E-power and improvements for e-tests

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Agenda

1. Thresholds for e-values
2. Thresholds under distributional assumptions
3. Comonotonic e-variables
4. Improving the e-BH procedure
5. E-power

Based on joint work with Christopher Blier-Wong (Waterloo–Toronto)

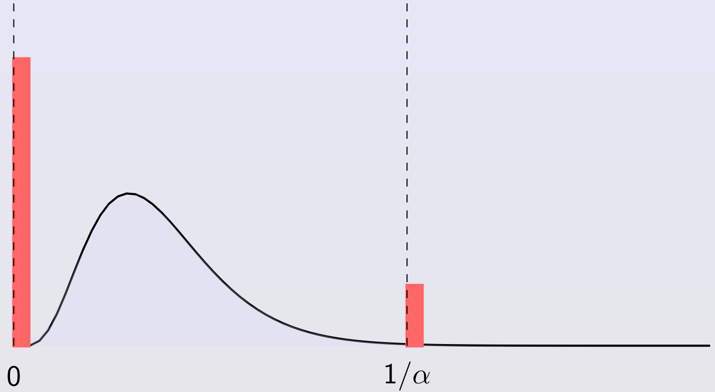
Markov's inequality

- Let E be an e-variable for $\mathcal{H}$
- Rejection threshold based on Markov's inequality: for $P \in \mathcal{H}$,

$$P\left(E \geq \frac{1}{\alpha}\right) \leq \alpha$$

- Markov's inequality is attainable
  - $P(E = 1/\alpha) = \alpha = 1 - P(E = 0)$
- Often too conservative
- The attending e-variable is very special
- Almost sharp if E is a obtained by an $\mathcal{H}$-martingale first hitting $1/\alpha$

Markov's inequality



It has to look like red and $1/\alpha$ has to precisely fall on the right leg to make Markov sharp

Rejection probability bounds for e-values

We will assume

- No access to the underlying data
- Only having the e-values
- Some side information on the underlying e-variables that the tester trusts

Rejection probability bounds for e-values

- For this work it suffices to consider $\mathcal{H} = \{\mathbb{P}\}$
- $\mathcal{E}$: a set of e-variables (to be specified later)
- $\mathcal{E}_0$: the set of all e-variables
- For $\gamma > 0$, define the quantity

$$R_\gamma(\mathcal{E}) = \sup_{E \in \mathcal{E}} \mathbb{P}(E \geq 1/\gamma)$$

- $R_\gamma(\mathcal{E})$ is the worst-case type I error of e-tests based on thresholds of $1/\gamma$
- $R_\gamma(\mathcal{E}_0) = \gamma$ for $\gamma \in (0, 1]$

Rejection probability bounds for e-values

Our goal:

- Compute $R_\gamma(\mathcal{E})$ for $\gamma \in (0, 1]$ and various $\mathcal{E}$

Thresholds for e-values

Lemma 1

For $\alpha \in (0,1)$, the quantity

$$T_\alpha(\mathcal{E}) := \inf\{t \geq 1 : R_{1/t}(\mathcal{E}) \leq \alpha\}$$

satisfies

$$T_\alpha(\mathcal{E}) = \left(\sup_{E \in \mathcal{E}} \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq 1-\alpha\}\right) \vee 1.$$

If $\gamma \mapsto R_\gamma(\mathcal{E})$ is continuous, then $T_\alpha(\mathcal{E})$ is the smallest real number $t \geq 1$ such that $\mathbb{P}(E \geq t) \leq \alpha$ for all $E \in \mathcal{E}$.

