Star-shaped and Quasi-star-shaped Risk Measures

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Agenda

1. Recap: Coherent and convex risk measures
2. Star-shaped risk measures
3. Quasi-star-shaped risk measures

Based on the following joint work

- Castagnoli†/Cattelan/Maccheroni/Tabaldi/W., Star-shaped risk measures. Working paper, 2021, arXiv:2103.15790
- Han/Wang/W./Xia, Cash-subadditive risk measures without quasi-convexity. Working paper, 2021, arXiv:2110.12198

Coherent risk measures

Artzner/Delbaen/Eber/Heath'99

A risk measure $\rho : \mathcal{X} \to \mathbb{R}$

- $\mathcal{X}$: a convex cone of random losses
- Default choice: the set of bounded rvs on $(\Omega, \mathcal{F}, \mathbb{P})$

A coherent risk measure satisfies

- Monotonicity: $\rho(X) \le \rho(Y)$ if $X \le Y$
- Cash additivity: $\rho(X + c) = \rho(X) + c$ for all $c \in \mathbb{R}$
- Subadditivity: $\rho(X + Y) \le \rho(X) + \rho(Y)$
- Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ for all $\lambda > 0$

$\Longrightarrow$ Normalization: $\rho(0) = 0$

Convex risk measures

Föllmer/Schied'02; Frittelli/Rosazza Gianin'02

A (normalized) monetary risk measure satisfies

- monotonicity, cash additivity, and normalization

A convex risk measure is monetary and convex

- $\rho(\lambda X + (1-\lambda)Y) \le \lambda\rho(X) + (1-\lambda)\rho(Y)$ for all $\lambda \in [0,1]$
- With positive homogeneity (PH): convexity $\iff$ subadditivity

Motivations for convexity relaxed from coherence

- Liquidity risk: $\rho(\lambda X) > \lambda\rho(X)$ for $\lambda > 1$ (violating PH)
- A merge may create extra risk: violating subadditivity

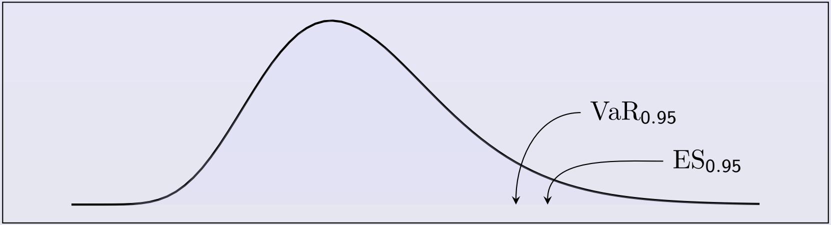
Acceptance sets

Acceptance set of a monetary risk measure

$$\mathcal{A}_\rho = \{X \in \mathcal{X} : \rho(X) \le 0\}$$

- $0 \in \partial\mathcal{A}_\rho =$ boundary of $\mathcal{A}_\rho$ (normalization)
- $X \le Y$ and $Y \in \mathcal{A}_\rho \implies X \in \mathcal{A}_\rho$ (monotonicity)
- $\mathcal{A}_\rho$ is convex $\iff \rho$ is convex
- $\mathcal{A}_\rho$ is convex and conic $\iff \rho$ is coherent
- $\rho(X) = \inf\{m \in \mathbb{R} : X - m \in \mathcal{A}_\rho\},\ X \in \mathcal{X}$ (cash additivity)

VaR and ES



Value-at-Risk (VaR), $p \in (0,1)$

$\mathrm{VaR}_p^Q : L^0 \to \mathbb{R}$,

$$\mathrm{VaR}_p^Q(X) = F_X^{-1}(p)$$
$$= \inf\{x \in \mathbb{R} : Q(X \le x) \ge p\}.$$

(left-quantile)

Expected Shortfall (ES), $p \in (0,1)$

$\mathrm{ES}_p^Q : L^1 \to \mathbb{R}$,

$$\mathrm{ES}_p^Q(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q^Q(X)\mathrm{d}q$$

(also: TVaR/CVaR/AVaR/CTE)

Progress

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2. Star-shaped risk measures
3. Quasi-star-shaped risk measures

Star-shaped risk measures

A star-shaped risk measure is monetary and star-shaped

- Star-shapedness: $\rho(\lambda X) \geq \lambda\rho(X)$ for all $\lambda > 1$

Equivalent conditions:

- $\rho(\lambda X) \leq \lambda\rho(X)$ for all $\lambda \in (0, 1)$ ($\Leftrightarrow$ convexity at 0)
- The risk-to-exposure ratio $\rho(\lambda X)/\lambda$ is increasing in $\lambda > 0$
- $\mathcal{A}_\rho$ is star-shaped at 0: $X \in \mathcal{A}_\rho \Longrightarrow \lambda X \in \mathcal{A}_\rho$ for all $\lambda \in [0, 1]$

Star-shaped risk measures

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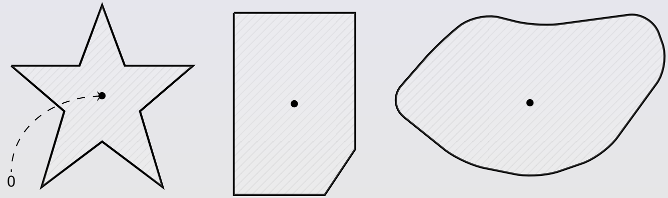


Star-shaped risk measures

Star-shapedness is

- weaker than convexity or positive homogeneity
- satisfied by all monetary risk measures in practice
  - distortion risk measures: VaR, ES, RVaR, GS, Wang's premium
  - convex risk measures: expectiles, entropic, entropic-VaR, shortfall, optimized certainty equivalents, higher-moment
  - benchmark-adjusted VaR and ES
  - robustification of risk measures
  - scenario-based risk measures

For a subadditive monetary risk measure

- Star-shapedness $\Leftrightarrow$ convexity $\Leftrightarrow$ positive homogeneity $\Leftrightarrow$ coherence

Motivation I: Liquidity risk

A dealer needs to clear a position X with some central clearing counterparties (CCPs)

- n CCPs with price functions $\rho_1, \dots, \rho_n$
- Liquidity cost $\Longrightarrow \rho_i(\lambda X)/\lambda$ increases in $\lambda > 0$
- $\mathcal{C}$: possible compositions of CCPs (subsets of $\{1, \dots, n\}$)

Dealer's optimal clearing problem (Glasserman/Moallemi/Yuan'16)

$$\min_{A \in \mathcal{C}} \min_{\substack{X_i \in \mathcal{X} \\ \text{s.t. } i \in A}} \left\{ \sum_{i \in A} \rho_i(X_i) \;\middle|\; \sum_{i \in A} X_i = X \right\} =: \rho(X)$$

- ρ is star-shaped but not convex (even if $\rho_1, \dots, \rho_n$ are convex)

Motivation II: Aggregation of opinions or prices

- $\rho_i,\ i \in I$: convex risk measures, expert opinions/prices
- Most conservative (convex)

$$\max_{i \in I} \rho_i(X)$$

- Most competitive (star-shaped but non-convex)

$$\min_{i \in I} \rho_i(X)$$

- α-max-min (star-shaped but non-convex)

$$\alpha \max_{i \in I} \rho_i(X) + (1-\alpha) \min_{i \in I} \rho_i(X)$$

- Median (star-shaped but non-convex)

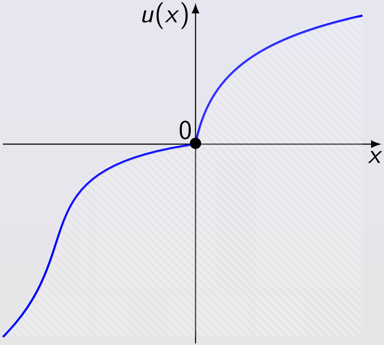
$$\text{median}\{\rho_i(X) \mid i \in I\}$$

Motivation III: Non-concave utilities

Utility-based shortfall risk measures (Föllmer/Schied'02)

$$\rho_u(X) = \inf\{m \in \mathbb{R} \mid \mathbb{E}_{\mathbb{P}}[u(m - X)] \geq u(0)\}, \qquad X \in \mathcal{X}$$

- u is concave $\Leftrightarrow$ ρ_u is convex $\Leftrightarrow$ strong risk aversion (empirically challengeable)
- Star-shaped at 0 utility functions (Landsberger/Meilijson'90): $\lambda \mapsto u(\lambda)/\lambda$ is decreasing on $(0, \infty)$ and $(-\infty, 0)$; $u(0) = 0$
- u star-shaped $\Leftrightarrow$ ρ_u star-shaped



Representation I

Theorem

For a mapping $\rho : \mathcal{X} \to \mathbb{R}$, the following are equivalent:

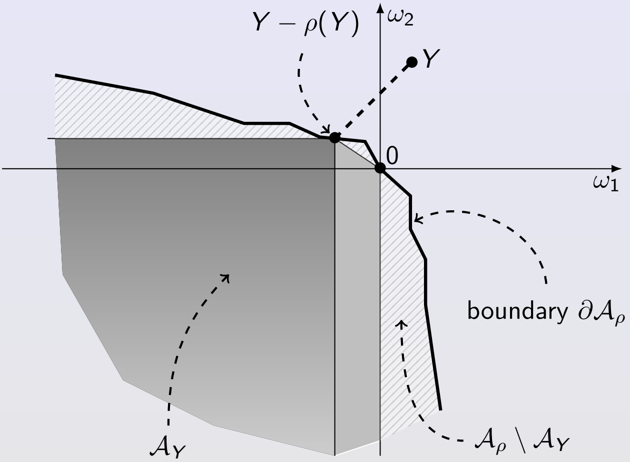
(i) *ρ is a star-shaped (resp. positively homogeneous and monetary) risk measure;*

(ii) *there exists a collection Γ of convex (resp. coherent) risk measures such that*

$$\rho(X) = \min_{\gamma \in \Gamma} \gamma(X), \qquad X \in \mathcal{X}.$$

Proof: Any star-shaped acceptance set $\mathcal{A}$ (with $0 \in \partial\mathcal{A}$) is the union of convex acceptance sets $\mathcal{A}_\gamma$ (with $0 \in \partial\mathcal{A}_\gamma$)

Proof sketch



The intuition behind the representation in case $\Omega = \{\omega_1, \omega_2\}$, where

$$\mathcal{A}_Y = \{X \in \mathcal{X} : X \leq \lambda(Y - \rho(Y)) \text{ for some } \lambda \in [0,1]\}$$

Representation II

- $\mathcal{P}$: probability measures $Q \ll \mathbb{P}$ on $(\Omega, \mathcal{F})$; $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$
- A normalized panelty is $\alpha_\gamma : \mathcal{P} \to [0, \infty]$ with $\inf_{Q \in \mathcal{P}} \alpha_\gamma(Q) = 0$
- A convex risk measure γ on $\mathcal{X}$ satisfying Fatou continuity has representation (Föllmer-Schied'02), for some normalized penalty α_γ,

$$\gamma(X) = \sup_{Q \in \mathcal{P}} \{\mathbb{E}_Q[X] - \alpha_\gamma(Q)\}, \qquad X \in \mathcal{X}$$

Proposition

A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is a star-shaped risk measure if and only if there exists a collection $\{\alpha_\gamma\}_{\gamma \in \Gamma}$ of normalized penalties such that

$$\rho(X) = \min_{\gamma \in \Gamma} \sup_{Q \in \mathcal{P}} \{\mathbb{E}_Q[X] - \alpha_\gamma(Q)\}, \qquad X \in \mathcal{X}.$$

Closure under operations

Theorem

For a collection of star-shaped risk measures, their average, supremum, infimum, and inf-convolution (when defined) are star-shaped risk measures.

- A closure property useful for many operations in finance
- This closure property also holds for
  - law-invariant star-shaped risk measures
  - SSD-consistent star-shaped risk measures
  - positively homogeneous risk measures

  but not for convex or coherent risk measures

Example I: scenario-based VaR and ES

- Scenario-based risk measures of W./Ziegel'21
- $\mathcal{Q}$: a finite collection $\mathcal{Q}$ of probability measures

$$\mathrm{MaxVaR}_\beta^{\mathcal{Q}}(X) = \max\{\mathrm{VaR}_\beta^Q(X) \mid Q \in \mathcal{Q}\}$$
$$\mathrm{MaxES}_\beta^{\mathcal{Q}}(X) = \max\{\mathrm{ES}_\beta^Q(X) \mid Q \in \mathcal{Q}\}$$
$$\mathrm{MedVaR}_\beta^{\mathcal{Q}}(X) = \mathrm{median}\{\mathrm{VaR}_\beta^Q(X) \mid Q \in \mathcal{Q}\}$$
$$\mathrm{MedES}_\beta^{\mathcal{Q}}(X) = \mathrm{median}\{\mathrm{ES}_\beta^Q(X) \mid Q \in \mathcal{Q}\}$$

- MaxVaR, MedVaR and MedES are star-shaped but not convex
- MaxES is star-shaped and convex

Example II: benchmark VaR

The benchmark-loss VaR of Bignozzi/Burzoni/Munari'20

$$\mathrm{LVaR}_g^Q(X) = \sup_{\alpha\in(0,1)} \{\mathrm{VaR}_\alpha^Q(X) - g(\alpha)\}$$

where $g : (0,1) \to \mathbb{R}$ is increasing with $\sup_{g\in\mathcal{G}} g(0+) = 0$

