A tale of two risk measures

Contrasting Value-at-Risk and Expected Shortfall

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Actuarial Research Conference 2021
DePaul University, Chicago August 2021 (online)

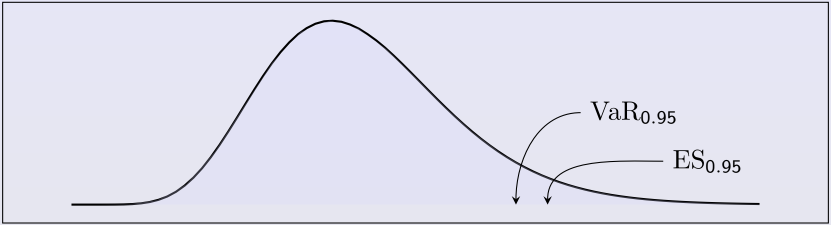
Agenda

Risk measures

A risk measure ρ assigns a real number to each risk (via a model)

- regulatory capital calculation
- insurance pricing
- decision making, optimization, and portfolio selection
- performance analysis and capital allocation

VaR and ES



Value-at-Risk (VaR), $p \in (0,1)$

$\mathrm{VaR}_p : L^0 \to \mathbb{R}$,

$$\mathrm{VaR}_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) \ge p\}.$$

(left-quantile)

Expected Shortfall (ES), $p \in (0,1)$

$\mathrm{ES}_p : L^1 \to \mathbb{R}$,

$$\mathrm{ES}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q(X)\mathrm{d}q$$

(also: TVaR/CVaR/AVaR/CTE)

VaR and ES

If X is continuously distributed, then

$$\mathrm{VaR}_p(X) = x_p \text{ where } \mathbb{P}(X \geq x_p) = \mathbb{P}(X > x_p) = 1 - p;$$

$$\mathrm{ES}_p(X) = \mathbb{E}[X|X > x_p] = \mathbb{E}[X|X \geq x_p].$$

Empirical estimators

- Let $n_p = \lfloor n(1-p) \rfloor$
- $\widehat{\mathrm{VaR}}_p$: empirical p-quantile (the n_p-th largest order statistic)
- $\widehat{\mathrm{ES}}_p$: average of the largest n_p observations

VaR and ES

The ongoing co-existence of VaR and ES:

- Basel III/IV - ES for market risk, VaR for backtest and OpRisk
- Solvency II - VaR
- Swiss Solvency Test - ES
- US Solvency (NAIC ORSA) - different system

General question

Question

What is a "good" risk measure? VaR, ES, or another?

- Regulator's and firm manager's perspectives can be different or even conflicting
  - well-being of the society versus interest of the shareholders
  - stability of a system versus sustainability of a firm
- Many practical questions on these risk measures

Theoretical properties

Theoretical properties

Artzner/Delbaen/Eber/Heath'99

A monetary risk measure satisfies two properties

- Monotonicity: $\rho(X) \leq \rho(Y)$ if $X \leq Y$
- Translation invariance: $\rho(X + c) = \rho(X) + c$ for $c \in \mathbb{R}$

for all X, Y in the domain $\mathcal{X}$ of ρ

A coherent risk measure satisfies, in addition,

- Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$
- Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ for $\lambda > 0$

Coherence

- VaR is monotone, translation invariant, positively homogenous, but not subadditive
- ES is coherent
  - also a convex risk measure (Fölmer/Schied'02)
- For elliptical risk vectors, VaR is subadditive
  - The elliptical family includes normal and t distributions
  - Excludes financial options, insurance losses, credit risks, ...
  - Fundamental theorems of QRM (as per Embrechts'19)
- VaR and ES are law invariant, i.e., $\rho(X) = \rho(Y)$ if $X \stackrel{d}{=} Y$

Capturing the tail risk

- Tail event: $X > x_p = \mathrm{VaR}_p(X)$
- VaR is blind about the loss magnitude when $X > x_p$
  - "ignoring the tail risk"; "only frequency"
- ES is the expected loss when $X > x_p$
  - "capturing the tail risk"; "frequency and severity"
- The Basel Committee on Banking Supervision (BCBS) Fundamental Review of the Trading Book (FRTB), Jan 2016
  - $\mathrm{ES}_{0.975}$ replaces $\mathrm{VaR}_{0.99}$ as the standard tool for market risk
  - Page 1, Executive Summary:
    "*Use of ES will help to ensure a more prudent capture of "tail risk" and capital adequacy ...*"