- This result and the next have nothing to do with e-variables

Calibrators

- The function $\gamma \mapsto R_{1/\gamma}(\mathcal{E})$ serves as a refinement of e-to-p calibrators
- For a subset $\mathcal{E}$ of e-variables, we say that a function $f : [0, \infty] \to [0, \infty)$ is an e-to-p calibrator on $\mathcal{E}$ if f is decreasing and $f(E)$ is a p-variable for all $E \in \mathcal{E}$
- $x \mapsto (1/x) \wedge 1$ is the only admissible e-to-p calibrator on $\mathcal{E}_0$
  - Weakly smaller than any other e-to-p calibrator — Vovk/W.'21
- For various subsets $\mathcal{E}$ of $\mathcal{E}_0$, we can find better e-to-p calibrators based on $R_{1/\gamma}(\mathcal{E})$

Calibrators

Theorem 1

The function $x \mapsto R_{1/x}(\mathcal{E})$ on $[0, \infty]$ is an e-to-p calibrator on $\mathcal{E}$, and it is the smallest such calibrator.

- By computing $R_\gamma(\mathcal{E})$ or an upper bound on it, we can convert e to p better than $x \mapsto (1/x) \wedge 1$
- Useful in case p-values are needed (e.g., BH procedure with other p-values)
- always have a smallest element (not true for p-to-e calibrators)

1. Thresholds for e-values
2. **Thresholds under distributional assumptions**
3. Comonotonic e-variables
4. Improving the e-BH procedure
5. E-power

Decreasing densities

All distributional descriptions are based on $\mathbb{P}$

$$\mathcal{E}_{\mathrm{D}} = \{E \in \mathcal{E}_0 : E \text{ has a decreasing density on its support}\}$$

- Allow point-mass at the left end-point of the support
- Examples: Exponential, Pareto

Decreasing densities

All distributional descriptions are based on $\mathbb{P}$

$$\mathcal{E}_{\mathrm{D}} = \{E \in \mathcal{E}_0 : E \text{ has a decreasing density on its support}\}$$

- Allow point-mass at the left end-point of the support
- Examples: Exponential, Pareto

Theorem 2

For $\gamma \in (0,1)$, $R_\gamma(\mathcal{E}_{\mathrm{D}}) = \gamma/2$ and $R_1(\mathcal{E}_{\mathrm{D}}) = 1$.

- Worst-case type I error with threshold $1/\gamma$ is halved
- $T_\alpha(\mathcal{E}_{\mathrm{D}}) = 1/(2\alpha)$

Unimodal densities

$$\mathcal{E}_{\mathrm{U}} = \{E \in \mathcal{E}_0 : E \text{ has a unimodal density on } \mathbb{R}\}$$

- Allow point-mass at the model
- Examples: log-normal, gamma

Unimodal densities

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- Examples: log-normal, gamma

Theorem 3

For $\gamma \in (0, 1]$, $R_\gamma(\mathcal{E}_{\mathrm{U}}) = (\gamma/2) \wedge (2\gamma - 1)$.

- Worst-case type I error with threshold $1/\gamma$ is halved if $\gamma \le 2/3$ (the most practical situation)

Log-concavity

$$\mathcal{E}_{\mathrm{LCD}} = \{E \in \mathcal{E}_0 : E \text{ has a log-concave density on } \mathbb{R}\}$$

$$\mathcal{E}_{\mathrm{LCS}} = \{E \in \mathcal{E}_0 : E \text{ has a log-concave survival function on } \mathbb{R}\}$$

$$\mathcal{E}_{\mathrm{LCF}} = \{E \in \mathcal{E}_0 : E \text{ has a log-concave distribution function on } \mathbb{R}\}$$

- Density can be 0 on $(-\infty, a)$ or (a, ∞), or both
- $\mathcal{E}_{\mathrm{LCD}} \subseteq \mathcal{E}_{\mathrm{LCS}}$; $\mathcal{E}_{\mathrm{LCD}} \subseteq \mathcal{E}_{\mathrm{LCF}}$
- $\mathcal{E}_{\mathrm{LCD}} \subseteq \mathcal{E}_{\mathrm{U}}$ and $\mathcal{E}_{\mathrm{D}} \subseteq \mathcal{E}_{\mathrm{LCF}}$
- LCD: normal, uniform, Laplace, exponential, gamma with shape parameter ≥ 1 (all confined to $[0, \infty)$ for us)
- Log-normal and Pareto distributions are LCF but not LCS

