Star-shaped and Quasi-star-shaped Risk Measures

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- **Economics of risk sharing and insurance**: New economic models for risk sharing and (re)insurance mechanisms, especially in connection to decision theory, game theory and contract theory, will enhance understanding of risk sharing markets and methodologies in sophisticated and advanced settings that are practically more realistic than traditional models.
- **Mathematics and data science of risk sharing**: Novel mathematical and statistical tools are needed to address challenging problems in risk sharing that emerge as the field quickly develops.
- **Model uncertainty in risk sharing**: In real world, risk sharing decisions are often made under uncertain circumstances, leading to the need of quantifying and understanding uncertainty arising from lack of data, model misspecification, computational limitations, or behavioral issues. Risk sharing models and mechanisms handling uncertainty in the decision-making process are useful tools in such a context.

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The deadline for submissions is **November 15, 2024**. All submissions that pass the pre-check stage will undergo peer review. The special issue is targeted to be published in **September 2025.**

Editors

- **An Chen**, Ulm University
- **Steven Vanduffel,** Vrije Universiteit Brussel
- **Ruodu Wang,** University of Waterloo

Agenda

Based on the following joint work

- Castagnoli[†]/Cattelan/Maccheroni/Tabaldi/W., Star-shaped risk measures. Operations Research, 2022
- Han/Wang/W./Xia, Cash-subadditive risk measures without quasi-convexity. Working paper, 2022, arXiv:2110.12198

Coherent risk measures

Artzner/Delbaen/Eber/Heath'99 MF

A risk measure $\rho : \mathcal{X} \to \mathbb{R}$

- $\mathcal{X}$: a convex cone of random losses
- Default choice: the set of bounded rvs on $(\Omega, \mathcal{F}, \mathbb{P})$

A coherent risk measure satisfies

- Monotonicity: $\rho(X) \le \rho(Y)$ if $X \le Y$
- Cash additivity: $\rho(X + c) = \rho(X) + c$ for all $c \in \mathbb{R}$
- Subadditivity: $\rho(X + Y) \le \rho(X) + \rho(Y)$
- Positive homogeneity (PH): $\rho(\lambda X) = \lambda \rho(X)$ for all $\lambda > 0$

  $\implies$ Normalization: $\rho(0) = 0$

Convex risk measures

Föllmer/Schied'02 FS; Frittelli/Rosazza Gianin'02 JBF

A normalized monetary risk measure satisfies

- monotonicity, cash additivity, and normalization

A convex risk measure is monetary and convex

- $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ for all $\lambda \in [0,1]$
- With positive homogeneity (PH): convexity $\iff$ subadditivity

Motivations for convexity relaxed from coherence

- Liquidity risk: $\rho(\lambda X) > \lambda\rho(X)$ for $\lambda > 1$ (violating PH)
- A merge may create extra risk: violating subadditivity

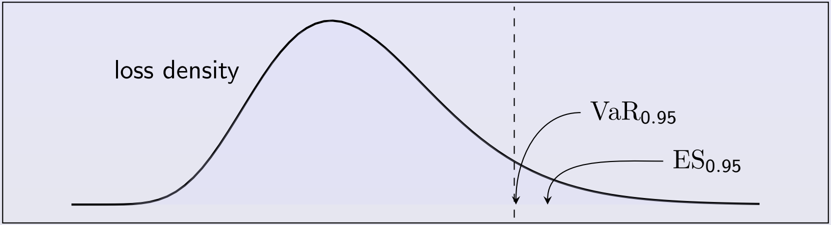
Acceptance sets

Acceptance set of a normalized monetary risk measure

$$\mathcal{A}_\rho = \{X \in \mathcal{X} : \rho(X) \le 0\}$$

- $0 \in \partial \mathcal{A}_\rho =$ boundary of $\mathcal{A}_\rho$ (normalization)
- $X \le Y$ and $Y \in \mathcal{A}_\rho \implies X \in \mathcal{A}_\rho$ (monotonicity)
- $\mathcal{A}_\rho$ is convex $\iff \rho$ is convex
- $\mathcal{A}_\rho$ is convex and conic $\iff \rho$ is coherent
- $\rho(X) = \inf\{m \in \mathbb{R} : X - m \in \mathcal{A}_\rho\}, \ X \in \mathcal{X}$ (cash additivity)

