Infinite-mean Pareto distributions in decision making

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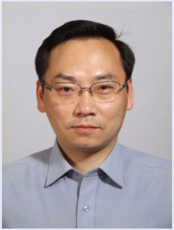
Content






Yuyu Chen (Melbourne)	Paul Embrechts (ETH Zurich)	Taizhong Hu (UST China)	Zhenzheng Zou (UST China)

- Chen/Embrechts/W., An unexpected stochastic dominance: Pareto distributions, dependence, and diversification Operations Research, 2024
- Chen/Embrechts/W., Risk exchange under infinite-mean Pareto models Working paper, 2024, arXiv:2403.20171
- Chen/Hu/W./Zou, Diversification for infinite-mean Pareto distributions Working paper, 2024, arXiv:2404.18467

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Simple probabilistic question

- Suppose that X and X' are identically distributed
- Is it possible that

$$\mathbb{P}(X < X') = 1?$$

Simple probabilistic question

- Suppose that X and X' are identically distributed
- Is it possible that

$$\mathbb{P}(X < X') = 1?$$

NO ... because if it holds true then there exists $x \in \mathbb{R}$ such that

$$\mathbb{P}(X < x) > \mathbb{P}(X' < x),$$

violating the assumption of identical distribution

Simple probabilistic question

- Suppose that X, Y, X', Y' are identically distributed
- Is it possible that

$$\mathbb{P}(X + Y < X' + Y') = 1?$$

Simple probabilistic question

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NO ... if X has finite mean ... because

$$\mathbb{E}[X + Y] = \mathbb{E}[X' + Y']$$

Simple probabilistic question

- Suppose that X, Y, X', Y' are identically distributed
- Is it possible that

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NO ... if X has finite mean ... because

$$\mathbb{E}[X + Y] = \mathbb{E}[X' + Y']$$

What if X does not have finite mean?

Pareto distribution

For $\theta, \alpha > 0$, the Pareto distribution is given by the cdf

$$P_{\alpha,\theta}(x) = 1 - \left(\frac{\theta}{x}\right)^{\alpha}, \quad x \geq \theta$$

- θ: scale parameter
- α: tail parameter
- $\mathrm{Pareto}(\alpha) = P_{\alpha,1}$
- $\mathrm{Pareto}(\alpha)$ has an infinite mean $\iff$ $\alpha \in (0, 1]$
  - extremely heavy-tailed
- the most common heavy-tailed distribution used in actuarial science

Infinite-mean models

Data from insurance, natural catastrophes, finance, and operational risk

- aircraft insurance — FINMA'21
- fire insurance — Beirlant/Dierckx/Goegebeur/Matthys'99
- commercial property insurance — Biffis/Chavez'14
- earthquakes — Ibragimov/Jaffee/Walden'09
- wind catastrophes — Rizzo'09
- nuclear power accidents — Hofert/Wüthrich'12; Sornette/Maillart/Kröger'13
- operational risk — Moscadelli'04
- cyber risk — Eling/Wirfs'19; Eling/Schnell'20
- returns from technological innovations — Silverberg/Verspagen'07

Our goals

Setup

- losses $X_1, \dots, X_n \sim \mathrm{Pareto}(\alpha)$; particular interest: $\alpha \leq 1$
- exposure vector $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$
- $\Delta_n = \{\boldsymbol{\theta} \in [0,1]^n : \sum_{i=1}^n \theta_i = 1\}$: standard n-simplex
- $[n] = \{1, \dots, n\}$
- a non-diversified portfolio: X_1
- a diversified portfolio: $\sum_{i=1}^n \theta_i X_i$

Questions:

- Which of X_1 and $\sum_{i=1}^n \theta_i X_i$ is more dangerous?
- What is the implication on a risk exchange economy?

Stochastic dominance

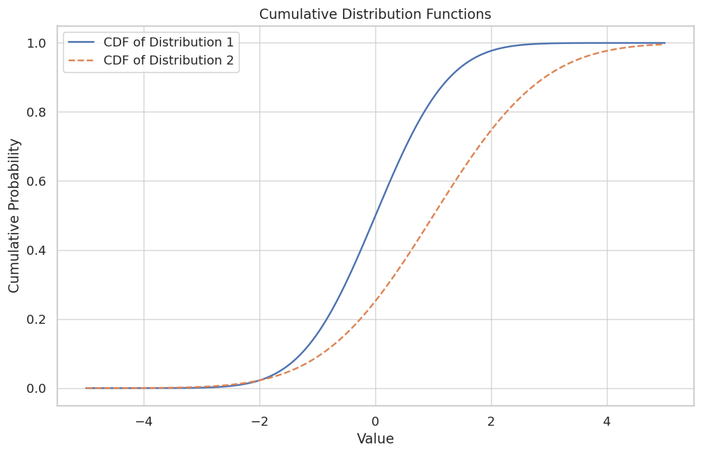
Stochastic dominance

Definition 1 (Stochastic order and convex order)

For two random variables X and Y:

- stochastic order $X \le_{\mathrm{st}} Y$ holds if $\mathbb{P}(X > x) \le \mathbb{P}(Y > x)$ for all $x \in \mathbb{R}$;
- convex order $X \le_{\mathrm{cx}} Y$ holds if $\mathbb{E}[u(X)] \le \mathbb{E}[u(Y)]$ for all convex functions u such that the two expectations exist;
- strict stochastic order $X <_{\mathrm{st}} Y$ holds if $\mathbb{P}(X > x) < \mathbb{P}(Y > x)$ for all $x > \text{ess-inf} X$.

