Joint Disappointment and Duet Expectile Preferences

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Agenda

Based on joint work with Fabio Bellini, Tiantian Mao and Qinyu Wu

Basic setup

- A measurable space $(\Omega, \mathcal{F})$
- $\mathcal{C}$ is a convex set of consequences (e.g., lotteries)
  - In most results in this talk, $\mathcal{C}$ will be $\mathbb{R}$ for simplicity
- $\mathcal{X}$ is the set of all acts, which are simple functions $(\Omega, \mathcal{F})$ to $\mathcal{C}$
- The decision maker has a preference relation $\succsim$ on $\mathcal{X}$

Basic setup

- The disappointment event of $X \in \mathcal{X}$: Gul'91 ECMA

$$D_X = \{\omega \in \Omega : X(\omega) \prec X\}$$

- D_X is the set of "unlucky" states of the world
- We can also consider the elation event of X

$$E_X = \{\omega \in \Omega : X(\omega) \succ X\}$$

- Example: If $\mathcal{C} = \mathbb{R}$, and a unique certainty equivalent c_X exists and $\succsim$ is strictly increasing on constants, then

$$D_X = \{\omega \in \Omega : X(\omega) < c_X\}$$

Disco acts

- Two acts X and Y are disappointment-concordant ('disco') if

$$D_X = D_Y$$

- They share the same unlucky states of the world deemed by $\succsim$
- Clearly a subjective notion

Disappointment-concordant aversion

Disappointment-concordant (disco) aversion

- A preference relation $\succsim$ satisfies disco aversion if

$$X \text{ and } Y \text{ are disco and } Y' \sim Y \implies X + Y' \succsim X + Y$$

- Adding a disco Y with X is the least favoured among all equally favoured choices
- The DM does not like events of misfortune to happen together

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Misfortune comes together

Example

- Consider $\Omega = \{0, 1\}$ and $\mathcal{C} = \mathbb{R}$
- $\succsim$: a strictly monotone preference such that $(a, b) \sim (b, a)$
- Let $X = Y = (a, b)$, $a < b$, and $Y' = (b, a) \sim Y$
- $a \prec X = Y \prec b$ and X and Y are disco
- Disco aversion implies

$$a + b = X + Y' \succsim X + Y = (2a, 2b)$$

- Clearly connected to risk aversion (e.g., $\mathbb{P}(0) = \mathbb{P}(1) = 1/2$)
- If $\succsim$ were EU, then the utility function would be concave

Concordance aversion

Which preference relations satisfy disco aversion?

An obvious example:

$$X \succsim Y \iff \mathbb{E}^P[u(X)] \geq \mathbb{E}^P[u(Y)], \quad \text{where } u : \mathcal{C} \to \mathbb{R} \text{ is linear}$$

(Neutrality $\Rightarrow$ not so interesting)

Progress

Mathematical setup

- For now, $\mathcal{C}$ is $\mathbb{R}$
- All uncertainty/randomness is in Ω
  - ... acts are random variables
- No probability assigned a priori
  - ... but will be generated by $\succsim$
- This setting will connect better to statistics and finance
- General $\mathcal{C}$: the Anscombe–Aumann (AA) framework

The finite case

- Finite setting: $\Omega = \{1, \ldots, n\}$ with $n \geq 4$, $\mathcal{F} = 2^{\Omega}$, $\mathcal{X} = \mathbb{R}^n$
- $\mathcal{M}$: measures on $(\Omega, \mathcal{F})$ with set-wise order
- $\mathcal{M}_1$: probability measures on $(\Omega, \mathcal{F})$

Axiom (Strict monotonicity - SM)

If $X \geq Y$ and $X \neq Y$ then $X \succ Y$.

Axiom (Continuity - C)

If $X_n \succsim Y_n$ for each $n \in \mathbb{N}$ and $X_n \to X$ and $Y_n \to Y$, then $X \succsim Y$.

Axiom (Disco aversion - DA)

If X and Y are disco and $Y' \sim Y$, then $X + Y' \succsim X + Y$.