Log-concavity

Theorem 4

For $\gamma \in (0,1)$,

$$R_\gamma(\mathcal{E}_{\mathrm{LCS}}) = \exp\{s_\gamma/\gamma\} \le \exp\{1 - 1/\gamma\},$$

where s_γ is the unique solution to the equation $\exp\{s/\gamma\} = s + 1$. Further, $R_1(\mathcal{E}_{\mathrm{LCS}}) = 1$.

- Huge improvement from Markov's bound for small γ

Log-concavity

Proposition 1

For $\gamma \in (0,1]$, *we have*

$$e^{-1/\gamma} \le R_\gamma(\mathcal{E}_{\mathrm{LCD}}) \le R_\gamma(\mathcal{E}_{\mathrm{U}}) \wedge R_\gamma(\mathcal{E}_{\mathrm{LCS}}) \le e^{1-1/\gamma}.$$

- $e^{-1/\gamma}$ is the case of exponential with mean 1

Log-concavity

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For $\gamma \in (0,1]$, *we have*

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- $e^{-1/\gamma}$ is the case of exponential with mean 1

Proposition 2

For $\gamma \in (0,1]$, $R_\gamma(\mathcal{E}_{\mathrm{LCF}}) = \gamma$.

- The assumption of LCF is too weak

Log-transformed random variables

$$\mathcal{E}_{\mathrm{LS}} = \{E \in \mathcal{E}_0 : \log E \text{ has a symmetric distribution}\}$$

$$\mathcal{E}_{\mathrm{LU}} = \{E \in \mathcal{E}_0 : \log E \text{ has a unimodal distribution}\}$$

$$\mathcal{E}_{\mathrm{LUS}} = \{E \in \mathcal{E}_0 : \log E \text{ has a unimodal and symmetric distribution}\}$$

$$\mathcal{E}_{\mathrm{LD}} = \{E \in \mathcal{E}_0 : \log E \text{ has a decreasing density on its support}\}$$

$$\mathcal{E}_{\mathrm{LN}} = \{E \in \mathcal{E}_0 : E \text{ has a log-normal distribution}\}$$

- E-variables are often multiplicative ...
- Require $\mathbb{P}(E = 0) = 0$, so that $\log E$ is real-valued
- $\mathcal{E}_{\mathrm{LN}} \subseteq \mathcal{E}_{\mathrm{LUS}} \subseteq \mathcal{E}_{\mathrm{LS}}$
- The point-mass distributions $x \in (0, 1]$ are included

Log-transformed random variables

Proposition 3

For $\gamma \in (0,1)$, $R_\gamma(\mathcal{E}_{\mathrm{LS}}) = \gamma \wedge (1/2)$ *and* $R_1(\mathcal{E}_{\mathrm{LS}}) = 1$.

Proposition 4

For $\gamma \in (0,1]$, $R_\gamma(\mathcal{E}_{\mathrm{LU}}) = \gamma$.

Log-transformed random variables

Proposition 3

For $\gamma \in (0,1)$, $R_\gamma(\mathcal{E}_{\mathrm{LS}}) = \gamma \wedge (1/2)$ *and* $R_1(\mathcal{E}_{\mathrm{LS}}) = 1$.

Proposition 4

For $\gamma \in (0,1]$, $R_\gamma(\mathcal{E}_{\mathrm{LU}}) = \gamma$.

Proposition 5

For $\gamma \in (0,1)$, $R_\gamma(\mathcal{E}_{\mathrm{LN}}) = \Phi(-\sqrt{-2\log\gamma})$, *where* Φ *is the standard normal cdf, and* $R_1(\mathcal{E}_{\mathrm{LN}}) = 1$.