- LVaR_g^Q is a star-shaped risk measure
- neither positively homogeneous nor convex

The adjusted ES of Burzoni/Munari/W.'22

$$\mathrm{ES}_g^Q(X) = \sup_{\alpha\in(0,1)} \{\mathrm{ES}_\alpha^Q(X) - g(\alpha)\}$$

is convex (hence star-shaped)

Consistent risk measures

- A risk measure is SSD-consistent if $\rho(X) \le \rho(Y)$ whenever $X \preceq_2 Y$
  - $X \preceq_2 Y$: $\mathbb{E}_{\mathbb{P}}[f(X)] \le \mathbb{E}_{\mathbb{P}}[f(Y)]$ for all increasing convex f
- A SSD-consistent monetary risk measure ρ on $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ has representation (Mao/W.'20) as an infimum of adjusted ES

$$\rho(X) = \inf_{g \in \mathcal{G}} \sup_{\alpha \in (0,1)} \{\mathrm{ES}_\alpha^{\mathbb{P}}(X) - g(\alpha)\} \qquad X \in \mathcal{X} \qquad \text{(MW)}$$

for some set $\mathcal{G}$ of increasing $g : (0,1) \to \mathbb{R}$ with $\sup_{g \in \mathcal{G}} g(0+) = 0$

Theorem

A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is an SSD-consistent star-shaped risk measure if and only if its has a representation (MW) in which $\mathcal{G}$ is a star-shaped set.

Progress

Cash-subadditive risk measures

El Karoui/Ravanelli'09

(Always assume monotonicity for a risk measure)

- Cash additivity: \$1 more loss ⇒ \$1 more capital (time 0)
- No problem if interest rate is a constant
- Stochastic discount factor $D \le 1$
- $\rho(X) = \rho_0(DX)$ with cash-additive ρ_0
- $\rho(X+c) = \rho_0(DX+Dc) \le \rho_0(DX+c) = \rho(X)+c$

Giving rise to

- Cash subadditivity: $\rho(X+c) \le \rho(X)+c$ for all $c \ge 0$

Quasi-convexity

Cerreia-Vioglio/Maccheroni/Marinacci/Montrucchio'11

For cash-subadditive risk measures, convexity is no longer natural

- Quasi-convexity: $\rho(\lambda X + (1-\lambda)Y) \le \max\{\rho(X), \rho(Y)\}$ for all $\lambda \in [0,1]$
- For monetary ρ, convexity $\Leftrightarrow$ quasi-convexity
- Representation ($\mathcal{P}_f$: set of finitely additive probabilities)

$$\rho(X) = \max_{Q \in \mathcal{P}_f} R\left(\mathbb{E}_Q[X], Q\right), \quad X \in \mathcal{X}$$

for some $R : \mathbb{R} \times \mathcal{P}_f \to \mathbb{R}$ satisfying some conditions

Example I: Expected insured claim

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be 1-Lipschitz (insured or retained loss)

- $\rho(X) = \mathbb{E}_{\mathbb{P}}[f(X)]$ is a cash-subadditive risk measure

Example: $\rho(X) = \mathbb{E}_{\mathbb{P}}[\min\{(X-d)_+, L\}]$

- insured loss with deductible and limit
- cash subadditive
- not quasi-convex or quasi-concave
- its range $D_\rho = [0, L]$
- $\mathbb{E}_{\mathbb{P}}$ can be replaced by any monetary risk measure

Example II: Λ-VaR

Λ-Value-at-Risk (Frittelli/Maggis/Peri'14)

$$\Lambda\mathrm{VaR}(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) \ge \Lambda(x)\}, \quad X \in \mathcal{X}$$

for some decreasing function $\Lambda : \mathbb{R} \to [0,1]$ not constantly 0

- Λ is a constant $\alpha \in (0,1) \implies \Lambda\mathrm{VaR} = \mathrm{VaR}^{\mathbb{P}}_{\alpha}$
- Cash subadditive, not cash additive
- Not quasi-convex

Quasi-star-shapedness and quasi-normalization

For a monetary ρ,

- Star-shapedness: $\rho(\lambda X) \leq \lambda\rho(X)$ for all $\lambda \in [0,1]$

is equivalent to convexity at each constant $t \in \mathbb{R}$

- $\rho(\lambda X + (1-\lambda)t) \leq \lambda\rho(X) + (1-\lambda)\rho(t)$ for all $\lambda \in [0,1]$