Main result - I

Theorem

In the finite setting, for a preference relation $\succsim$ on $\mathcal{X}$, Axioms SM, C and DA hold if and only if there exist two probabilities $P, Q \in \mathcal{M}_1$ and $\alpha \in (0, 1/2]$ with $0 < \alpha P \le (1-\alpha)Q$ such that

$$X \succsim Y \iff \mathrm{Ex}_\alpha^{P,Q}(X) \ge \mathrm{Ex}_\alpha^{P,Q}(Y),$$

where $\mathrm{Ex}_\alpha^{P,Q}(X)$ is the unique number $y \in \mathbb{R}$ such that

$$\alpha \mathbb{E}^P[(X-y)_+] = (1-\alpha)\mathbb{E}^Q[(y-X)_+].$$

In this case, P, Q, α are unique.

- $(x)_+ = \max\{x, 0\}$

Duet expectiles

Duet expectiles

Formally, for any $P, Q \in \mathcal{M}$, define

$$\mathrm{Ex}^{P,Q}(X) = \inf\left\{y \in \mathbb{R} : \mathbb{E}^P[(X-y)_+] \leq \mathbb{E}^Q[(y-X)_+]\right\}.$$

Without restricting P, Q, α

- $\mathrm{Ex}^{P,Q} = \mathrm{Ex}_\alpha^{\widetilde{P},\widetilde{Q}}$ holds with

$$\widetilde{P} = \frac{P}{P(\Omega)}; \quad \widetilde{Q} = \frac{Q}{Q(\Omega)}; \quad \alpha = \frac{P(\Omega)}{P(\Omega)+Q(\Omega)}$$

- We call $\mathrm{Ex}^{P,Q}$ or $\mathrm{Ex}_\alpha^{\widetilde{P},\widetilde{Q}}$ a duet expectile
- $\mathrm{Ex}_\alpha^{P,P}$ is the solo expectile — Newey/Powell'87 ECMA
- Replacing disco aversion with disco propensity gives the same thesis with $\alpha P \geq (1-\alpha)Q > 0$ with $\alpha \in [1/2, 1)$

Probabilistic sophistication

- We say that $\succsim$ is (P, Q)-based: W./Ziegel'21 FS

$$X \overset{\mathrm{d}}{=}_P Y,\ X \overset{\mathrm{d}}{=}_Q Y \implies X \sim Y$$

- P and Q are endogenous $\implies$ 2-probabilistic sophistication
- In a financial market, 2-probabilistic sophistication arises
  - P and Q may be the physical and pricing probability measures

The case of an infinite Ω

Infinite, atomless, setting

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a standard probability space[1]
  - The only role played by $\mathbb{P}$ is to specify the null sets of $\mathcal{F}$, that are relevant for continuity and for strict monotonicity
- $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ with $\|\cdot\|_\infty$
- $\mathcal{M}$: non-zero σ-additive measures on $(\Omega, \mathcal{F})$
- $P \overset{\text{ac}}{\sim} Q$: P and Q are mutually absolutely continuous

[1] $(\Omega, \mathcal{F}, \mathbb{P})$ is a standard probability space if there exists a uniform random variable V on $(0, 1)$ such that $\sigma(V) = \mathcal{F}$

Weak monotone continuity

An additional continuity assumption on $\precsim$ is needed to ensure that P and Q are σ-additive

Axiom (Weak monotone continuity - WMC)

For each $m \in \mathbb{R}$ and descending chain $\{A_n\}_{n\in\mathbb{N}}$ of events with $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$, there exists $n_0 \in \mathbb{N}$ such that $m\mathbb{1}_{A_{n_0}} + \mathbb{1}_{A^c_{n_0}} \succ 0$.

- A weaker version of monotone continuity used to restrict finitely additive measures to σ-additive ones: Arrow'70

For each $X, Y \in \mathcal{X}$, $m \in \mathbb{R}$ and descending chain $\{A_n\}_{n\in\mathbb{N}}$ of events with $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$, $X \succ Y$ implies that there exists $n_0 \in \mathbb{N}$ such that $m\mathbb{1}_{A_{n_0}} + X\mathbb{1}_{A^c_{n_0}} \succ Y$.