Comonotonic addititivity

- Comonotonicity of (X, Y): X and Y are both increasing functions of a common random variable Z
- Comonotonic addititivity: $\rho(X+Y) = \rho(X) + \rho(Y)$ if (X, Y) is comonotonic
  - Economic theory: Yaari'87; Schmeidler'89
  - Actuarial Science: Wang/Young/Panjer'97; Denneberg'94
  - Mathematical Finance: Kusuoka'01
- No diversification for comonotonic portfolios
- Both VaR and ES are comonotonic additive

Numéraire invariance

Numéraire invariance

- $R \geq 0$ is a random exchange rate (e.g., EUR/CHF)
- If X is acceptable, i.e., $\rho(X) \leq 0$, then so should be RX
- Numéraire invariance: $\rho(X) \leq 0 \Rightarrow \rho(RX) \leq 0$ for any random variable $R \geq 0$
- VaR is numéraire invariant; ES is not
  - Koch-Medina/Munari'15
  - He/Peng'18

Surplus invariance

Surplus invariance

- Whether X is acceptable depends only on potential loss but not surplus
- Surplus invariance: $\rho(X) \le 0 \;\Leftrightarrow\; \rho(X_+) \le 0$
- VaR is surplus invariant; ES is not
  - Cont/Deguest/He'13
  - Koch-Medina/Moreno-Bromberg/Munari'15

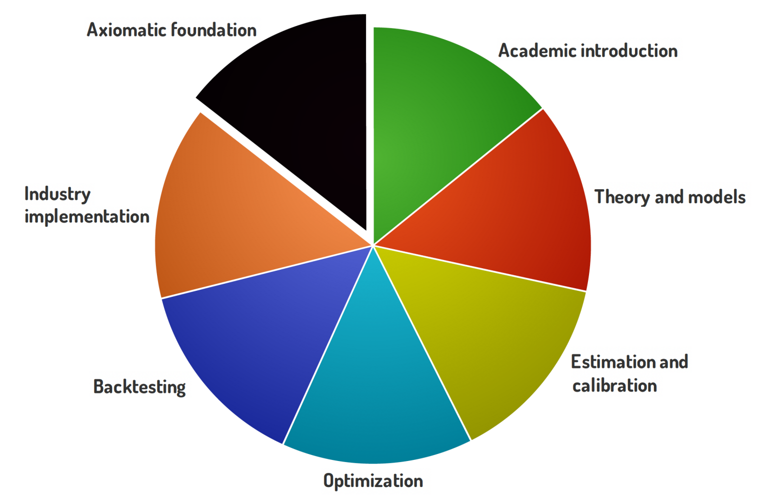
Domain

	VaR	ES
Capturing tail risk	no	yes
Coherence	no	yes
Numéraire invariance	yes	no
Surplus invariance	yes	no
Domain	all	integrable

- ES is finite for loss X with $\mathbb{E}[X_+] < \infty$
  - Suitable for losses from financial assets and most insurance businesses
  - Catastrophe risk? Operational risk?

Axiomatic theory

Axiomatic theory



Axiomatic theories for VaR

Axiomatic characterizations of VaR (quantile):

- Chambers'09: ordinal covariance + law invariance
- Kou/Peng'16: elicitability + comonotonic additivity + non-linearity
- He/Peng'18: surplus invariance + law invariance + pos. homog.
- Liu/W.'21: elicitability + tail relevance + pos. homog.

all + monetary + some form of continuity

- Consider $\mathcal{X} = L^\infty$

An axiomatic theory of VaR

- Ordinal covariance: $\rho(\phi(X)) = \phi(\rho(X))$ for all strictly increasing and continuous ϕ
  - e.g., $\mathrm{VaR}_p(\exp(X)) = \exp(\mathrm{VaR}_p(X))$
- Lower semicontinuity: with respect to convergence in distribution

Theorem (Chambers'09 MF)

A functional $\rho : \mathcal{X} \to \mathbb{R}$ satisfies law invariance, monotonicity, lower semicontinuity and ordinal covariance if and only if $\rho = \mathrm{VaR}_p$ for some $p \in (0, 1)$.