Log-transformed random variables

Theorem 5

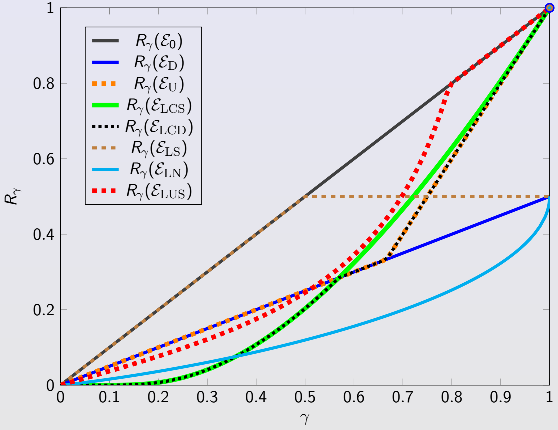
For $\gamma \in (0,1]$,

$$\frac{\gamma}{e} \le R_\gamma(\mathcal{E}_{\mathrm{LD}}) = R_\gamma(\mathcal{E}_{\mathrm{LUS}}) \le \frac{\gamma}{e}\left(\frac{1}{1-\gamma^2} \vee e\right).$$

- Improvable with a factor of $\approx e$

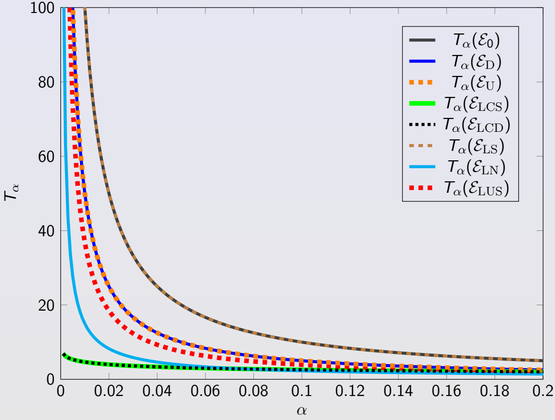
Summary of worse-case type I errors

Comparison of worse-case type I errors for different conditions on the shapes of the e-variable distributions



Summary of improved thresholds

Comparison of thresholds different conditions on the shapes of the e-variable distributions



Summary of improved thresholds

	α					
	0.001	0.01	0.02	0.05	0.1	0.2
$\mathcal{E}_0, \mathcal{E}_{\mathrm{LS}}, \mathcal{E}_{\mathrm{LU}}$	1000	100	50	20	10	5
$\mathcal{E}_{\mathrm{D}}, \mathcal{E}_{\mathrm{U}}$	500	50	25	10	5	2.5
$\mathcal{E}_{\mathrm{D}}, \mathcal{E}_{\mathrm{LUS}}$	368	36.82	18.45	7.49	3.93	2.28
$\mathcal{E}_{\mathrm{LN}}$	118	14.97	8.24	3.87	2.27	1.42
$\mathcal{E}_{\mathrm{LCD}}, \mathcal{E}_{\mathrm{LCS}}$	6.91	4.65	4	3.15	2.56	2

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Comonotonic e-variables

- A set of random variables is comonotonic if each element is an increasing function of a common random variable (e.g., data)
- For testing Q_{θ_0} against Q_θ, a common e-variable is

$$E_\theta = \frac{\mathrm{d}Q_\theta}{\mathrm{d}Q_{\theta_0}}$$

- For testing $\{Q_{\theta_0}\}$ against $\{Q_\theta : \theta \in \Theta_1\}$, one can use the mixture e-variable

$$E_\nu = \int_{\Theta_1} \frac{\mathrm{d}Q_\theta}{\mathrm{d}Q_{\theta_0}} \nu(\mathrm{d}\theta)$$

  where ν is a distribution on Θ_1
- $(E_\theta)_{\theta\in\Theta_1}$ may be comonotonic, e.g., one-sided Gaussian