Stochastic dominance



Stochastic dominance

- We mainly interpret X as loss
- Stochastic order $\iff$ first-order stochastic dominance
  - $\mathbb{E}[u(X)] \le \mathbb{E}[u(Y)]$ for all increasing loss functions u
  - $\rho(X) \le \rho(Y)$ for all increasing risk measures ρ

Equivalence — e.g., Theorem 1.A.1 of Shaked/Shantikumar'07

- $X \le_{\mathrm{st}} Y \iff \mathbb{P}(X' \le Y') = 1$ for some $X' \stackrel{\mathrm{d}}{=} X$ and $Y' \stackrel{\mathrm{d}}{=} Y$
- $X <_{\mathrm{st}} Y \iff \mathbb{P}(X' < Y') = 1$ for some $X' \stackrel{\mathrm{d}}{=} X$ and $Y' \stackrel{\mathrm{d}}{=} Y$

Finite-mean case

Proposition 1

Let $\theta_1, \ldots, \theta_n > 0$ such that $\sum_{i=1}^n \theta_n = 1$ and $X, X_1, \ldots, X_n$ be identically distributed random variables with finite mean and any dependence structure. Then, $X \leq_{\mathrm{st}} \sum_{i=1}^n \theta_i X_i$ holds if and only if $X_1 = \cdots = X_n$ almost surely.

- No non-trivial dominance in case of finite mean

An unexpected stochastic dominance

Theorem 1

Let $X, X_1, \dots, X_n$ be iid $\mathrm{Pareto}(\alpha)$ random variables, $\alpha \in (0,1]$. For $(\theta_1, \dots, \theta_n) \in \Delta_n$, we have

$$X \le_{\mathrm{st}} \sum_{i=1}^{n} \theta_i X_i.$$

Moreover, $X <_{\mathrm{st}} \sum_{i=1}^{n} \theta_i X_i$ if $\theta_i > 0$ for at least two $i \in [n]$.

- EVT: $\mathbb{P}(\sum_{i=1}^{n} X_i/n > t) \ge \mathbb{P}(X > t)$ for t large enough
- Known case: $n = 2$, $\theta_1 = \theta_2 = \alpha = 1/2$

Example 2.18 in the lecture slides of McNeil/Frey/Embrechts'15

Special thanks to Wenhao Zhu and Yuming Wang, who provided a first proof

An unexpected stochastic dominance

"Unexpected"

- The strict dominance

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i < \sum_{i=1}^{n} X_i'\right) = 1$$

  can happen even if $X_i \overset{\mathrm{d}}{=} X_i'$ for $i \in [n]$

- For Pareto, dominance $\iff$ no finite expectation

Generalizations

- This result has many generalizations
- Notably it holds for weak negative association, a form of negative dependence

Dominance relation between two diversified portfolios

Definition 2 (Majorization order)

For $\boldsymbol{\theta} \in (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ and $\boldsymbol{\eta} \in (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$, $\boldsymbol{\theta}$ is dominated by $\boldsymbol{\eta}$ in majorization order, denoted by $\boldsymbol{\theta} \preceq \boldsymbol{\eta}$, if

$$\sum_{i=1}^{n} \theta_i = \sum_{i=1}^{n} \eta_i \quad \text{and} \quad \sum_{i=1}^{k} \theta_{(i)} \geq \sum_{i=1}^{k} \eta_{(i)} \text{ for } k \in [n-1],$$

where $\theta_{(i)}$ is the i-th order statistic of $\boldsymbol{\theta}$ from the smallest.

- Write $\boldsymbol{\theta} \prec \boldsymbol{\eta}$ if $\boldsymbol{\theta} \preceq \boldsymbol{\eta}$ and $\boldsymbol{\theta} \neq \boldsymbol{\eta}$
- $\boldsymbol{\theta} \preceq \boldsymbol{\eta} \iff$ components of $\boldsymbol{\theta}$ are less spread out than $\boldsymbol{\eta}$
- $(1/n, \dots, 1/n) \preceq \boldsymbol{\theta} \preceq (1, 0, \dots, 0)$ for $\boldsymbol{\theta} \in \Delta_n$
- Discrete version of convex order $\leq_{\text{cx}}$ Marshall/Olkin/Arnold'11

Stochastic dominance: Majorization

Theorem 2

Suppose that $\boldsymbol{\theta}, \boldsymbol{\eta} \in \mathbb{R}_+^n$ *satisfy* $\boldsymbol{\theta} \preceq \boldsymbol{\eta}$. *Let* $\mathbf{X}$ *be a vector of n iid* $\mathrm{Pareto}(\alpha)$ *random variables,* $\alpha \in (0, 1]$. *We have*

$$\boldsymbol{\eta} \cdot \mathbf{X} \le_{\mathrm{st}} \boldsymbol{\theta} \cdot \mathbf{X}.$$

Moreover, if $\boldsymbol{\theta} \prec \boldsymbol{\eta}$, *then* $\boldsymbol{\eta} \cdot \mathbf{X} <_{\mathrm{st}} \boldsymbol{\theta} \cdot \mathbf{X}$.