VaR and ES



Value-at-Risk (VaR), $p \in (0,1)$

$\mathrm{VaR}_p^Q : L^0 \to \mathbb{R}$,

$$\mathrm{VaR}_p^Q(X) = F_X^{-1}(p)$$
$$= \inf\{x \in \mathbb{R} : Q(X \le x) \ge p\}.$$

(left-quantile)

Expected Shortfall (ES), $p \in (0,1)$

$\mathrm{ES}_p^Q : L^1 \to \mathbb{R}$,

$$\mathrm{ES}_p^Q(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q^Q(X)\mathrm{d}q$$

(also: TVaR/CVaR/AVaR/CTE)

Progress

Star-shaped risk measures

In this part we will always assume normalization

$$\rho(0) = 0;$$

we include it in the definition of a monetary risk measure

Star-shaped risk measures

A star-shaped risk measure is monetary and star-shaped

- Star-shapedness: $\rho(\lambda X) \geq \lambda \rho(X)$ for all $\lambda > 1$

Equivalent conditions:

- $\rho(\lambda X) \leq \lambda \rho(X)$ for all $\lambda \in (0,1)$ $\iff$ convexity at 0
- The risk-to-exposure ratio $\rho(\lambda X)/\lambda$ is increasing in $\lambda > 0$
- $\mathcal{A}_\rho$ is star-shaped at 0: $X \in \mathcal{A}_\rho \implies \lambda X \in \mathcal{A}_\rho$ for all $\lambda \in [0,1]$

Star-shaped risk measures

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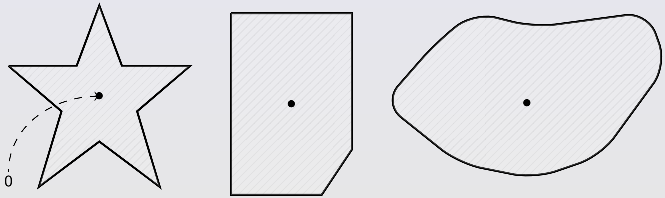


Star-shaped risk measures

Star-shapedness is

- weaker than convexity or positive homogeneity
- satisfied by all monetary risk measures in practice

For a subadditive monetary risk measure

$$\text{Star-shapedness} \iff \text{convexity} \iff \text{PH} \iff \text{coherence}$$

Motivation I: Liquidity risk

A dealer needs to clear a position X with some central clearing counterparties (CCPs)

- n CCPs with price functions $\rho_1, \dots, \rho_n$
- Liquidity cost $\implies \rho_i(\lambda X)/\lambda$ increases in $\lambda > 0$
- $\mathcal{C}$: possible compositions of CCPs (subsets of $\{1, \dots, n\}$)

Dealer's optimal clearing problem Glasserman/Moallemi/Yuan'16 OR

$$\min_{A \in \mathcal{C}} \min_{\substack{X_i \in \mathcal{X} \\ \text{s.t. } i \in A}} \left\{ \sum_{i \in A} \rho_i(X_i) \,\middle|\, \sum_{i \in A} X_i = X \right\} =: \rho(X)$$

- ρ is star-shaped but not convex (even if $\rho_1, \dots, \rho_n$ are convex)

Motivation II: Aggregation of opinions or prices

- $\rho_i, \ i \in I$: convex risk measures, expert opinions/prices
- Most conservative (convex)

$$\max_{i \in I} \rho_i(X)$$

- Most competitive (star-shaped but non-convex)

$$\min_{i \in I} \rho_i(X)$$

- α-max-min (star-shaped but non-convex)

$$\alpha \max_{i \in I} \rho_i(X) + (1-\alpha) \min_{i \in I} \rho_i(X)$$

- Median (star-shaped but non-convex)

$$\mathrm{median}\{\rho_i(X) \mid i \in I\}$$

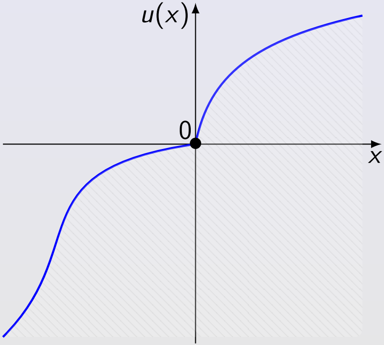
Motivation III: Non-concave utilities

Utility-based shortfall risk measures

$$\rho_u(X) = \inf\{m \in \mathbb{R} \mid \mathbb{E}_{\mathbb{P}}[u(m-X)] \geq u(0)\}, \qquad X \in \mathcal{X}$$

