon} \quad \text{for } c \in [1, 1/\epsilon],$$

where $\widehat{\mathrm{ES}}$ and $\widehat{\mathrm{VaR}}$ are the empirical ES and VaR, respectively

- Let $X_{[1]} \le \cdots \le X_{[n]}$ be the order statistics of $X_1, \dots, X_n$.

$$\widehat{\mathrm{VaR}}_p = X_{[i]} \;\text{ for }\; p \in \left(\frac{i-1}{n}, \frac{i}{n}\right],\; i = 1, \dots, n,$$

$$\widehat{\mathrm{ES}}_p = \frac{1}{1-p}\int_p^1 \widehat{\mathrm{VaR}}_q \mathrm{d}q, \quad p \in (0,1).$$

- Safely assume that c above is unique

Empirical PELVE estimators

Smoothed empirical PELVE estimator $\widetilde{\Pi}_\epsilon(n)$:

- $\widehat{\mathrm{ES}}_p$ is continuous in p
- $\widehat{\mathrm{VaR}}_p$ has jumps
- Quick fix: use $\widetilde{\mathrm{VaR}}_p$, the standard linearly interpolated quantile (McNeil-Frey-Embrechts'15, Section 9.2.6)
- Calculate $\widetilde{\mathrm{ES}}_p$ based on $\widetilde{\mathrm{VaR}}_p$

Assumption 4 (Regularity)

$X \sim F$ has a density function f which is positive and continuous at $F^{-1}(1-\epsilon)$, and $\mathbb{E}[|X|^{2+\delta}] < \infty$ for some $\delta > 0$.

Asymptotic normality

Theorem 4 (Asymptotic normality)

Suppose that $X, X_1, X_2, \dots \in L^1$ *are iid,* $\epsilon \in (0,1)$*, and Assumptions 1 and 4 hold. Let* $c = \Pi_\epsilon(X)$ *and* $\hat{c}_n = \widehat{\Pi}(n)$ *or* $\hat{c}_n = \widetilde{\Pi}(n)$*. Then*

$$\sqrt{n}(\hat{c}_n - c) \xrightarrow{\mathrm{p}} \frac{1}{b}\left(\int_q^1 \frac{W(1-t)}{\epsilon f(F^{-1}(t))}\mathrm{d}t - aW(\epsilon)\right) \sim \mathrm{N}(0, \sigma^2),$$

where W *is a standard Brownian bridge on* $[0,1]$, $p = 1-\epsilon$, $q = 1 - c\epsilon$, $a = c/f(F^{-1}(p))$, $b = F^{-1}(p) - F^{-1}(q)$, *and* σ^2 *can be computed as*

$$\sigma^2 = \frac{1}{b^2}\left(a^2(\epsilon - \epsilon^2) + \frac{2}{\epsilon^2}\int_{F^{-1}(q)}^{\infty} E_{F(x)}F(x)\mathrm{d}x - \frac{2a}{\epsilon}E_p + 2a(E_q - b)\right),$$

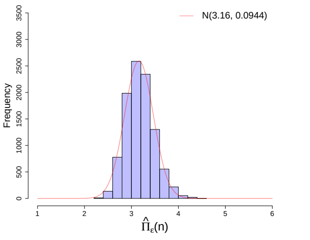
where $E_t = \mathbb{E}[(X - F^{-1}(t))_+]$ *for* $t \in (0,1)$.

Asymptotic normality

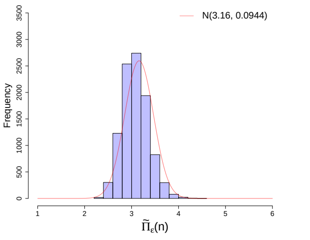
Remarks.