Diversification pays or not

- $\mathbf{X}$: a vector of iid $\mathrm{Pareto}(\alpha)$ components
- $\boldsymbol{\theta} \preceq \boldsymbol{\eta} \implies \boldsymbol{\theta}$ is more diversified

Classic result — Theorem 3.A.35 of Shaked/Shantikumar'07

$$\alpha > 1 \implies \boldsymbol{\eta} \cdot \mathbf{X} \geq_{\mathrm{cx}} \boldsymbol{\theta} \cdot \mathbf{X}$$

- All risk-averse decision makers prefer the more diversified
- Diversification pays — Samuelson'67

Our result

$$\alpha \leq 1 \implies \boldsymbol{\eta} \cdot \mathbf{X} \leq_{\mathrm{st}} \boldsymbol{\theta} \cdot \mathbf{X}$$

- All rational decision makers prefer the less diversified
- Diversification hurts — Ibragimov/Jaffee/Walden'11

Stochastic dominance: Majorization

Corollary 1

For $k, \ell \in \mathbb{N}$ *such that* $k \le \ell$, *let* $X_1, \dots, X_\ell$ *be iid* $\mathrm{Pareto}(\alpha)$ *random variables,* $\alpha \in (0, 1]$. *We have*

$$\frac{1}{k}\sum_{i=1}^{k} X_i \le_{\mathrm{st}} \frac{1}{\ell}\sum_{i=1}^{\ell} X_i.$$

A risk exchange market

A risk exchange market

Notation

- $\mathcal{X}$: the set of random variables
- $\mathcal{X}_\rho \subseteq \mathcal{X}$: the set of financial losses
- $\rho : \mathcal{X}_\rho \to \mathbb{R}$ is a risk measure

Monotonicity

- Weak monotonicity: $\rho(X) \le \rho(Y)$ for $X, Y \in \mathcal{X}_\rho$ if $X \le_{\mathrm{st}} Y$
- Mild monotonicity: ρ is weakly monotone and $\rho(X) < \rho(Y)$ if $\mathbb{P}(X < Y) = 1$

Examples of risk measures

For $X \sim F$,

- Value-at-Risk (VaR):

$$\mathrm{VaR}_q(X) = F^{-1}(q) = \inf\{t \in \mathbb{R} : F(t) \geq q\}, \;\; q \in (0,1]$$

- Expected Shortfall (ES):

$$\mathrm{ES}_p(X) = \frac{1}{1-p} \int_p^1 \mathrm{VaR}_u(F)\mathrm{d}u, \;\; p \in (0,1)$$

- Range-VaR (RVaR):

$$\mathrm{RVaR}_{p,q}(X) = \frac{1}{q-p} \int_p^q \mathrm{VaR}_u(F)\mathrm{d}u, \;\; 0 \leq p < q < 1$$

VaR, ES and RVaR are mildly monotone

A risk exchange market

Risk exchange market

- n agents
- Pareto risks
- risk measures

Distortion risk measures

For a random variable Y, a distortion risk measure ρ is defined as

$$\rho(Y) = \int_{-\infty}^{0} (h(\mathbb{P}(Y > x)) - 1)\mathrm{d}x + \int_{0}^{\infty} h(\mathbb{P}(Y > x))\mathrm{d}x,$$

where $h : [0,1] \to [0,1]$, called the distortion function, is a nondecreasing function with $h(0) = 0$ and $h(1) = 1$

- The class includes VaR, ES, and RVaR
- Any distortion risk measure is mildly monotone unless it is a mixture of $\mathrm{ess\text{-}sup}$ and $\mathrm{ess\text{-}inf}$

A risk exchange market

A Pareto risk exchange market with $n \geq 2$ agents:

- $\mathbf{X} = (X_1, \dots, X_n)$ and $X, X_1, \dots, X_n \overset{\text{iid}}{\sim} \mathrm{Pareto}(\alpha)$ with $\alpha > 0$
- the initial exposure vector of agent i is $\mathbf{a}^i = a_i \mathbf{e}_{i,n}$ with $a_i > 0$
- $\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}_+^n$ is the premium vector
- $\mathbf{w}^i \in \mathbb{R}_+^n$ is the exposure vector of agent i over $\mathbf{X}$ after exchanging risks

The total loss of agent $i \in [n]$ after risk sharing is

$$L_i(\mathbf{w}^i, \mathbf{p}) = \mathbf{w}^i \cdot \mathbf{X} - (\mathbf{w}^i - \mathbf{a}^i) \cdot \mathbf{p}.$$

A risk exchange market

Agent $i \in [n]$ is equipped with

- a risk measure ρ_i on $\mathcal{X}$
  - $\mathcal{X}$ is the convex cone generated by $X_1, \dots, X_n$ and constants
- a cost function $c_i(\|\mathbf{w}^i\| - \|\mathbf{a}^i\|)$
  - $\mathcal{X}$ c_i is a non-negative convex function satisfying $c_i(0) = 0$

The risk assessment for agent $i \in [n]$ is

$$\rho_i(L_i(\mathbf{w}^i, \mathbf{p})) + c_i(\|\mathbf{w}^i\| - \|\mathbf{a}^i\|)$$

Equilibrium analysis in a risk exchange economy

An equilibrium of the market is $\left(\mathbf{p}^*, \mathbf{w}^{1*}, \dots, \mathbf{w}^{n*}\right) \in (\mathbb{R}_+^n)^{n+1}$ if the following two conditions hold:

(a) Individual optimality:

$$\mathbf{w}^{i*} \in \underset{\mathbf{w}^i \in \mathbb{R}_+^n}{\arg\min} \left\{\rho_i\left(L_i(\mathbf{w}^i, \mathbf{p}^*)\right) + c_i(\|\mathbf{w}^i\| - \|\mathbf{a}^i\|)\right\} \text{ for } i \in [n]$$