An axiomatic theory of ES

- A tail event A of X satisfies $0 < \mathbb{P}(A) < 1$ and $X(\omega) \geq X(\omega')$ for a.s. all $\omega \in A$ and $\omega' \in A^c$.
  - e.g., $A = \{X > x\}$
- No reward for concentration: There exists an event $A \in \mathcal{F}$ such that $\rho(X + Y) = \rho(X) + \rho(Y)$ holds for all risks X and Y sharing the tail event A.

Theorem (W./Zitikis'21 MS)

A functional $\rho : \mathcal{X} \to \mathbb{R}$ with $\rho(1) = 1$ satisfies law invariance, monotonicity, lower semicontinuity and no reward for concentration if and only if $\rho = \mathrm{ES}_p$ for some $p \in (0, 1)$.

Axiomatic theories

	VaR	ES
First axiom	monotonicity	monotonicity
Second axiom	law invariance	law invariance
Third axiom	lower semicontinuity	lower semicontinuity
Fourth axiom	ordinal covariance	no reward for concentration

Converting between VaR and ES

Converting between VaR and ES

- For all $p \in (0,1)$, $\mathrm{ES}_p(X) \geq \mathrm{VaR}_p(X)$
- For light-tailed distributions (e.g., normal or exponential)

$$\lim_{p\to 1} \frac{\mathrm{ES}_p(X)}{\mathrm{VaR}_p(X)} = 1$$

- For heavy-tailed distributions (e.g., Pareto or t)
  - $\mathbb{P}(X > x) = x^{-\alpha} L(x), \ \alpha > 1$; L slowly varying

it holds

$$\lim_{p\to 1} \frac{\mathrm{ES}_p(X)}{\mathrm{VaR}_p(X)} = \frac{\alpha}{\alpha - 1}$$

FRTB

Fundamental Review of the Trading Book (FRTB)

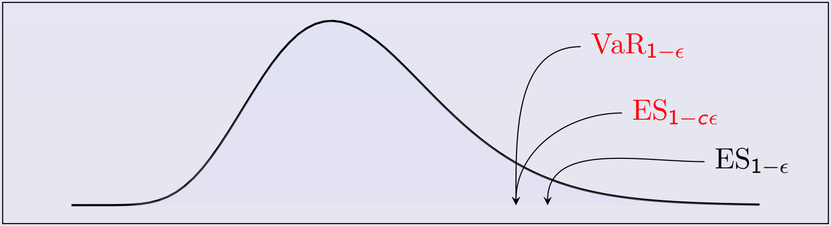
- Widely discussed since 2012, still not fully implemented

$$\mathrm{VaR}_{0.99} \implies \mathrm{ES}_{0.975}$$

- In a survey in 2015, 2/3 of banks reported higher capital charge under the (back-then) proposed FRTB
- General relationship between $\mathrm{VaR}_{0.99}$ and $\mathrm{ES}_{0.975}$?

PELVE

- A number $c \in [1, 1/\epsilon]$ such that $\mathrm{ES}_{1-c\epsilon}(X) = \mathrm{VaR}_{1-\epsilon}(X)$



- For $\epsilon = 0.01 \quad \longleftrightarrow \quad \mathrm{VaR}_{0.99}$ in the FRTB transition:
  - $c > 2.5 \Rightarrow \mathrm{ES}_{0.975} > \mathrm{VaR}_{0.99} \Rightarrow$ capital increases
  - $c \approx 2.5 \Rightarrow \mathrm{ES}_{0.975} \approx \mathrm{VaR}_{0.99} \Rightarrow$ little or no change in capital
- c is called the PELVE at level ϵ (Li/W.'19)
  - Probability Equivalent Level of VaR and ES

Typical values of PELVE

ϵ	Dirac	U	N	Exp	LN(σ^2) 0.04	LN(σ^2) 0.25	LN(σ^2) 1
0.100	1.00	2.00	2.46	2.72	2.56	2.76	3.23
0.050			2.51		2.61	2.79	3.19
0.010			2.58		2.66	2.81	3.13
0.005			2.59		2.67	2.81	3.10