- u is concave $\Leftrightarrow$ ρ_u is convex $\Leftrightarrow$ strong risk aversion (empirically challengeable)
- Star-shaped at 0 utility functions $\lambda \mapsto u(\lambda)/\lambda$ is decreasing on $(0,\infty)$ and $(-\infty,0)$; $u(0)=0$

  Landsberger/Meilijson'90 JET
- u star-shaped $\Leftrightarrow$ ρ_u star-shaped



Representation I

Theorem

For a mapping $\rho : \mathcal{X} \to \mathbb{R}$, the following are equivalent:

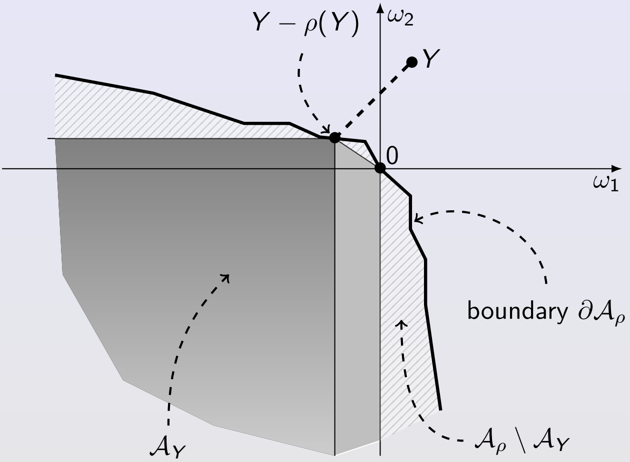
(i) *ρ is a star-shaped (resp. positively homogeneous and monetary) risk measure;*

(ii) *there exists a collection Γ of convex (resp. coherent) risk measures such that*

$$\rho(X) = \min_{\gamma \in \Gamma} \gamma(X), \qquad X \in \mathcal{X}.$$

Proof: Any star-shaped acceptance set $\mathcal{A}$ (with $0 \in \partial \mathcal{A}$) is the union of convex acceptance sets $\mathcal{A}_\gamma$ (with $0 \in \partial \mathcal{A}_\gamma$)

Proof sketch



The intuition behind the representation in case $\Omega = \{\omega_1, \omega_2\}$, where

$$\mathcal{A}_Y = \{X \in \mathcal{X} : X \leq \lambda(Y - \rho(Y)) \text{ for some } \lambda \in [0,1]\}$$

Representation II

- $\mathcal{P}$: probability measures $Q \ll \mathbb{P}$ on $(\Omega, \mathcal{F})$; $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$
- A normalized panelty is $\alpha_\gamma : \mathcal{P} \to [0, \infty]$ with $\inf_{Q\in\mathcal{P}} \alpha_\gamma(Q) = 0$
- A convex risk measure γ on $\mathcal{X}$ satisfying Fatou continuity has representation, for some normalized penalty α_γ,

$$\gamma(X) = \sup_{Q\in\mathcal{P}} \{\mathbb{E}_Q[X] - \alpha_\gamma(Q)\}, \qquad X \in \mathcal{X}$$

Proposition

A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is a star-shaped risk measure if and only if there exists a collection $\{\alpha_\gamma\}_{\gamma\in\Gamma}$ of normalized penalties such that

$$\rho(X) = \min_{\gamma\in\Gamma} \sup_{Q\in\mathcal{P}} \{\mathbb{E}_Q[X] - \alpha_\gamma(Q)\}, \qquad X \in \mathcal{X}.$$

Closure under operations

Theorem

For a collection of star-shaped risk measures, their average, supremum, infimum, and inf-convolution (when defined) are star-shaped risk measures.