(b) Market clearance:

$$\sum_{i=1}^n \mathbf{w}^{i*} = \sum_{i=1}^n \mathbf{a}^i$$

In this case, the vector $\mathbf{p}^*$ is an equilibrium price, and $(\mathbf{w}^{1*}, \dots, \mathbf{w}^{n*})$ is an equilibrium allocation

Equilibrium analysis in a risk exchange economy

Theorem 3

In the Pareto risk sharing market, suppose that $\alpha \in (0,1]$, *and* $\rho_1, \dots, \rho_n$ *are mildly monotone.*

(i) *All equilibria* $\left(\mathbf{p}^*, \mathbf{w}^{1*}, \dots, \mathbf{w}^{n*}\right)$ *(if they exist) satisfy that* $\mathbf{p}^* = (p, \dots, p)$ *for some* $p \in \mathbb{R}_+$ *and* $\left(\mathbf{w}^{1*}, \dots, \mathbf{w}^{n*}\right)$ *is an* n*-permutation of* $(\mathbf{a}^1, \dots, \mathbf{a}^n)$.

(ii) *Suppose that* $\rho_1, \dots, \rho_n$ *are distortion risk measures on* $\mathcal{X}$. *The tuple* $\left((p, \dots, p), \mathbf{a}^1, \dots, \mathbf{a}^n\right)$ *is an equilibrium if* p *satisfies*

$$c'_{i+}(0) \ge p - \rho_i(X) \ge c'_{i-}(0) \quad \text{for } i \in [n].$$

- The condition in (ii) is almost necessary for $(p, \dots, p)$ to be an equilibrium price

Equilibrium analysis in a risk exchange economy

Conclusions

- No agent will hold two assets
- No risk sharing is beneficial
- Implication: In the presence of catastrophic losses, large insurance companies should not share losses with each other
- Similar results hold under trading or diversification constraints such as $\mathbf{w} \in V_b$ with $b \in [0,1)$ and

$$V_b = \left\{ (w_1, \dots, w_n) \in \mathbb{R}_+^n : w_j \geq b \sum_{i=1}^{n} w_i \text{ for } j \in [n] \right\}$$

Risk exchange with external agents

- n internal agents and $m = kn$ external agents
- Internal agents have the same mildly monotone distortion risk measure ρ_I, cost function c_I, and initial loss exposure a
- External agents have the same mildly monotone distortion risk measure ρ_E and cost function c_E
- c_I and c_E: strictly convex and continuously differentiable except at 0 with $c_I(0) = c_E(0) = 0$
- $\mathbf{u}^j \in \mathbb{R}^n_+$: exposure vector of external agent $j \in [m]$ after risk sharing
- For external agent j, the loss after risk sharing is

$$L_E(\mathbf{u}^j, \mathbf{p}) = \mathbf{u}^j \cdot \mathbf{X} - \mathbf{u}^j \cdot \mathbf{p},$$

- external agent $j \in [m]$ minimizes $\rho_E \left(L_E(\mathbf{u}^j, \mathbf{p})\right) + c_E(\|\mathbf{u}^j\|)$

Risk exchange with external agents

An equilibrium of this market is
$(\mathbf{p}^*, \mathbf{w}^{1*}, \dots, \mathbf{w}^{n*}, \mathbf{u}^{1*}, \dots, \mathbf{u}^{m*}) \in (\mathbb{R}_+^n)^{n+m+1}$ satisfying

(a) Individual optimality:

$$\mathbf{w}^{i*} \in \operatorname*{arg\,min}_{\mathbf{w}^i \in \mathbb{R}_+^n} \left\{ \rho_I \left(L_i(\mathbf{w}^i, \mathbf{p}^*) \right) + c_I(\|\mathbf{w}^i\| - \|\mathbf{a}^i\|) \right\} \text{ for } i \in [n];$$

$$\mathbf{u}^{j*} \in \operatorname*{arg\,min}_{\mathbf{u}^j \in \mathbb{R}_+^n} \left\{ \rho_E \left(L_E(\mathbf{u}^j, \mathbf{p}^*) \right) + c_E(\|\mathbf{u}^j\|) \right\} \text{ for } j \in [m]$$

(b) Market clearance:

$$\sum_{i=1}^n \mathbf{w}^{i*} + \sum_{j=1}^m \mathbf{u}^{j*} = \sum_{i=1}^n \mathbf{a}^i$$

Equilibrium analysis in a risk exchange economy

Conclusions

- This model can be completely solved
- No agent will hold two assets
- Risk sharing is beneficial among internal and external agents under a mild cost-benefit inequality
- A necessary condition for a non-trivial equilibrium is $\rho_E(X) < \rho_I(X)$ (external agents have a lower risk premium)
- Implication: In the presence of catastrophic losses, a large insurance company may seek reinsurance from external reinsurers

Risk exchange with external agents

Quadratic cost

Suppose that $c_I(x) = \lambda_I x^2$, and $c_E(x) = \lambda_E x^2$, $x \in \mathbb{R}$, where $\lambda_I, \lambda_E > 0$. We can compute the equilibrium price

$$p = \frac{k\lambda_I}{k\lambda_I + \lambda_E}\rho_E(X) + \frac{\lambda_E}{k\lambda_I + \lambda_E}\rho_I(X).$$

We also have the equlibrium allocations $\mathbf{u}^* = (u, \ldots, u)$ and $\mathbf{w}^* = (w, \ldots, w)$ where

$$u = \frac{\rho_I(X) - \rho_E(X)}{2(k\lambda_I + \lambda_E)} \quad \text{and} \quad w = \frac{k(\rho_E(X) - \rho_I(X))}{2(k\lambda_I + \lambda_E)} + a.$$

A small conjecture

A small conjecture

Our main result $\Longrightarrow$ for independent losses $Y_1, \ldots, Y_n$ following GPD with the same tail parameter $\alpha = 1/\xi \leq 1$, it holds that

$$\sum_{i=1}^{n} \mathrm{VaR}_p(Y_i) \leq \mathrm{VaR}_p\left(\sum_{i=1}^{n} Y_i\right), \text{ for all } p \in (0,1)$$

- With strict inequality
- This also holds under weighted sums and majorization

Conjecture

This holds in case of different tail parameters as well.