Main result - II

Theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard probability space. For a preference relation $\succsim$ on $\mathcal{X}$, Axioms SM, C, WMC and DA hold if and only if there exist $P, Q \in \mathcal{M}_1$ and $\alpha \in (0, 1/2]$ with $\alpha P \le (1-\alpha)Q$ and $P \overset{\mathrm{ac}}{\sim} Q \overset{\mathrm{ac}}{\sim} \mathbb{P}$ such that

$$X \succsim Y \iff \mathrm{Ex}_{\alpha}^{P,Q}(X) \ge \mathrm{Ex}_{\alpha}^{P,Q}(Y).$$

In this case, P, Q, α are unique.

- Only the null sets of $\mathbb{P}$ are relevant

Anscombe-Aumann framework

- $\mathcal{C}$ is not $\mathbb{R}$ but the space of simple lotteries (i.e., finitely supported distributions)
- Acts f are mappings from $(\Omega, \mathcal{F})$ to $\mathcal{C}$

The MBA (Monotonic, Bernoullian, Archimedean) axioms

- A1-A4 of Cerreia-Vioglio/Ghirardato/Maccheroni/Marinacci/Siniscalchi'11 ET

Proposition 1 of CGMMS11

The MBA Axioms yield: There exist an affine utility $u : \mathcal{C} \to \mathbb{R}$ and a risk measure ρ such that

$$f \succsim g \iff \rho(u(f)) \geq \rho(u(g)).$$

Anscombe-Aumann framework

Theorem

In the finite setting, the MBA Axioms and Axioms SM, C and DA (adapted to the AA framework) hold if and only if there exist an affine utility function $u:\mathcal{C}\to\mathbb{R}$, two probabilities $P,Q\in\mathcal{M}_1$, and $\alpha\in(0,1/2]$ with $0<\alpha P\le(1-\alpha)Q$ such that

$$f \succsim g \iff \mathrm{Ex}_\alpha^{P,Q}(u(f)) \ge \mathrm{Ex}_\alpha^{P,Q}(u(g)).$$

- $\implies$ the Gul disappointment model

$$X \succsim Y \iff \mathrm{Ex}_\alpha^{P}(u(X)) \ge \mathrm{Ex}_\alpha^{P}(u(Y))$$

when $P=Q$ and acts are Dirac-valued

- $\implies$ the Savage subjective expected utility when $\alpha=1/2$

Progress

1. Disappointment concordance
2. Axioms and characterization
3. **Solo expectiles**
4. Risk aversion and convexity

The solo expectiles

For $X \in L^1(\Omega, \mathcal{F}, P)$, the (solo) expectile at level $\alpha \in (0,1)$ is the unique y such that

$$\alpha \mathbb{E}^P[(X-y)_+] = (1-\alpha)\mathbb{E}^P[(y-X)_+]$$

Original motivation: asymmetric least square   Newey/Powell'87 ECMA

$$\mathrm{Ex}_\alpha^P(X) = \arg\min_{y \in \mathbb{R}} \left\{ \mathbb{E}^P \left[\alpha(X-y)_+^2 + (1-\alpha)(y-X)_+^2 \right] \right\}$$

- Connection to the expectation: $\alpha = 1/2$
- Connection to the quantile: quadratic loss $\to$ linear loss
- A coherent risk measure if $\alpha \geq 1/2$

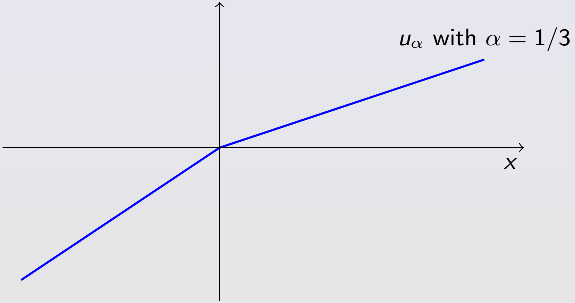
The solo expectiles

Expectiles are related to loss aversion for $\alpha \in (0, 1/2]$:

$$\mathrm{Ex}_\alpha^P(X) = \max\{m \in \mathbb{R} : \mathbb{E}[u_\alpha(X - m)] \geq u(0)\}$$

where $u_\alpha(x) = (1 - \alpha)x + (2\alpha - 1)x_+$

- (utility-indifferent price)