- A closure property useful for many operations in finance
- This closure property also holds for
  - law-invariant star-shaped risk measures
  - SSD-consistent star-shaped risk measures
  - positively homogeneous risk measures

  but not for convex or coherent risk measures

Example I: scenario-based VaR and ES

Scenario-based risk measures W./Ziegel'21 FS

- $\mathcal{Q}$: a finite collection $\mathcal{Q}$ of probability measures

$$\mathrm{MaxVaR}_\beta^{\mathcal{Q}}(X) = \max\{\mathrm{VaR}_\beta^{Q}(X) \mid Q \in \mathcal{Q}\}$$
$$\mathrm{MaxES}_\beta^{\mathcal{Q}}(X) = \max\{\mathrm{ES}_\beta^{Q}(X) \mid Q \in \mathcal{Q}\}$$
$$\mathrm{MedVaR}_\beta^{\mathcal{Q}}(X) = \mathrm{median}\{\mathrm{VaR}_\beta^{Q}(X) \mid Q \in \mathcal{Q}\}$$
$$\mathrm{MedES}_\beta^{\mathcal{Q}}(X) = \mathrm{median}\{\mathrm{ES}_\beta^{Q}(X) \mid Q \in \mathcal{Q}\}$$

- MaxVaR, MedVaR and MedES are star-shaped but not convex
- MaxES is star-shaped and convex

Example II: benchmark VaR

The benchmark-loss VaR — Bignozzi/Burzoni/Munari'20 JRI

$$\mathrm{LVaR}_g^Q(X) = \sup_{\alpha\in(0,1)} \{\mathrm{VaR}_\alpha^Q(X) - g(\alpha)\}$$

where $g : (0,1) \to \mathbb{R}$ is increasing with $\sup_{g\in\mathcal{G}} g\,(0+) = 0$

- LVaR_g^Q is a star-shaped risk measure
- neither positively homogeneous nor convex

The adjusted ES of — Burzoni/Munari/W.'22 JBF

$$\mathrm{ES}_g^Q(X) = \sup_{\alpha\in(0,1)} \{\mathrm{ES}_\alpha^Q(X) - g(\alpha)\}$$

is convex (hence star-shaped)

Consistent risk measures

- ρ is SSD-consistent if $\rho(X) \le \rho(Y)$ whenever $X \preceq_{\text{icx}} Y$
  - $X \preceq_{\text{icx}} Y$: $\mathbb{E}_{\mathbb{P}}[f(X)] \le \mathbb{E}_{\mathbb{P}}[f(Y)]$ for all increasing convex f
- Classic definition of risk aversion — Rothschild/Stiglitz'70 JET
- Satisfied by all finite-valued law-invariant convex risk measures

Consistent risk measures

A SSD-consistent monetary risk measure ρ on $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ has representation as an infimum of adjusted ES   Mao/W.'20 SIFIN

$$\rho(X) = \inf_{g \in \mathcal{G}} \sup_{\alpha \in (0,1)} \{\mathrm{ES}^{\mathbb{P}}_\alpha(X) - g(\alpha)\} \qquad X \in \mathcal{X} \qquad \text{(MW)}$$

for some set $\mathcal{G}$ of increasing $g : (0,1) \to \mathbb{R}$ with $\sup_{g \in \mathcal{G}} g(0+) = 0$

Theorem

A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is an SSD-consistent star-shaped risk measure if and only if its has a representation (MW) in which $\mathcal{G}$ is a star-shaped set.

- This result can be generalized without cash additivity
  - e.g., return risk measures   Laeven/Rosazza Gianin/Zullino'24 IME

Progress

Cash-subadditive risk measures

El Karoui/Ravanelli'09 MF

(No longer assume normalization)

- Cash additivity: \$1 more loss $\Rightarrow$ \$1 more capital (time 0)
- No problem if interest rate is a constant
- Stochastic discount factor $D \le 1$
- $\rho(X) = \rho_0(DX)$ with monetary ρ_0
- $\rho(X+c) = \rho_0(DX + Dc) \le \rho_0(DX + c) = \rho(X) + c$