Infinite-mean Pareto models with different tail parameters

Estimated parameters of infinite-mean GPDs Moscadelli'04

i	1	2	3	4	5	6
ξ_i	1.19	1.17	1.01	1.39	1.23	1.22
β_i	774	254	233	412	107	243

Table: The estimated parameters ξ_i and β_i, $i \in [6]$

- GPD is parametrized by $G_{\xi,\beta}(x) = 1 - (1 + \xi x/\beta)^{-1/\xi}$ for $x \geq 0$, where $\xi \geq 0$ and $\beta > 0$

Infinite-mean Pareto models with different tail parameters

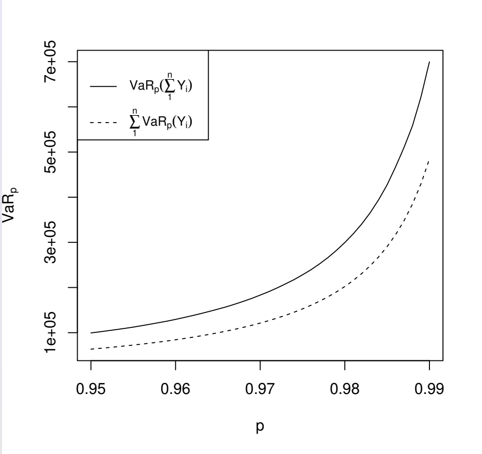


Figure: Curves of $\mathrm{VaR}_p(\sum_{i=1}^n Y_i)$ and $\sum_{i=1}^n \mathrm{VaR}_p(Y_i)$ for the $n = 6$ GPD losses

Summary

Main results

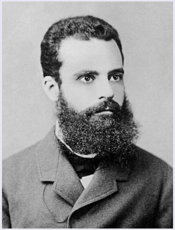
- Diversification penalty exists in many infinite-mean setups
  - The conclusion flips for infinite-mean gains instead of losses (e.g., entrepreneurship)
- Pareto risk exchange markets
  - Infinite-mean with only internal agents — no trade
  - Finite-mean with only internal agents — trade
  - Infinite-mean with external agents — trade only externally

Many open questions

- Majorization with negative association
- Different tail parameters
- Other extremely heavy-tailed distributions

Vilfredo FD Pareto (1848–1923)

Thank you for your kind attention



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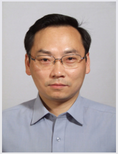

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An unexpected stochastic dominance

Proof sketch for $\boldsymbol{\theta} = (1/n, \dots, 1/n)$ and non-strict dominance.

- Define $S : (u_1, \dots, u_n) \mapsto \min_{i \in [n]} \frac{n}{i} u_{(i)}$ where $u_{(i)}$ is the i-th order statistic of $(u_1, \dots, u_n)$ from the smallest
- The Simes theorem: Simes'86

$$\textit{If } U_1, \dots, U_n \textit{ are iid } \mathrm{U}[0,1] \textit{ then } S(U_1, \dots, U_n) \textit{ is } \mathrm{U}[0,1]$$

- Comparison: for $u_1, \dots, u_n > 0$, Chen/Liu/Tan/W.'23

$$u_1^{-1} + \dots + u_n^{-1} \geq j u_{(j)}^{-1} \quad \text{for all } j \in [n]$$

$$\implies \quad u_1^{-1} + \dots + u_n^{-1} \geq n(S(u_1, \dots, u_n))^{-1}$$

$$\implies \quad \underbrace{U_1^{-1}}_{\mathrm{Pa}(1)} + \dots + \underbrace{U_n^{-1}}_{\mathrm{Pa}(1)} \geq n(S(U_1, \dots, U_n))^{-1} \overset{\mathrm{d}}{=} n \underbrace{U_1^{-1}}_{\mathrm{Pa}(1)}$$

- Inequality for generalized means: for $\alpha < 1$, Hardy-Littlewood-Pólya'34

$$\left(\frac{1}{n}(u_1^{-1/\alpha} + \dots + u_n^{-1/\alpha})\right)^{-\alpha} \leq \left(\frac{1}{n}(u_1^{-1} + \dots + u_n^{-1})\right)^{-1}$$

$$\implies \underbrace{U_1^{-\alpha}}_{\mathrm{Pa}(\alpha)} + \dots + \underbrace{U_n^{-\alpha}}_{\mathrm{Pa}(\alpha)} \geq n(S(U_1, \dots, U_n))^{-\alpha} \overset{\mathrm{d}}{=} n \underbrace{U_1^{-\alpha}}_{\mathrm{Pa}(\alpha)}$$

The Simes theorem and its impact

An improved Bonferroni procedure for multiple tests of significance

RJ Simes - Biometrika, 1986 - academic.oup.com

… , the **Bonferroni procedure** is still … the **procedure** is conservative and lacks power if several highly correlated tests are undertaken. This paper introduces a modified **Bonferroni procedure**, …

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Controlling the false discovery rate: a practical and powerful approach to multiple testing

Y Benjamini, Y Hochberg - Journal of the Royal statistical …, 1995 - Wiley Online Library

… From this point of view, a desirable error **rate** to control may be the expected proportion of errors among the rejected hypotheses, which we term the **false discovery rate** (FDR). This …

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BH'95 Theorem: For iid $\mathrm{U}[0,1]$ p-values, the BH procedure at level α has false discovery rate $\alpha K_0/K$.