ϵ	t(ν) 2	t(ν) 10	t(ν) 30	Pareto(α) 2	Pareto(α) 4	Pareto(α) 10
0.100	3.60	2.58	2.49	4.00	3.16	2.87
0.050	3.80	2.65	2.55			
0.010	3.96	2.74	2.63			
0.005	3.98	2.77	2.65			

Implications of PELVE

Theoretical conclusions

- For heavy-tailed risks, $c > 2.7$ (more capital)
- For light-tailed risks, $c \in [2.5, 2.7]$ (roughly similar capital)
- For portfolios, ES rewards diversification more than VaR
  - not related to coherence

Empirical observations

- For individual asset log-returns, $c \approx 3$ (heavy)
- For well-diversified portfolios (such as $1/N$), $c \approx 2.7$ (light)

Estimation of VaR vs ES (cf. Danielson/Zhou'16)

- $\widehat{\mathrm{VaR}}_{0.99}$ has a smaller error if tail is quite heavy (roughly $c > 2.9$)
- $\widehat{\mathrm{ES}}_{0.975}$ has a smaller error if tail is not too heavy (roughly $c < 2.9$)

Optimization, capital allocation, and risk aggregation

Optimization, capital allocation, and risk aggregation

Rockafellar/Uryasev'02

$$\mathrm{VaR}_p(X) \in \underset{x\in\mathbb{R}}{\arg\min}\left\{x + \frac{1}{1-p}\mathbb{E}[(X-x)_+]\right\}$$

$$\mathrm{ES}_p(X) = \min_{x\in\mathbb{R}}\left\{x + \frac{1}{1-p}\mathbb{E}[(X-x)_+]\right\}$$

- Minimizing ES as an objective
  - ⇒ minimizing an expected convex function ✓
- Optimization with ES as constraints
  - ⇒ can be solved via convex programming ✓
- VaR does not have any of the above features

Capital allocation

- n individual business lines (desks) with losses $X_1, \dots, X_n$
- Total loss $S = \sum_{i=1}^n X_i$, assumed continuous
- Total capital $C^\rho = \rho(S)$ where ρ is VaR_p or ES_p
- Allocate $C_1^\rho, \dots, C_n^\rho$ to each desk such that $C^\rho = \sum_{i=1}^n C_i^\rho$

The classic Euler capital allocation (RORAC compatibility)

$$C_i^{\mathrm{VaR}_p} = \mathbb{E}[X_i | S = \mathrm{VaR}_p(S)]$$

$$C_i^{\mathrm{ES}_p} = \mathbb{E}[X_i | S > \mathrm{VaR}_p(S)]$$

- $C_i^{\mathrm{VaR}_p}$ is much harder to estimate, compute, or simulate
  - e.g., Tasche'08; Scaillet'04; Asmit/Peng/W./Yu'19
  - $\Rightarrow$ a large literature on sensitivity analysis of quantiles

Risk aggregation

- Because ES is subadditive, with unknown dependence

$$\mathrm{ES}_p\left(\sum_{i=1}^{n} X_n\right) \le \sum_{i=1}^{n} \mathrm{ES}_p(X_i)$$

- Marginal information provides bounds on the portfolio
- Worst-case ES : $\overline{\mathrm{ES}}_p = \sum_{i=1}^{n} \mathrm{ES}_p(X_i)$
- VaR: not subadditive!
- $\overline{\mathrm{VaR}}_p$, $\underline{\mathrm{VaR}}_p$, and $\underline{\mathrm{ES}}_p$: generally open questions for $n \ge 3$
  - Embrechts/Puccetti'06; W./Peng/Yang'13; Embrechts/Puccetti/Rüschendorf'13; Embrechts/Wang/W.'15

Example: Pareto risks

Bounds on VaR and ES for the sum of n Pareto(2) distributed rvs for $p = 0.999$; VaR_p^+ corresponds to the sum of individual VaR_p.