Giving rise to

- Cash subadditivity: $\rho(X+c) \le \rho(X) + c$ for all $c \ge 0$

Quasi-convexity

Cerreia-Vioglio/Maccheroni/Marinacci/Montrucchio'11 MF

For cash-subadditive risk measures, convexity is no longer natural

- Quasi-convexity: $\rho(\lambda X + (1-\lambda)Y) \le \max\{\rho(X), \rho(Y)\}$ for all $\lambda \in [0,1]$
- For monetary ρ, convexity $\Leftrightarrow$ quasi-convexity
- Representation ($\mathcal{P}_f$: set of finitely additive probabilities)

$$\rho(X) = \max_{Q \in \mathcal{P}_f} R\left(\mathbb{E}_Q[X], Q\right), \quad X \in \mathcal{X}$$

for some $R : \mathbb{R} \times \mathcal{P}_f \to \mathbb{R}$ satisfying some conditions

Example I: Expected insured claim

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be 1-Lipschitz (insured or retained loss)

- $\rho(X) = \mathbb{E}_{\mathbb{P}}[f(X)]$ is a cash-subadditive risk measure

Example: $\rho(X) = \mathbb{E}_{\mathbb{P}}[\min\{(X-d)_+, L\}]$

- insured loss with deductible and limit
- cash subadditive
- not quasi-convex or quasi-concave
- its range $D_\rho = [0, L]$
- $\mathbb{E}_{\mathbb{P}}$ can be replaced by any monetary risk measure

Example II: Risk measures based on eligible assets

- An acceptance set $\mathcal{A} \subseteq \mathcal{X}$
- A reference asset $S = (S_0, S_T) \in \mathcal{X}^2$, where $S_T \geq 0$
- Define Falka/Koch-Medina/Munari'14 FS

$$\rho_{\mathcal{A},S}(X) = \inf\left\{m \in \mathbb{R} : X - \frac{m}{S_0} S_T \in \mathcal{A}\right\}, \quad X \in \mathcal{X}$$

- The minimal needed capital S to meet the acceptability specified by $\mathcal{A}$
- $\rho_{\mathcal{A},S}$ is cash subadditive if $\mathbb{P}(S_T < S_0) = 0$
- In general not quasi-convex

Quasi-star-shapedness and quasi-normalization

For a normalized monetary ρ,

- Star-shapedness: $\rho(\lambda X) \leq \lambda \rho(X)$ for all $\lambda \in [0,1]$

is equivalent to convexity at each constant $t \in \mathbb{R}$

- $\rho(\lambda X + (1-\lambda)t) \leq \lambda \rho(X) + (1-\lambda)\rho(t)$ for all $\lambda \in [0,1]$

Quasi-star-shapedness (QSS): quasi-convexity at each constant $t \in \mathbb{R}$

- QSS: $\rho(\lambda X + (1-\lambda)t) \leq \max\{\rho(X), \rho(t)\}$ for all $\lambda \in [0,1]$

Normalization in this context: $\rho(t) = t$ for all $t \in \mathbb{R}$

- Quasi-normalization: $\rho(t) = t$ for all t in the range D_ρ of ρ
  - e.g., $\rho(X) = \mathbb{E}[X \wedge K]$

Quasi-star-shapedness and quasi-normalization

Proposition

For monotone cash-additive risk measures,

(i) *normalization* $\iff$ *quasi-normalization;*

(ii) *star-shapedness* $\iff$ *quasi-star-shapedness;*

(iii) *convexity* $\iff$ *quasi-convexity.*

In contrast, for monotone cash-subadditive risk measures, the above equivalence does not hold.

Representation I

	a (...) risk measure	is an infimum of (...) risk measures
Mao/W.'20	CA, SSD-consistent	CA, convex, law-invariant
Jia/Xia/Zhao'20	CA	CA, convex
Castagnoli et al.'22	CA, star-shaped, normalized	CA, convex, normalized
Han et al.'22	CS, SSD-consistent	CS, quasi-convex, law-invariant
	CS	CS, quasi-convex
	CS, QSS, normalized	CS, quasi-convex, normalized
	CS, QSS, quasi-normalized	CS, quasi-convex, quasi-normalized
Laeven et al.'24	QSS, SSD-consistent	convex, SSD-consistent