The solo expectiles

Which property, not ex-ante related to an exogenously given $\mathbb{P}$, characterize the (solo) expectiles among duet expectiles?

Axiom (Event independence - EI)

For all $A, B, C \in \mathcal{F}$ *disjoint,* $\mathbb{1}_A \succsim \mathbb{1}_B \implies \mathbb{1}_{A\cup C} \succsim \mathbb{1}_{B\cup C}$.

Axiom EI was used to rationalize subjective probability — de Finetti'31

Theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a standard probability space. For a preference relation* $\succsim$ *on* $\mathcal{X}$*, Axioms SM, C, EI, WMC and DA hold if and only if there exist* $P \in \mathcal{M}_1$ *and* $\alpha \in (0, 1/2]$ *with* $P \overset{\mathrm{ac}}{\sim} \mathbb{P}$ *such that* $\succsim$ *is represented by* Ex^P_α.

- In the finite case, adding Axiom EI is not enough to pin down solo expectiles

Expectile characterizations in the literature

A comparison with other expectile axiomatizations

- Elicitability + coherence — Weber'06 MF; Ziegel'16 MF
  - Based on statistical properties
  - Interpreted as risk measures — regulatory capital requirement
- The Gul disappointment model can be represented by — Gul'91

$$X \mapsto \mathrm{Ex}_{\alpha}^{\mathbb{P}}(u(X))$$

  - Assuming probability (choice over lotteries)
- Disco aversion + standard axioms — This paper
  - Not assuming probability

Probabilistic interpretation

- Dis-concordance is a notion of positive correlation
- Different from comonotonicity defined by

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0 \text{ for all } \omega, \omega' \in \Omega$$

- Concordance is subjective to $\succsim$, while comonotonicity is objective, and neither implies the other

Probabilistic interpretation

We say that aversion to a relation D holds if

$$(X,Y)\in D,\ \ Y'\sim Y \implies X+Y'\succsim X+Y$$

- No distribution
- Different from dependence aversion:

$$(X,Y)\in D,\ \ Y'\overset{\mathrm{d}}{=}Y \implies X+Y'\succsim X+Y$$

  - Risk aversion — Maccheroni/Marinacci/W./Wu'23 wp

Probabilistic interpretation

Aversion to a relation D holds:

$$(X, Y) \in D,\ \ Y' \sim Y \implies X + Y' \succsim X + Y$$

Aversion to ...	characterizes ...	in
Comonotonicity (objective)	concave Choquet integrals	Wakker'90 JET
Risk concentration (objective)	Expected Shortfall (ES)	W./Zitikis'21 MS Han/Wang/W./Wu'24 MF
Anticomonotonicity (objective)	the mean	Principi/Wakker/W.'23 wp
Disappointment concordance (subjective)	concave expectiles	this paper

Axiomatizing ES

Wang and Zitikis: *Axiomatic Foundation for the Expected Shortfall*
Management Science, 2021, vol. 67, no. 3, pp. 1413–1429, © 2020 INFORMS

Theorem 1. *A functional $\rho: L^1 \to \mathbb{R}$ with $\rho(1) = 1$ satisfies Axioms M, LI, P, and NRC if and only if $\rho = \mathrm{ES}_p$ for some $p \in (0,1)$. Moreover, in the forward direction of the above statement, the value of p is uniquely given by $1 - \mathbb{P}(A)$, where A is any stress event in Axiom NRC.*

Axiom LI (Law Invariance). *The risk value depends on the loss via its distribution; that is, $\rho(X) = \rho(Y)$ whenever $X \overset{d}{=} Y$.*