	$n = 8$	$n = 56$
$\underline{\mathrm{VaR}}_p$	31	53
$\underline{\mathrm{ES}}_p$	178	472
VaR_p^+	245	1715
$\overline{\mathrm{VaR}}_p$	465	3454
$\overline{\mathrm{ES}}_p$	498	3486
$\overline{\mathrm{VaR}}_p/\mathrm{VaR}_p^+$	1.898	2.014
$\overline{\mathrm{ES}}_p/\overline{\mathrm{VaR}}_p$	1.071	1.009

Dependence-uncertainty spread

ES and VaR of $S_n = X_1 + \cdots + X_n$, where

- $X_i \sim \mathrm{Pareto}(2 + 0.1i), \; i = 1, \ldots, 5;$
- $X_i \sim \mathrm{Exp}(i - 5), \; i = 6, \ldots, 10;$
- $X_i \sim \mathrm{Log\text{–}Normal}(0, (0.1(i - 10))^2), \; i = 11, \ldots, 20.$

	$n = 5$			$n = 20$		
	best	worst	spread	best	worst	spread
$\mathrm{ES}_{0.975}$	22.48	44.88	22.40	29.15	102.35	73.20
$\mathrm{VaR}_{0.975}$	9.79	41.46	31.67	21.44	100.65	79.21
$\mathrm{VaR}_{0.99}$	12.96	62.01	49.05	22.29	136.30	114.01
$\overline{\mathrm{ES}}_{0.975}/\overline{\mathrm{VaR}}_{0.975}$		1.08			1.02	

- VaR_p has a larger spread than ES_q, $p \geq q$, under mild conditions (Embrechts/Wang/W.'15)

Robustness

Robustness

Statistical robustness addresses the question of "what if the data is compromised with small error?"

- Originally robustness is defined on estimators (estimation procedures)
- Models are at most "approximately correct" $\Rightarrow$ robustness
- Hampel'71 identified robustness of a statistical functional with continuity with respect to some metric
  - Huber/Ronchetti'07

Robustness of risk measures

- With respect to convergence in distribution:
  - VaR_p is continuous at distributions whose quantile is continuous at p. VaR_p is argued as being almost robust.
  - ES_p is not continuous for any $\mathcal{X} \supset L^\infty$ (similar to the mean)
- ES_p is continuous w.r.t. some other (stronger) metric, e.g., the L^q metric for $q \geq 1$ (or the Wasserstein-L^q metric)

Robustness in a static setting (Cont/Deguest/Scandilo'10):

$$\mathrm{ES} \prec \mathrm{VaR}$$

However, one cannot decouple the properties of a risk measure from the incentives it creates

Robustness in risk allocation

Risk sharing, risk exchange, and market equilibria

$$X \longmapsto (X_1, \ldots, X_n) \text{ s.t. } \sum_{i=1}^{n} X_i = X$$

- Optimality: aggregate risk ⇔ collaborative ⇐ competitive
- Robustness: small model misspecification of X does not lead to very different individual risk values

Robustness in risk allocation (Embrechts/Liu/W.'18):

$$\text{VaR} \prec \text{ES}$$

Robustness in optimization

"The optimization problem"

to minimize $\rho(g(X_1, \ldots, X_n))$ over $g \in \mathcal{G}$

- Robustness: small model misspecification of $(X_1, \ldots, X_n)$ does not lead to very different optimized risk values

Robustness in optimization (Embrechts/Schied/W.'21):

$$\mathrm{VaR} \prec \mathrm{ES}$$

- The non-robustness of VaR comes from the fact that optimizing VaR is "too greedy": always ignores tail risk, and hopes that the probability of the tail risk is correctly modelled