Simes'86 Theorem: For iid $\mathrm{U}[0,1]$ p-values, if $K_0 = K$, then the BH procedure at level α has false discovery rate α.

Stochastic dominance: Generalizations

Diversification penalty also exists in the following setups, all with infinite mean

- Negative dependence
- Super-Pareto distributions
- Insurance portfolios: Random number and weights
- Tail risks: Tail distributions being infinite-mean Pareto
- Truncated risks: Pareto losses truncated at high levels
- Catastrophe losses: Pareto losses triggered by catastrophes
- Different indices: Pareto losses with different tail parameters

Stochastic dominance: Negative dependence

Definition 3 (Joag-Dev/Proschan'83)

A random vector $\mathbf{Z} = (Z_1, \ldots, Z_n)$ is negatively associated (NA) if for every pair of disjoint sets A, B of $[n]$,

$$\mathrm{cov}(f(\mathbf{Z}_A), g(\mathbf{Z}_B)) \le 0,$$

where $\mathbf{Z}_A = (Z_k)_{k \in A}$, $\mathbf{Z}_B = (Z_k)_{k \in B}$, and f and g are both increasing coordinatewise.

- One of the most popular notions of negative dependence
- Invariant under transforms (marginal-free, copula)

Stochastic dominance: Negative dependence

Definition 4

A set $S \subseteq \mathbb{R}^k$, $k \in \mathbb{N}$ is decreasing if $\mathbf{x} \in S$ implies $\mathbf{y} \in S$ for all $\mathbf{y} \leq \mathbf{x}$. Random variables $X_1, \dots, X_n$ are weakly negatively associated (WNA) if for any $i \in [n]$, decreasing set $S \subseteq \mathbb{R}^{n-1}$, and $x \in \mathbb{R}$ with $\mathbb{P}(X_i \leq x) > 0$,

$$\mathbb{P}(\mathbf{X}_{-i} \in S \mid X_i \leq x) \leq \mathbb{P}(\mathbf{X}_{-i} \in S),$$

where $\mathbf{X}_{-i} = (X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n)$.

- Weaker than NA in general
- Gaussian: NA $\iff$ WNA $\iff$ nonpositive correlations

Super-Pareto distribution

Definition 5

A random variable X with essential infimum $z_X \in \mathbb{R}$ is super-Pareto (or has a super-Pareto distribution) if the function $g : x \mapsto 1/\mathbb{P}(X > x)$ is strictly increasing and concave on $[z_X, \infty)$. Moreover, X is regular if $z_X > 0$ and $g(x) \leq x/z_X$ for $x \geq z_X$.

- For $\alpha \in (0, 1]$, $g : x \mapsto 1/(1 - P_{\alpha,\theta}(x)) = (x/\theta)^{\alpha} \vee 1$ is strictly increasing, concave, and bounded by x/θ on $[\theta, \infty)$ $\Longrightarrow$ all extremely heavy-tailed Pareto distributions are super-Pareto and regular
- The super-Pareto property is preserved under increasing, convex, and non-constant transforms

Stochastic dominance: Negative dependence

- WNAID: WNA and identically distributed

Theorem 4

Suppose that $X_1, \dots, X_n$ are super-Pareto and WNAID, and $X \overset{\mathrm{d}}{=} X_1$. For $(\theta_1, \dots, \theta_n) \in \Delta_n$, we have

$$X \le_{\mathrm{st}} \sum_{i=1}^{n} \theta_i X_i.$$

Moreover, $X <_{\mathrm{st}} \sum_{i=1}^{n} \theta_i X_i$ holds if $\theta_i > 0$ for at least two $i \in [n]$.

- Intuition: Negative dependence makes large losses less likely to happen together, but our first result shows that it less risky if large losses happen together

Stochastic dominance: Insurance risks

Proposition 2

Let $X, X_1, X_2, \ldots$ *be iid* $\mathrm{Pareto}(\alpha)$, $\alpha \in (0,1]$, $W_j > 0$ *for* $j = 1, 2, \ldots,$ *and* N *be a counting random variable, such that* X, $\{X_i\}_{i\in\mathbb{N}}$, $\{W_i\}_{i\in\mathbb{N}}$, *and* N *are independent. We have*

$$X\mathbb{1}_{\{N\geq 1\}} \leq_{\mathrm{st}} \frac{\sum_{i=1}^{N} W_i X_i}{\sum_{i=1}^{N} W_i} \quad \text{and} \quad \sum_{i=1}^{N} W_i X \leq_{\mathrm{st}} \sum_{i=1}^{N} W_i X_i.$$

- Classic collective risk model: $W_1 = W_2 = \cdots = 1$

$$X_1\mathbb{1}_{\{N\geq 1\}} \leq_{\mathrm{st}} \frac{1}{N}\sum_{i=1}^{N} X_i \quad \text{and} \quad NX_1 \leq_{\mathrm{st}} \sum_{i=1}^{N} X_i$$