- $\sigma^2 \approx O(\epsilon^{-1})$ as $\epsilon \downarrow 0$
  - $\widehat{\mathrm{ES}}_{1-c\epsilon}$ and $\widehat{\mathrm{VaR}}_{1-\epsilon}$ effectively use $O(n\epsilon)$ data points
- Very small value of $\epsilon \Rightarrow$ large estimation error of Π_ϵ
- Π_ϵ typically stable in $\epsilon \Rightarrow$ no need ϵ too small
- $\widetilde{\Pi}_\epsilon(n)$ seems to have a negative bias whereas $\widehat{\Pi}_\epsilon(n)$ does not
- Similar results on α-mixing data
  - proof is based on Asimit-Peng-W.-Yu'19 MF

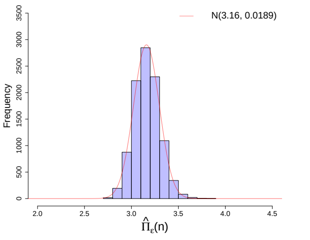
Simulation of PELVE estimators for Pareto(4), $\epsilon = 0.05$



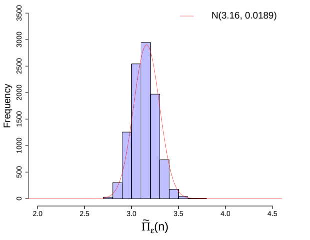
(a) $\widehat{\Pi}_\epsilon(n)$, $n = 1000$



(b) $\widetilde{\Pi}_\epsilon(n)$, $n = 1000$



(c) $\widehat{\Pi}_\epsilon(n)$, $n = 5000$



(d) $\widetilde{\Pi}_\epsilon(n)$, $n = 5000$

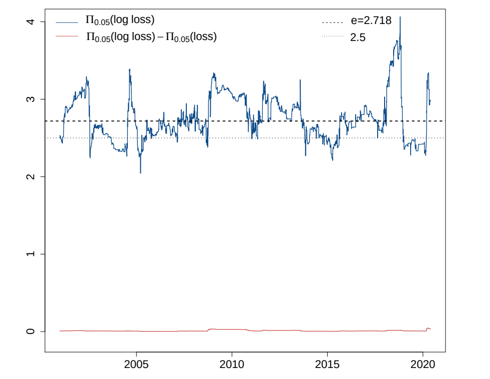
Progress

Data description

Data description

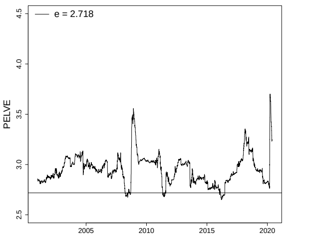
- Price data of S&P 500 constituents
- Daily log-losses (log-transform is negligible)
- 01/04/1999 to 05/30/2020 ($\approx$ 21.4 years)
- Empirical estimators based on a moving window of 500 days
- We report sector averages of S&P 500

Empirical PELVE of S&P 500

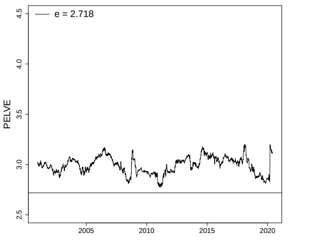


5% PELVE of log-loss vs linear loss, S&P 500, Jan 01 - May 20

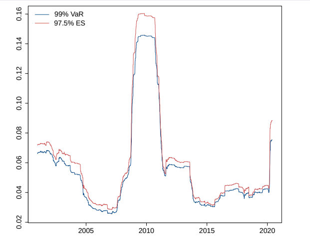
Empirical PELVE, VaR and ES for S&P Sectors



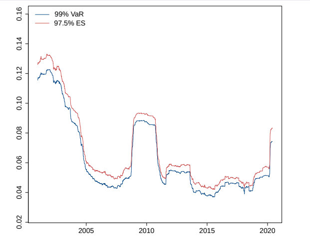
(a) 5% PELVE of S&P 500 Financials



(b) 5% PELVE of S&P 500 IT

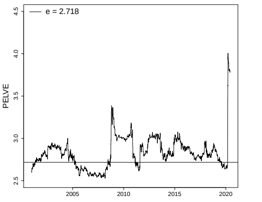


(c) VaR and ES of S&P 500 Financials

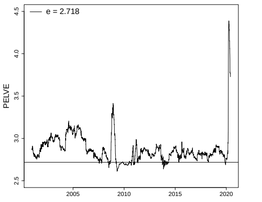


(d) VaR and ES of S&P 500 IT

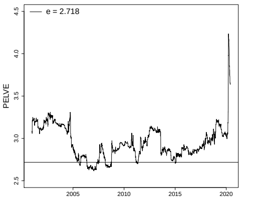
Empirical PELVE, VaR and ES for S&P Sectors