Supremum of comonotonic e-variables

Proposition 6

Suppose that $\{E_\theta : \theta \in \Theta\}$ is a collection of comonotonic e-variables for a hypothesis $\mathcal{Q}$. Then

$$\sup_{Q \in \mathcal{Q}} Q\left(\sup_{\theta \in \Theta} E_\theta \geq 1/\alpha\right) \leq \alpha.$$

- If $\{Q_\theta : \theta \in \Theta\}$ is a collection of comonotonic e-variables, then we can take the supremum e-variables instead of a mixture

Example

- We are testing $\mathrm{N}(0,1)$ against $\mathrm{N}(\mu,1)$ for $\mu \neq 0$
- We have n independent observations $X_1, \dots, X_n$
- Likelihood ratio e-variable:

$$E_\mu = \exp(\mu S_n - n\mu^2/2),$$

where $S_n = \sum_{i=1}^n X_i$

- $\{E_\mu : \mu > 0\}$ is a collection of comonotonic e-variables

Example

- If the prior ν is $\mathrm{N}(0,1)$, then the mixture test (two-sided alternative) is

$$E_\nu(n) = \frac{1}{\sqrt{n+1}} \exp\left(\frac{S^2}{2n+2}\right)$$

- Suppose the alternative is $\mu > 0$, we can use

$$Y(n) = \sup_{\mu>0} \exp\left(\mu S_n - n\mu^2/2\right) = \exp\left(\frac{(S_n)_+^2}{2n}\right)$$

- Taking the supremum does not necessarily generalize to optional stopping: $Y(n)$ is only a valid test for fixed n

Example

- Data from $\mathrm{N}(0.3, 1)$
- Null hypothesis $\mathrm{N}(0, 1)$
  - 10,000 replications
  - $\alpha = 0.05$
  - average sample needed to archive power

Example

Test	Threshold	n 10	50	100	500	β 0.5	0.9	0.95	0.99
E_μ with $\mu = 0.3$	$\mathcal{T}_\alpha(\mathcal{E}_0)$	0	0.36	0.69	1	67	179	227	361
	$\mathcal{T}_\alpha(\mathcal{E}_{\mathrm{U}})$	0.03	0.49	0.77	1	52	158	206	321
	$\mathcal{T}_\alpha(\mathcal{E}_{\mathrm{LUS}})$	0.05	0.54	0.80	1	45	145	193	312
	$\mathcal{T}_\alpha(\mathcal{E}_{\mathrm{LN}})$	0.17	0.67	0.86	1	31	124	171	286
	OS	0	0.46	0.80	1	54	138	177	272
E_μ with $\mu = 0.4$	$\mathcal{T}_\alpha(\mathcal{E}_0)$	0.02	0.36	0.59	0.97	75	283	391	681
	$\mathcal{T}_\alpha(\mathcal{E}_{\mathrm{U}})$	0.07	0.46	0.66	0.98	59	257	362	647
	$\mathcal{T}_\alpha(\mathcal{E}_{\mathrm{LUS}})$	0.10	0.50	0.69	0.98	51	247	347	633
	$\mathcal{T}_\alpha(\mathcal{E}_{\mathrm{LN}})$	0.23	0.59	0.75	0.98	34	219	322	604
	OS	0.03	0.49	0.75	0.99	51	179	246	451
Supremum	$\mathcal{T}_\alpha(\mathcal{E}_0)$	0.07	0.37	0.71	1	67	153	183	253
	$\mathcal{T}_\alpha(\mathcal{E}_{\mathrm{U}})$	0.12	0.49	0.80	1	52	128	159	221
	$\mathcal{T}_\alpha(\mathcal{E}_{\mathrm{LN}})$	0.24	0.68	0.91	1	31	95	119	174
Mixture	$\mathcal{T}_\alpha(\mathcal{E}_0)$	0.02	0.14	0.39	1	122	240	280	371
	$\mathcal{T}_\alpha(\mathcal{E}_{\mathrm{U}})$	0.03	0.20	0.48	1	104	217	254	338