Axiom P (Prudence). *The risk value is not underestimated by approximations; that is, the bound $\lim_{k\to\infty} \rho(\xi_k) \ge \rho(X)$ holds whenever $\xi_k \to X$ (pointwise) and the limit $\lim_{k\to\infty} \rho(\xi_k)$ exists.*

Axiom M (Monotonicity). *A surely larger or equal loss leads to a larger or equal risk value; that is, $\rho(X) \le \rho(Y)$ whenever $X \le Y$.*

Axiom NRC (No Reward for Concentration). *There exists an event $A \in \mathcal{F}$ such that $\rho(X + Y) = \rho(X) + \rho(Y)$ holds for all risks X and Y sharing the tail event A.*

Axiomatizing ES

国家金融监督管理总局规章

商业银行资本管理办法

(2023年10月26日国家金融监督管理总局令2023年第4号公布 自2024年1月1日起施行)

附件15——市场风险内部模型法监管要求.docx

四、可建模风险因子的资本要求

（一）商业银行可使用任何能够反映其所有主要风险的模型方法计算市场风险资本要求，包括但不限于方差-协方差法、历史模拟法和蒙特卡罗模拟法等。

（二）商业银行应在每个交易日计算全行层面和交易台层面预期尾部损失，使用单尾 97.5%的置信区间。

（三）计算预期尾部损失时，商业银行应对基于基准 10 天持有期的预期尾部损失进行放大。公式如下：

$$ES=\sqrt{\left(ES_T(P)\right)^2+\sum_{j\geq 2}\left(ES_T(P,j)\sqrt{\frac{(LH_j-LH_{j-1})}{T}}\right)^2}$$

Progress

Risk aversion and convexity

- Weak risk aversion under $P \in \mathcal{M}_1$:

$$\mathbb{E}^P[X] \succsim X \text{ for all } X \in \mathcal{X}$$

- Strong risk aversion under $P \in \mathcal{M}_1$:[2]

$$X \geq_{\rm ssd}^P Y \text{ implies } X \succsim Y$$

- Convexity of $\succsim$:

$$X \sim Y \text{ implies } \lambda X + (1-\lambda) Y \succsim X \text{ for all } \lambda \in [0,1]$$

[2] $X \geq_{\rm ssd}^P Y$ means that $\mathbb{E}^P[\phi(X)] \geq \mathbb{E}^P[\phi(Y)]$ for all increasing and concave utility functions $\phi : \mathbb{R} \to \mathbb{R}$

Risk aversion and convexity

Theorem

Suppose that $\succsim$ *is represented by* $\mathrm{Ex}_\alpha^{P,Q}$ *for some* $P, Q \in \mathcal{M}_1$ *and* $\alpha \in (0,1)$, *and Axiom SM holds. The following are equivalent:*

(i) $\succsim$ *is weakly risk averse under* P;

(ii) $\succsim$ *is weakly risk averse under* Q;

(iii) $\succsim$ *is convex;*

(iv) $\mathrm{Ex}_\alpha^{P,Q}$ *is concave;*

(v) $\alpha P \le (1-\alpha) Q$.

Risk aversion and convexity

Consequences of the axioms in terms of risk aversion:

- SM, C and DA imply convexity of $\succsim$
- SM, C and DA imply weak risk aversion under both P and Q
- SM, C, WMC, DA and EI imply strong risk aversion under P
- If $P \neq Q$, then $\mathrm{Ex}_{\alpha}^{P,Q}$ is not law-based under either P or Q; it is not strongly risk averse under either P or Q

Dual representation

Proposition

Let $P, Q \in \mathcal{M}_1$ and $\alpha \in (0,1)$ and assume that SM *holds. If $\alpha P \le (1-\alpha)Q$, then the following dual representation holds:*

$$\mathrm{Ex}_{\alpha}^{P,Q}(X) = \inf_{R \in \mathcal{R}} \mathbb{E}^R[X] \quad \textit{for } X \in \mathcal{X},$$

where

$$\mathcal{R} = \left\{ R \in \mathcal{M}_1 : \alpha \operatorname*{ess\text{-}sup}_Q \frac{\mathrm{d}R}{\mathrm{d}Q} \le (1-\alpha) \operatorname*{ess\text{-}inf}_P \frac{\mathrm{d}R}{\mathrm{d}P} \right\}.$$