Robustness in optimization

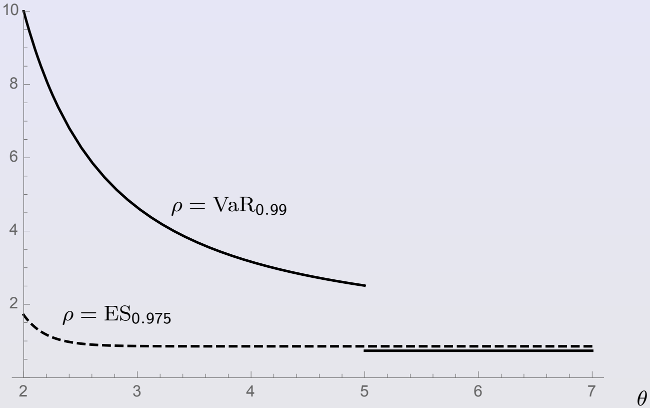


Figure: $\rho(g_X(Z))$ for $Z \sim \mathrm{Pareto}(\theta)$ and $X \sim \mathrm{Pareto}(\widehat{\theta} = 5)$. The function g_X minimizes $\rho(g(X))$ within the class of all measurable functions g satisfying $0 \le g(x) \le x$ and $\mathbb{E}[Xg(X)] \ge 1$.

Optimization, capital allocation, and risk aggregation

	VaR	ES
Optimization	non-convex	convex
Capital allocation	difficult	straightforward
Risk aggregation	difficult	straightforward
Uncertainty spread	relatively large	relatively small
Robustness (static)	$\mathrm{VaR} \succ \mathrm{ES}$	
Robustness (optimization)	$\mathrm{VaR} \prec \mathrm{ES}$	

Elicitability

Elicitability

Definition (Osband'85)

A functional $\rho : \mathcal{X} \to \mathbb{R}^d$ is elicitable on $\mathcal{X}$ if there exists a loss function $L : \mathbb{R}^{d+1} \to \mathbb{R}$ such that for all $X \in \mathcal{X}$,

$$\rho(X) = \arg\min_{\mathbf{y}\in\mathbb{R}^d} \mathbb{E}[L(\mathbf{y}, X)].$$

If $(\rho_1, \rho_2) : \mathcal{X} \to \mathbb{R}^2$ is elicitable, then ρ_1 is co-elicitable with ρ_2.

- Elicitability $\Rightarrow$ empirical risk minimization (ERM)

Elicitability

Examples for $d = 1$. ($L^q(p)$: rvs in L^q with a unique p-quantile)

- The mean is elicitable on L^2 with
$$L(y, X) = (y - X)^2.$$
- The median is elicitable on $L^1(1/2)$ with
$$L(y, X) = |y - X|.$$
- The p-quantile VaR_p is elicitable on $L^1(p)$ with
$$L(y, X) = (1 - p)y + (X - y)_+.$$
- The p-expectile e_p is elicitable on L^2 with
$$L(y, X) = (1 - p)(y - X)_+^2 + p(X - y)_+^2.$$

Elicitability

$\mathbb{E}[L(\hat{\rho}, X)]$ can be seen as an average error for an estimate $\hat{\rho}$

- Good estimate $\Rightarrow$ smaller average error (empirically)
- Forecast comparison
- Model selection
- Learning theory

Theorem (Gneiting'11 JASA)

For $p \in (0,1)$, on $L^\infty(p)$, VaR_p is elicitable whereas ES_p is not.

- Ziegel'16, Bellini/Bignozzi'15, Kou/Peng'16, Liu/W.'21, ...

Co-elicitability

Theorem (Fissler/Ziegel'16 AoS)

For $p \in (0,1)$, ES_p is co-elicitable with VaR_p on $L^\infty(p)$.

- ES is "second-order" elicitable
- Forecast comparison of ES can be carried out with VaR
- Similarly, the variance is co-elicitable with the mean

Theorem (Wang/W.'20 MF)

A coherent, lower semicontinuous, and comonotonic additive risk measure ρ is co-elicitable with VaR_p on $L^\infty(p)$ if and only if $\rho = \mathrm{ES}_p$.

Backtesting

Backtesting

- Risk measure ρ to backtest
- Define

$$\mathcal{F}_{t-1} := \sigma(X_s : s \leq t-1)$$

- Daily observations
  - risk measure forecast r_t for $\rho(X_t)$
  - realized loss X_t

Hypothesis to test

$$H_0 : \begin{array}{c} \text{conditional on } \mathcal{F}_{t-1}\text{:} \\ r_t \geq \rho(X_t|\mathcal{F}_{t-1}) \end{array} \quad \text{for } t = 1, \ldots, T$$

Backtesting VaR

Information

- Daily prediction $r_t = \widehat{\mathrm{VaR}}_p(X_t)$
- Daily realization X_t

Backtesting for fixed T

- Under H_0: $Y_t = \mathbb{1}_{\{X_t > r_t\}}$ are independent Bernoulli sample with mean at most $1 - p$
- $S_T = \sum_{t=1}^{T} Y_t \le_{\mathrm{st}} \mathrm{Binom}(T, 1-p)$
- Easy to construct p-values (reject if S_t large)
- Completely model free

Such a simple procedure does not exist for ES!