- If $\mathbb{P}(N \geq 2) \neq 0$, then strict dominance holds

Stochastic dominance: Tail risks

- For $\alpha > 0$, we say that Y has a $\mathrm{Pareto}(\alpha)$ distribution beyond $x \geq 1$ if $\mathbb{P}(Y > t) = t^{-\alpha}$ for $t \geq x$

Proposition 3

Let $Y, Y_1, \ldots, Y_n$ be iid random variables distributed as $\mathrm{Pareto}(\alpha)$ beyond $x \geq 1$ and $\alpha \in (0, 1]$. Assume $Y \geq_{\mathrm{st}} X \sim \mathrm{Pareto}(\alpha)$. For $(\theta_1, \ldots, \theta_n) \in \Delta_n$ and $t \geq x$, $\mathbb{P}\left(\sum_{i=1}^n \theta_i Y_i > t\right) \geq \mathbb{P}\left(Y > t\right)$, and the inequality is strict if $t > 1$ and $\theta_i > 0$ for at least two $i \in [n]$.

- If $X, X_1, \ldots, X_n$ are $\mathrm{Pareto}(\alpha)$ beyond m, then

$$X \vee m \leq_{\mathrm{st}} \sum_{i=1}^n \theta_i (X_i \vee m); \quad (X - m)_+ \leq_{\mathrm{st}} \sum_{i=1}^n \theta_i (X_i - m)_+$$

Stochastic dominance: Truncated risks

Proposition 4

Let $X, X_1, \ldots, X_n$ *be iid* $\mathrm{Pareto}(\alpha)$ *random variables,* $\alpha \in (0,1]$*, and* $Y_i = X_i \wedge c_i$ *where* $c_i \geq 1$ *for each* $i \in [n]$*. Suppose that* $(\theta_1, \ldots, \theta_n) \in \Delta_n$ *with* $\theta_i > 0$ *for* $i \in [n]$*, and denote by* $c = \min\{c_1\theta_1, \ldots, c_n\theta_n\}$*. We have*

$$\mathbb{P}\left(\sum_{i=1}^{n} \theta_i Y_i > t\right) = \mathbb{P}\left(\sum_{i=1}^{n} \theta_i X_i > t\right) > \mathbb{P}(X > t)$$

for $t \in (1, c]$*.*

Stochastic dominance: Catastrophe losses

Theorem 5

Let $X_1, \ldots, X_n$ be iid $\mathrm{Pareto}(\alpha)$ random variables, $\alpha \in (0,1]$, and $A_1, \ldots, A_n$ be any events independent of $(X_1, \ldots, X_n)$. For $(\theta_1, \ldots, \theta_n) \in \Delta_n$, we have

$$\lambda X \mathbb{1}_A \le_{\mathrm{st}} \sum_{i=1}^{n} \theta_i X_i \mathbb{1}_{A_i},$$

where $\lambda \geq 1$, $X \sim \mathrm{Pareto}(\alpha)$, and A is independent of X satisfying $\lambda \mathbb{P}(A) = \sum_{i=1}^{n} \theta_i \mathbb{P}(A_i)$.

- larger losses with low frequency is better than smaller losses with high frequency

Majorization: Catastrophe losses and tail risks

Theorem 6

Let $X_1, \dots, X_n$ be iid $\mathrm{Pareto}(\alpha)$ random variables with $\alpha \in (0,1]$, and $A_1, \dots, A_n$ be events with equal probability that are independent of $X_1, \dots, X_n$. Let $\boldsymbol{Y} = (X_1 \mathbb{1}_{A_1}, \dots, X_n \mathbb{1}_{A_n})$. If $\boldsymbol{\theta}, \boldsymbol{\eta} \in \mathbb{R}_+^n$ satisfy $\boldsymbol{\theta} \preceq \boldsymbol{\eta}$, then $\boldsymbol{\theta} \cdot \boldsymbol{Y} \geq_{\mathrm{st}} \boldsymbol{\eta} \cdot \boldsymbol{Y}$.

- $\|(\theta_1, \dots, \theta_n)\| = \sum_{i=1}^n |\theta_i|$

Proposition 5

Let $\boldsymbol{Y} = (Y_1, \dots, Y_n)$ be a vector of iid $\mathrm{Pareto}(\alpha)$ random variables beyond $c \geq 1$ with $\alpha \in (0,1]$ and $\boldsymbol{\theta}, \boldsymbol{\eta} \in \mathbb{R}_+^n$ satisfy $\boldsymbol{\theta} \prec \boldsymbol{\eta}$. Then $\mathbb{P}(\boldsymbol{\theta} \cdot \boldsymbol{Y} > x) > \mathbb{P}(\boldsymbol{\eta} \cdot \boldsymbol{Y} > x)$ for $x > c\|\boldsymbol{\theta}\|$.

Different tail parameters

- $\boldsymbol{\theta}^{\uparrow} = (\theta_{(1)}, \dots, \theta_{(n)})$: increasing rearrangement of $\boldsymbol{\theta}$

Proposition 6

Suppose that $\boldsymbol{\theta}, \boldsymbol{\eta} \in \mathbb{R}_+^n$ satisfy $\boldsymbol{\theta} \preceq \boldsymbol{\eta}$. Let $\mathbf{X} = (X_1, \dots, X_n)$ be a vector of independent components with $X_i \sim \mathrm{Pareto}(\alpha_i)$ with $0 < \alpha_1 \le \cdots \le \alpha_n \le 1$. We have

$$\boldsymbol{\eta}^{\uparrow} \cdot \mathbf{X} \le_{\mathrm{st}} \boldsymbol{\theta}^{\uparrow} \cdot \mathbf{X}.$$

Moreover, if $\boldsymbol{\theta} \prec \boldsymbol{\eta}$, then $\boldsymbol{\eta}^{\uparrow} \cdot \mathbf{X} <_{\mathrm{st}} \boldsymbol{\theta}^{\uparrow} \cdot \mathbf{X}$.