If $\alpha P \ge (1-\alpha)Q$:

- A coherent risk measure
- A sublinear expectation

Dual representation

By writing $X = u(f)$ (with standard axioms) this leads to a Gilboa-Schmeidler form

$$f \succsim g \iff \inf_{R\in\mathcal{R}} \int u(f)\mathrm{d}R \geq \inf_{R\in\mathcal{R}} \int u(g)\mathrm{d}R$$

In the finite case

$$\mathcal{R} = \left\{ R \in \mathcal{M}_1 : \frac{\mathrm{d}R/\mathrm{d}Q(\omega_1)}{\mathrm{d}R/\mathrm{d}P(\omega_2)} \leq \frac{1-\alpha}{\alpha} \text{ for all } \omega_1, \omega_2 \in \Omega \right\}$$

- $\mathcal{R}$: All probabilities R 'sandwiched' by P and Q
- $\alpha = 1/2$ forces $P = Q = R$

Further results on duet expectiles

Proposition

Let $P, Q \in \mathcal{M}$. The duet expectile $\mathrm{Ex}^{P,Q}$ has the following properties.

(i) *Monotonicity:* $\mathrm{Ex}^{P,Q}(Y) \geq \mathrm{Ex}^{P,Q}(X)$ *whenever* $Y \geq X$.

(ii) *Translation invariance:* $\mathrm{Ex}^{P,Q}(X+c) = \mathrm{Ex}^{P,Q}(X) + c$ *for* $c \in \mathbb{R}$.

(iii) *Positive homogeneity:* $\mathrm{Ex}^{P,Q}(\lambda X) = \lambda \mathrm{Ex}^{P,Q}(X)$ *for* $\lambda \geq 0$.

(iv) *Symmetry:* $\mathrm{Ex}^{P,Q}(X) = -\mathrm{Ex}^{Q,P}(-X)$.

(v) *Continuity in the following senses:*

a) $|\mathrm{Ex}^{P,Q}(X) - \mathrm{Ex}^{P,Q}(Y)| \leq \|X - Y\|_\infty^{P \vee Q}$.

b) *If* $P \wedge Q \neq 0$, *then*

$$|\mathrm{Ex}^{P,Q}(X) - \mathrm{Ex}^{P,Q}(Y)| \leq \frac{\|X - Y\|_1^{P \vee Q}}{(P \wedge Q)(\Omega)}.$$

Conclusion

What do we learn?

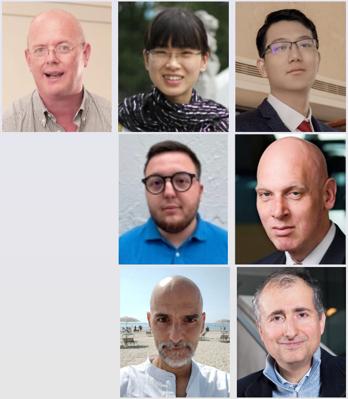
Disco, solo, duet.

Conclusion

- New concepts of disappointment-concordant (disco) acts and disco aversion
- Disco aversion is a negative attitude toward "events of misfortune come together"
- Disco aversion and other standard axioms characterize duet expectile preferences
- Two endogenous probabilities are implied
- An axiomatization of the classic solo expectile preferences
- Connection to various notions of risk aversion under two probabilities and 2-probabilistic sophistication

Thank you

Thank you for your kind attention



- Bellini/Mao/W./Wu
  Joint disappointment and duet expectile preferences (new title)
  arXiv:2404.17751, 2024
- Principi/Wakker/W.
  Anticomonotonicity for preference axioms: The natural counterpart to comonotonicity
  arXiv:2307.08542, 2023
- Maccheroni/Marinacci/W./Wu
  Risk aversion and insurance propensity
  arXiv:2310.09173, 2023