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E-BH procedure

- K hypotheses
- $e_1, \dots, e_K$: e-values
- $e_{[1]} \geq \cdots \geq e_{[K]}$: order statistics

E-BH procedure

The e-BH procedure $\mathcal{G}(\alpha) : [0, \infty]^K \to 2^{\mathcal{K}}$ for $\alpha > 0$ rejects hypotheses with the largest k^* e-values, where

$$k^* = \max \left\{ k \in \mathcal{K} : \frac{k e_{[k]}}{K} \geq \frac{1}{\alpha} \right\}.$$

- The e-BH procedure applied to arbitrarily dependent (AD) e-values has FDR at most $K_0 \alpha / K$

Boosting e-values in the e-BH procedure

- Under AD, find constant $b \geq 1$ such that

$$\mathbb{E}[T(\alpha b E)] \leq \alpha,$$

where $K/\mathcal{K} := \{K/k : k \in \mathcal{K}\}$, and

$$T(x) = \frac{K}{\lceil K/x \rceil} \mathbb{1}_{\{x \geq 1\}} \text{ with } T(\infty) = K.$$

- Under positive regression dependence on a subset (PRDS), find $b \geq 1$ such that

$$\max_{x \in K/\mathcal{K}} x \mathbb{P}(\alpha b E \geq x) \leq \alpha$$

Boosting e-values in the e-BH procedure

- To avoid reliance on K, consider the relaxed conditions:
  - under AD, $\mathbb{E}\left[\alpha b E \mathbb{1}_{\{\alpha b E \geq 1\}}\right] \leq \alpha$
  - under PRDS, $\max_{x \geq 1} x\mathbb{P}(\alpha b E \geq x) \leq \alpha$
- Assuming continuity, define the boosting factor for null hypotheses $E \in \mathcal{E}$
  - under AD as $B_\alpha^{\mathrm{AD}}(\mathcal{E})$, where

$$B_\alpha^{\mathrm{AD}}(\mathcal{E}) = \inf_{E \in \mathcal{E}} \sup\left\{c \geq 1 : \mathbb{E}\left[\alpha c E \mathbb{1}_{\{\alpha c E \geq 1\}}\right] \leq \alpha\right\}$$

  - under PRDS as $B_\alpha^{\mathrm{PR}}(\mathcal{E})$, where

$$B_\alpha^{\mathrm{PR}}(\mathcal{E}) = \inf_{E \in \mathcal{E}} \sup\left\{c \geq 1 : \max_{x \geq 1} x\mathbb{P}(\alpha c E \geq x) \leq \alpha\right\}$$

Boosting e-values in the e-BH procedure

Theorem 6

For some $\alpha \in (0,1)$, *let* $c_1^{\mathrm{AD}}(\alpha)$ *be the unique constant* $b \geq 1$ *such that*

$$e^{-1/(\alpha b)}(1+\alpha b) = \alpha / e$$

and $c_2^{\mathrm{AD}}(\alpha)$ *be the unique constant* $b' \geq 1$ *such that*

$$e^{-1/(\alpha b')}(1+\alpha b') = \alpha$$

Then,

$$c_1^{\mathrm{AD}}(\alpha) \leq B_\alpha^{\mathrm{AD}}(\mathcal{E}_{\mathrm{LCS}}) \leq c_2^{\mathrm{AD}}(\alpha).$$

- Under AD with nulls $E \in \mathcal{E}_{\mathrm{LCS}}$, we can boost e-values by $17.35, 9.82, 4.74$ and 2.83 for $\alpha = 0.01, 0.02, 0.05$ and 0.1

Boosting e-values in the e-BH procedure

Theorem 7

Define

$$c_1^{\mathrm{PR}}(\alpha) = \frac{1}{\alpha - \alpha \log \alpha}; \qquad c_2^{\mathrm{PR}}(\alpha) = \begin{cases} e, & \alpha \geq 1/e \\ -\frac{1}{\alpha \log \alpha}, & \alpha \leq 1/e \end{cases}.$$