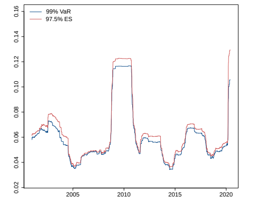
(a) 5% PELVE of S&P 500 Energy



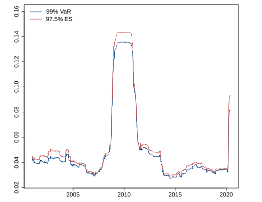
(b) 5% PELVE of S&P 500 Real Estate



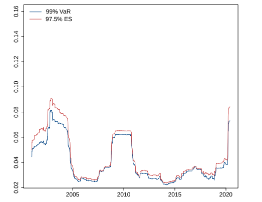
(c) 5% PELVE of S&P 500 Utilities



(d) VaR & ES of S&P 500 Energy

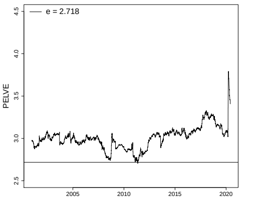


(e) VaR & ES of S&P 500 Real Estate



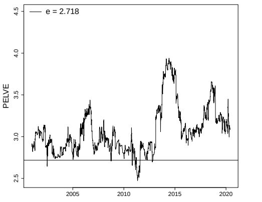
(f) VaR & ES of S&P 500 Utilities

Empirical PELVE for other S&P Sectors

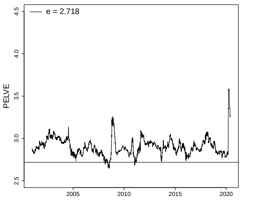


(a) S&P 500 Consumer Discretionary

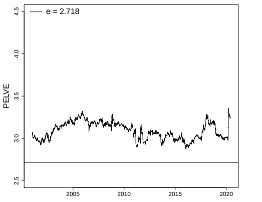
(b) S&P 500 Consumer Staples



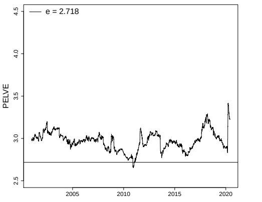
(c) S&P 500 Communication Services



(d) S&P 500 Materials



(e) S&P 500 Healthy Care



(f) S&P 500 Industrials

Summary of findings

Findings:

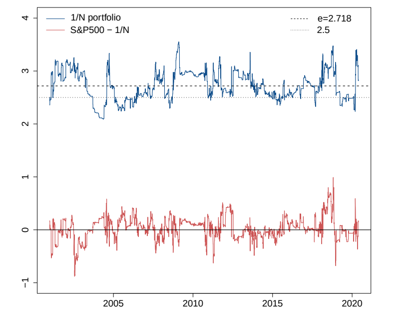
- Most PELVE are between 2.8 and 3.4 prior to COVID-19 and usually above $e \approx 2.72$ (average $= 2.98$) $\Rightarrow$ Heavy tails
  - Tail index between 3 and 5 (Jansen-De Vries'91 RES, Cont'01 QF) $\Rightarrow$ PELVE between 3.05 and 3.38 for small ϵ
- Overall PELVE values are quite stable during the past 20 years except for peaks around the two crises
- VaR/ES:
  - IT: dot-com bubble $\gg$ subprime crisis/covid crisis
  - Financials/Real Estate: subprime crisis $\gg$ covid crisis (so far)
- PELVE strongly disagrees with VaR/ES

Well-diversified portfolios

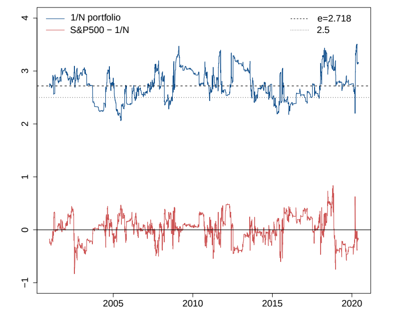
$1/N$ portfolio (DeMiguel-Garlappi-Uppal'09 RFS):