Backtesting ES

Model-free backtest for ES (on-going work)

- relies on e-values and e-tests
  - Definition of an e-value E: $\mathbb{E}[E] \le 1$ and $E \ge 0$
  - Vovk/W.'21, W./Ramdas'21, Shafer'21, ...
- relies on VaR forecasts

Define

$$e_p(x, r, z) = \frac{(x-z)_+}{(1-p)(r-z)}, \quad x \in \mathbb{R}, \ z \le r$$

- if $r \ge \mathrm{ES}_p(X)$ and $z = \mathrm{VaR}_p(X)$, then $\mathbb{E}[e_p(X, r, z)] \le 1$
- if $r < \mathrm{ES}_p(X)$, then $\mathbb{E}[e_p(X, r, z)] > 1$ for all z

Backtesting ES

The general protocol for $t \in \mathbb{N}$

- The firm supplies ES forecast r_t and VaR forecast z_t
- Decide a predictable $\lambda_t \in [0,1]$ ($\Rightarrow$ not shown to the firm)
- Observe realized loss X_t
- Obtain the e-value $x_t = e_p(X_t, r_t, z_t)$
- Compute the e-process ($E_0 = 1$)

$$E_t = E_{t-1}(1 - \lambda_t + \lambda_t x_t) = \prod_{s=1}^{t}(1 - \lambda_s + \lambda_s x_s).$$

Backtesting ES

$$H_0: \quad r_t \geq \mathrm{ES}_p(X_t|\mathcal{F}_{t-1}) \text{ and } z_t = \mathrm{VaR}_p(X_t|\mathcal{F}_{t-1}) \quad \text{for } t = 1, \ldots, T$$

$$H_0': \quad \begin{array}{c} r_t - z_t \geq \mathrm{ES}_p(X_t|\mathcal{F}_{t-1}) - \mathrm{VaR}_p(X_t|\mathcal{F}_{t-1}) \\ \text{and } z_t \geq \mathrm{VaR}_p(X_t|\mathcal{F}_{t-1}) \end{array} \quad \text{for } t = 1, \ldots, T$$

Theorem

Under H_0 or H_0', $(E_t)_{t=1,\ldots,T}$ is a supermartingale, and

$$\mathbb{P}\left(\sup_{t\geq 1} E_t \geq \frac{1}{\alpha}\right) \leq \alpha.$$

- model free; anytime valid (works for stopping times T)
- prudent regulation: one may reject if $E_T > 1 + \epsilon$

VaR versus ES: Summary

VaR versus ES: Summary

	Value-at-Risk	Expected Shortfall
Domain	always exists	needs first moment
Capturing tail risk	only frequency	frequency and severity
Estimation	comparably difficult	comparably difficult
Numéraire invariance	yes	no
Surplus invariance	yes	no
Diversification	non-coherent/non-NRC	coherent/NRC
Optimization	non-convex/non-robust	convex/robust
Capital allocation	difficult to estimate	straightforward (Euler)
Continuity	weak topology	L-metrics
Elicitability	first order	second order
Backtesting	straightforward	complicated (e-backtesting)

Some references

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W./Zitikis'21. An axiomatic foundation for the Expected Shortfall. Management Science

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Embrechts/Liu/W.'18. Quantile-based risk sharing. Operations Research

Embrechts/Schied/W.'21. Robustness in the optimization of risk measures. Operations Research

W./Wei'20. Risk functionals with convex level sets. Mathematical Finance

Vovk/W.'21. E-values: Calibration, combination, and applications. Annals of Statistics

W./Ramdas'21. False discovery rate control with e-values. arXiv:2009.02824

Thank you

Thank you for your kind attention

- Working papers series on the theory of risk measures: http://sas.uwaterloo.ca/~wang/pages/WPS1.html