Quasi-star-shapedness (QSS): quasi-convexity at each constant $t \in \mathbb{R}$

- QSS: $\rho(\lambda X + (1-\lambda)t) \leq \max\{\rho(X), \rho(t)\}$ for all $\lambda \in [0,1]$

Normalization ($\rho(t) = t$ for all $t \in \mathbb{R}$) is extended to

- Quasi-normalization: $\rho(t) = t$ for all t in the range D_ρ of ρ

Quasi-star-shapedness and quasi-normalization

Proposition

For cash-additive risk measures,

(i) *normalization* $\Leftrightarrow$ *quasi-normalization;*

(ii) *star-shapedness* $\Leftrightarrow$ *quasi-star-shapedness;*

(iii) *convexity* $\Leftrightarrow$ *quasi-convexity.*

In contrast, for cash-subadditive risk measures, the above equivalence does not hold.

Properties of Λ-VaR

Proposition

The risk measure $\Lambda\mathrm{VaR}$ *has the representation, for all* $X \in \mathcal{X}$,

$$\Lambda\mathrm{VaR}(X) = \inf_{x\in\mathbb{R}} \left\{ \mathrm{VaR}^{\mathbb{P}}_{\Lambda(x)}(X) \vee x \right\} = \sup_{x\in\mathbb{R}} \left\{ \mathrm{VaR}^{\mathbb{P}}_{\Lambda(x)}(X) \wedge x \right\}.$$

Moreover, $\Lambda\mathrm{VaR}$ *is*

- *cash subadditive but generally not cash additive;*
- *quasi-star-shaped but generally not star-shaped;*
- *quasi-normalized but generally not normalized.*

Representation I

	a (...) risk measure	is an infimum of (...) risk measures
Mao/W.'20	CA, SSD-consistent	CA, convex, law-invariant
Jia/Xia/Zhao'20	CA	CA, convex
Castagnoli et al.'21	CA, star-shaped, normalized	CA, convex, normalized
Han et al.'21	CS, SSD-consistent	CS, quasi-convex, law-invariant
	CS	CS, quasi-convex
	CS, QSS, normalized	CS, quasi-convex, normalized
	CS, QSS, quasi-normalized	CS, quasi-convex, quasi-normalized

CA: cash additive; CS: cash subadditive

Representation II

Proposition

A functional $\rho : \mathcal{X} \to \mathbb{R}$ *is a cash-subadditive risk measure if and only if there exists a set* $\mathcal{R}$ *of upper semi-continuous, quasi-concave, increasing and 1-Lipschitz in the first argument functions* $R : \mathbb{R} \times \mathcal{P}_f \to \mathbb{R}$ *such that*

$$\rho(X) = \min_{R \in \mathcal{R}} \max_{Q \in \mathcal{P}_f} R\left(\mathbb{E}_Q[X], Q\right), \quad \text{for all } X \in \mathcal{X}.$$

Some references

Artzner/Delbaen/Eber/Heath'99 Coherent measures of risk. MF

Föllmer/Schied'02 Convex measures of risk and trading constraints. F&S

Frittelli/Rosazza Gianin'02 Putting order in risk measures. JBF

Glasserman/Moallemi/Yuan'16 Hidden illiquidity with multiple central counterparties. OR

Landsberger/Meilijson'90 Lotteries, insurance, and star-shaped utility functions. JET

W./Ziegel'21 Scenario-based risk evaluation. F&S

Bignozzi/Burzoni/Munari'20 Risk measures based on benchmark loss distributions. JRI

Burzoni/Munari/W.'22 Adjusted Expected Shortfall. JBF

Mao/W.'20 Risk aversion in regulatory capital calculation. SIFIN

El Karoui/Ravanelli'09 Cash subadditive risk measures and interest rate ambiguity. MF

Cerreia-Vioglio/Maccheroni/Marinacci/Montrucchio'11 Risk measures: Rationality and diversification. MF

Frittelli/Maggis/Peri'14 Risk measures on $\mathcal{P}(\mathbb{R})$ and value at risk with probability/loss function. MF

Jia/Xia/Zhao'20 Monetary risk measures. arXiv: 2012.06751

Thank you

Thank you for your kind attention

Based on the following joint work

- Castagnoli[†]/Cattelan/Maccheroni/Tabaldi/W., Star-shaped risk measures.
  Working paper, 2021, arXiv:2103.15790
- Han/Wang/W./Xia, Cash-subadditive risk measures without quasi-convexity.
  Working paper, 2021, arXiv:2110.12198

Working papers series on the theory of risk measures

http://sas.uwaterloo.ca/~wang/pages/WPS1.html