- $N = 500$ constituents of S&P 500
- Monthly rebalanced
- The constituents of S&P 500 change over time $\Rightarrow$ two types
  - (a) with replacement, $N = 500$
  - (b) without replacement, $N \downarrow 186$ (May 2020)
- S&P 500 index is a diversified portfolio (with survival bias)
- Jan 1999 to May 2020

5% PELVE of $1/N$ portfolios



(g) $1/N$ portfolio with replacement



(h) $1/N$ portfolio without replacement

Summary of findings

	average PELVE	average log-return
S&P 500	2.75	3.94%
$1/N$ with repl.	2.72	8.15%
$1/N$ without repl.	2.74	9.02%

- All curves fluctuate around $e \approx 2.72$
- Well-diversified portfolios have a smaller PELVE, close to e.
- A rough explanation: a well-diversified portfolio is more "Gaussian-like" $\Rightarrow$ its PELVE is closer to Gaussian (≈ 2.5)
- High return and low PELVE of the $1/N$ portfolios

Implications for risk management

Implications for risk management

- Switching from 99% VaR to 97.5% ES
  - diversified portfolio $\Rightarrow$ somewhat fine
  - non-diversified portfolio $\Rightarrow$ significant capital increase
- For a well-diversified portfolio X and a non-diversified one Y:
  - $\mathrm{VaR}_{0.99}(X) = \mathrm{VaR}_{0.99}(Y) \Rightarrow \mathrm{ES}_{0.975}(X) < \mathrm{ES}_{0.975}(Y)$.
- ES rewards portfolio diversification more than the VaR
  - Hidden feature: not mentioned by FRTB or previous research
  - Subadditivity (coherence) does not imply this
    - elliptical risk factor models
    - time-series models with Gaussian white noise

Progress

Concluding remarks

PELVE in banking regulation

- Motivated by 2019 Basel FRTB
- $\mathrm{VaR}_{0.99} \Rightarrow \mathrm{ES}_{0.975}$ increases capital for heavy-tailed losses
- Loss distributions in the US equity market are heavy-tailed
- Well-diversified portfolios have lower PELVE

Theory

- Location-scale invariant and quasi-convex/concave
- Monotone in convex transform and in tail index (asymptotically)
- Well defined for all commonly used distributions
- Empirical estimation is standard and simple

Thank you

Thank you for your kind attention

The manuscript is available at SSRN:3489566

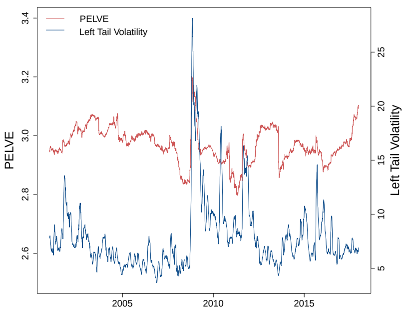
Comments are welcome

PELVE

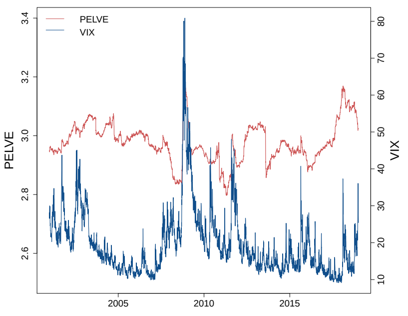
PELVE vs volatility measures

- The average PELVE of S&P 500 constituents
- Left Tail Volatility index (LTV) during the period Jan 2001 to Dec 2017 (Andersen-Todorov-Ubukata'20 JoE)
- CBOE Volatility index (VIX) during the period Jan 2001 to Dec 2018

PELVE of average S&P 500 constituents vs LTV and VIX



(i) 5% PELVE vs LTV (Jan 01 - Dec 17)



(j) 5% PELVE vs VIX (Jan 01 - Dec 18)