Aversion to comonotonicity

- Suppose that $\succsim$ is averse to comonotonicity
- Let $Z \mapsto c_Z$ be the unique certainty equivalent
- Let X and Y be comonotonic
- X and c_Y are also comonotonic
- Aversion to comonotonicity implies

$$c_X + c_Y \succsim X + c_Y \succsim X + Y \succsim X + c_Y \succsim c_X + c_Y$$

- $Z \mapsto c_Z$ is comonotonic additive
- Hence a Choquet integral Schmeidler'86 PAMS
- It is superadditive (concave) because

$$Y' \sim Y \implies c_{X+Y'} \geq c_{X+Y} = c_X + c_Y$$

Aversion to anticomonotonicity

- Suppose that $\succsim$ is averse to anticomonotonicity
- Let $Z \mapsto c_Z$ be the unique certainty equivalent
- Let X and Y be anticomonotonic
- X and c_Y are also anticomonotonic
- Aversion to anticomonotonicity implies

$$c_X + c_Y \succsim X + c_Y \succsim X + Y \succsim X + c_Y \succsim c_X + c_Y$$

- $Z \mapsto c_Z$ is anticomonotonic additive
- Hence it is additive

Principi/Wakker/W.'23 arXiv

Risk aversion and convexity

- Take a set $\mathcal{Q}$ of probabilities
- (Strong) $\mathcal{Q}$-risk aversion: Dentcheva/Ruszczyński'10 MP

$$X \geq_{\mathrm{ssd}}^{P} Y \text{ for all } P \in \mathcal{Q} \text{ implies } X \succsim Y$$

- $\mathrm{Ex}^{P,Q}$ is not $(\widetilde{P}, \widetilde{Q})$-risk averse in general

Proposition

If $\succsim$ is represented by $\mathrm{Ex}^{P,Q}$ for $P, Q \in \mathcal{M}$ satisfying $P \leq Q$ and $P \neq Q$, then $\succsim$ is $(\widetilde{P}, \widetilde{R})$-risk averse, where $R = Q - P$.

Further results on duet expectiles

Proposition

$\mathrm{Ex}^{P,Q}$ defined on $L^\infty(\Omega,\mathcal{F},\mathbb{P})$ satisfies SM if and only if $P \overset{\mathrm{ac}}{\sim} Q \overset{\mathrm{ac}}{\sim} \mathbb{P}$.

In the finite case $\mathrm{Ex}^{P,Q}$ satisfies SM $\iff P, Q > 0$.

Assumption (A)

There exist $S_1, S_2, S_3 \in \mathcal{F}$ such that $\{S_1, S_2, S_3\}$ is a partition of Ω, with $P(S_1)$, $P(S_2) > 0$ and $Q(S_1)$, $Q(S_3) > 0$.

- In the finite case with $n \geq 3$, $\mathrm{Ex}^{P,Q}$ satisfies SM $\implies$ $P, Q > 0$ and Assumption A holds
- In the standard probability space case, $\mathrm{Ex}^{P,Q}$ satisfies SM $\implies$ $P \overset{\mathrm{ac}}{\sim} Q \overset{\mathrm{ac}}{\sim} \mathbb{P}$ and Assumption A holds

Further results on duet expectiles

Proposition

For $P, Q \in \mathcal{M}$, assume that Assumption A holds. The duet expectile $\mathrm{Ex}^{P,Q}$ is concave (convex) if and only if $P \le Q$ ($P \ge Q$).

Proposition

Let $P, Q \in \mathcal{M}$ and suppose that Assumption A holds. The representation $\mathrm{Ex}^{P,Q} = \mathrm{Ex}_\alpha^{S,R}$ for $S, R \in \mathcal{M}_1$ and $\alpha \in (0,1)$ is uniquely given by $S = \widetilde{P}$, $R = \widetilde{Q}$ and $\alpha = P(\Omega)/(P(\Omega) + Q(\Omega))$.