Always assume monotonicity; CA: cash additive; CS: cash subadditive

Representation II

Proposition

A functional $\rho : \mathcal{X} \to \mathbb{R}$ is a monotone cash-subadditive risk measure if and only if there exists a set $\mathcal{R}$ of upper semi-continuous, quasi-concave, increasing and 1-Lipschitz in the first argument functions $R : \mathbb{R} \times \mathcal{P}_f \to \mathbb{R}$ such that

$$\rho(X) = \min_{R \in \mathcal{R}} \max_{Q \in \mathcal{P}_f} R\left(\mathbb{E}_Q[X], Q\right), \quad \text{for all } X \in \mathcal{X}.$$

Progress

Λ-VaR

Λ-Value-at-Risk — Frittelli/Maggis/Peri'14 MF

$$\Lambda\mathrm{VaR}(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) \ge \Lambda(x)\}, \quad X \in \mathcal{X}$$

for some decreasing function $\Lambda : \mathbb{R} \to [0,1]$ not constantly 0

- Λ is a constant $\alpha \in (0,1) \implies \Lambda\mathrm{VaR} = \mathrm{VaR}^{\mathbb{P}}_{\alpha}$
- Cash subadditive, not cash additive
- Not quasi-convex
- Axiomatized via locality — Bellini/Peri'22 SIFIN
- Root in political science — Moulin'80 Pub. Cho.

Properties of Λ-VaR

Theorem

The risk measure $\Lambda\mathrm{VaR}$ *has the representation, for all* $X \in \mathcal{X}$,

$$\Lambda\mathrm{VaR}(X) = \inf_{x\in\mathbb{R}} \left\{ \mathrm{VaR}^{\mathbb{P}}_{\Lambda(x)}(X) \vee x \right\} = \sup_{x\in\mathbb{R}} \left\{ \mathrm{VaR}^{\mathbb{P}}_{\Lambda(x)}(X) \wedge x \right\}.$$

Moreover, $\Lambda\mathrm{VaR}$ *is*

- *cash subadditive but generally not cash additive;*
- *quasi-star-shaped but generally not star-shaped;*
- *quasi-normalized but generally not normalized.*

The Moulin Theorem

On strategy-proofness and single peakedness

H Moulin - Public Choice, 1980 - Springer

Conclusion This paper investigates one of the possible weakening of the (too demanding) assumptions of the Gibbard-Satterthwaite theorem. Namely we deal with a class of voting …

2. The characterization theorem

Theorem

The following two statements are equivalent:

(i) the voting scheme π from $\mathcal{R}^n$ into $\mathcal{R}$ is strategy-proof, anonymous, and efficient;

(ii) there exist $(n-1)$ real numbers $\alpha_1, \ldots, \alpha_{n-1} \in \mathcal{R} \cup \{+\infty, -\infty\}$ such that:

$$\forall (x_1, \ldots, x_n) \in \mathcal{R}^n \quad \pi(x_1, \ldots, x_n) = m(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_{n-1})$$

↑ median

The Moulin Theorem

- $x_1, \ldots, x_n$: the most preferred policies by voters
- π: voting scheme
- $\pi(x_1, \ldots, x_n)$: the policy based on the disclosed $x_1, \ldots, x_n$
- Voters prefer the policy to be close to their most preferred
- Strategy-proof: No voters have incentives to disclose falsely
- Anonymous: All voters are treated equal
- Efficient: The policy cannot be Pareo-improved

The Moulin Theorem

- $\alpha_n = \infty$
- Δ: cdf of $\mathrm{Unif}\{\alpha_1, \dots, \alpha_n\}$ (some points are infinite)
- F_X: cdf of $X \sim \mathrm{Unif}\{x_1, \dots, x_n\}$
- $\Lambda = 1 - \Delta$

$$\begin{aligned}\Lambda\mathrm{VaR}(X) &= \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) \ge \Lambda(x)\} \\ &= \inf\{x \in \mathbb{R} : F_X(x) + \Delta(x) \ge 1\} \\ &= \mathrm{VaR}_{1/2}((F_X + \Delta)/2) \\ &= m(x_1, \dots, x_n, \alpha_1, \dots, \alpha_{n-1})\end{aligned}$$

- $\Lambda\mathrm{VaR}(X)$ is the median of the average of F_X and $1 - \Lambda$
- There is no loss of generality to think this way

Thank you

Thank you for your kind attention

Working papers series on the theory of risk measures

http://sas.uwaterloo.ca/~wang/pages/WPS1.html