Risk exchange with external agents

Let

$$L_E(b) = c_E'(b) + \rho_E(X) \quad \text{and} \quad L_I(b) = c_I'(b) + \rho_I(X), \quad b \in \mathbb{R},$$

and write $L_I^-(0) = c_{I-}'(0) + \rho_I(X)$ and $L_I^+(0) = c_{I+}'(0) + \rho_I(X)$.

- $L_E(0)$ and $L_I^-(0)$ are marginal cost and benefit of entering the market.
- To have internal and external agents participate in risk sharing, one needs

$$\rho_E(X) \le L_E(0) < p < L_I^-(0) \le \rho_I(X).$$

Result on risk exchange with external agents

In the Pareto risk sharing market of n internal and $m = kn$ external agents, let $\alpha \in (0,1]$ and $\mathcal{E} = (\mathbf{p}, \mathbf{w}^{1*}, \dots, \mathbf{w}^{n*}, \mathbf{u}^{1*}, \dots, \mathbf{u}^{m*})$.

(i) Suppose that $L_E(a/k) < L_I(-a)$. The tuple $\mathcal{E}$ is an equilibrium if and only if $\mathbf{p} = (p, \dots, p)$, $p = L_E(a/k)$, $(\mathbf{u}^{1*}, \dots, \mathbf{u}^{m*})$ is a permutation of $u^*(\mathbf{e}_{\lceil 1/k \rceil, n}, \dots, \mathbf{e}_{\lceil m/k \rceil, n})$, $u^* = a/k$, and $(\mathbf{w}^{1*}, \dots, \mathbf{w}^{n*}) = (\mathbf{0}_n, \dots, \mathbf{0}_n)$.

Result on risk exchange with external agents

(ii) Suppose that $L_E(a/k) \geq L_I(-a)$ and $L_E(0) < L_I^-(0)$. Let u^* be the unique solution to $L_E(u) = L_I(-ku)$, $u \in (0, a/k]$. The tuple $\mathcal{E}$ is an equilibrium if and only if $\mathbf{p} = (p, \dots, p)$, $p = L_E(u^*)$, $(\mathbf{u}^{1*}, \dots, \mathbf{u}^{m*}) = u^*(\mathbf{e}_{k_1,n}, \dots, \mathbf{e}_{k_m,n})$, and $(\mathbf{w}^{1*}, \dots, \mathbf{w}^{n*}) = (a - ku^*)(\mathbf{e}_{\ell_1,n}, \dots, \mathbf{e}_{\ell_n,n})$, where $k_1, \dots, k_m \in [n]$ and $\ell_1, \dots, \ell_n \in [n]$ such that $u^* \sum_{j=1}^m \mathbb{1}_{\{k_j=s\}} + (a - ku^*) \sum_{i=1}^n \mathbb{1}_{\{\ell_i=s\}} = a$ for each $s \in [n]$. Moreover, if $u^* < a/(2k)$, then the tuple $\mathcal{E}$ is an equilibrium if and only if $\mathbf{p} = (p, \dots, p)$, $p = L_E(u^*)$, $(\mathbf{u}^{1*}, \dots, \mathbf{u}^{m*})$ is a permutation of $u^*(\mathbf{e}_{\lceil 1/k \rceil,n}, \dots, \mathbf{e}_{\lceil m/k \rceil,n})$, and $(\mathbf{w}^{1*}, \dots, \mathbf{w}^{n*})$ is a permutation of $(a - ku^*)(\mathbf{e}_{1,n}, \dots, \mathbf{e}_{n,n})$.

Result on risk exchange with external agents

(iii) Suppose that $L_E(0) \geq L_I^-(0)$. The tuple $\mathcal{E}$ is an equilibrium if and only if $\mathbf{p} = (p, \dots, p)$, $p \in [L_I^-(0), L_E(0) \wedge L_I^+(0)]$, $(\mathbf{u}^{1*}, \dots, \mathbf{u}^{m*}) = (\mathbf{0}_n, \dots, \mathbf{0}_n)$, and $(\mathbf{w}^{1*}, \dots, \mathbf{w}^{n*})$ is a permutation of $a(\mathbf{e}_{1,n}, \dots, \mathbf{e}_{n,n})$.

Risk exchange market for $\alpha > 1$

Proposition 7

In the Pareto risk sharing market, suppose that $\alpha \in (1, \infty)$, *and* $\rho_1, \dots, \rho_n$ *are* ES_q *for some* $q \in (0,1)$. *Let*

$$\mathbf{w}^{i*} = \frac{a_i}{\sum_{j=1}^n a_j} \sum_{j=1}^n \mathbf{a}^j \text{ for } i \in [n] \text{ and } \mathbf{p}^* = (\mathbb{E}[X_1|A], \dots, \mathbb{E}[X_n|A]),$$

where $A = \{\sum_{i=1}^n a_i X_i \geq \mathrm{VaR}_q(\sum_{i=1}^n a_i X_i)\}$. *The tuple* $(\mathbf{p}^*, \mathbf{w}^{1*}, \dots, \mathbf{w}^{n*})$ *is an equilibrium.*

- If losses are not extremely heavy-tailed, then risk sharing is beneficial