We have that

$$c_1^{\mathrm{PR}}(\alpha) \leq B_\alpha^{\mathrm{PR}}(\mathcal{E}_{\mathrm{LCS}}) \leq c_2^{\mathrm{PR}}(\alpha).$$

- Under PRDS with nulls $E \in \mathcal{E}_{\mathrm{LCS}}$, we can boost e-values by $17.84, 10.18, 5.01$ and 3.03 for $\alpha = 0.01, 0.02, 0.05$ and 0.1

Example

- Assume null e-variable follows $\mathrm{Exp}(1)$
  - Remark that $E \in \mathcal{E}_{\mathrm{LCS}}$
- Assume alternative follows $\mathrm{Gamma}(1+\Theta, 1/(1+\Theta))$, where Θ follows $\mathrm{Exp}(1/b)$
  - If $b = 0$, the alternative reduces to the null
  - We set $b = 4$; mean under the alternative is 41
- Let $K = 1000$, $K_0 = 500$
- Simulate K e-values under the null and alternative with negative dependence between 500 pairs

Example

Number of discoveries and observed FDP for

- base e-BH
- boosted e-BH under AD with $E \in \mathcal{E}_{\mathrm{LCS}}$
- boosted e-BH under AD with $E \overset{\mathrm{d}}{\sim} \mathrm{Exp}(1)$ W./Ramdas'22
- p-BH procedure with $P = \exp(-E)$ (no FDR proof)

	e-BH	boosted $\mathcal{E}_{\mathrm{LCS}}$		boosted $\mathrm{Exp}(1)$		p-BH	
α	Discov.	Discov.	FDP	Discov.	FDP	Discov.	FDP
0.01	0	68.7	0	158.7	0	340.7	0.00505
0.02	0	135.0	0	201.7	0	356.0	0.01005
0.05	0	196.8	0	252.7	0	382.3	0.02497
0.10	0	235.7	0	292.2	0	411.9	0.05001

E-power

- Power of p-variable under an alternative Q:

$$Q(P \le \alpha)$$

E-power

- Power of p-variable under an alternative Q:

$$Q(P \leq \alpha)$$

Question

How do we e-valuate the power?

E-power

- Q: an alternative probability
- E: an e-variable testing P (against Q)

We have seen a lot about this object — Shafer, Grünwald, Ramdas, ...

$$\Psi^Q(E) = \mathbb{E}^Q[\log E]$$

which we call the e-power — Vovk/W.'24 NEJSDS

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Why $\Psi^Q(E)$?

E-power

About $\Psi^Q(E) = \mathbb{E}^Q[\log E]$ there are many nice things

- Relations to likelihood ratios, optimal growth rate, RIPr, ...

E-power

About $\Psi^Q(E) = \mathbb{E}^Q[\log E]$ there are many nice things

- Relations to likelihood ratios, optimal growth rate, RIPr, ...
- $\mathbb{E}^Q[E] \leq 1 \implies \Psi^Q(E) \leq 0$

E-power

About $\Psi^Q(E) = \mathbb{E}^Q[\log E]$ there are many nice things

- Relations to likelihood ratios, optimal growth rate, RIPr, ...
- $\mathbb{E}^Q[E] \le 1 \implies \Psi^Q(E) \le 0$
- $\Psi^Q(E_1) > 0$ and $\Psi^Q(E_2) > 0$, E_1, E_2 independent $\implies \Psi^Q(E_1 E_2) > 0$ and $\Psi^Q(E_1/2 + E_2/2) > 0$

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  $\implies \Psi^Q(E_1 E_2) > 0$ and $\Psi^Q(E_1/2 + E_2/2) > 0$

  (independence is not really needed)
- $\Psi^Q(E) > 0$ and $\lambda \in (0,1) \implies \Psi^Q((1-\lambda) + \lambda E) > 0$

E-power

There are also troubles ...

- It is not well defined for all E
  - An extreme example is $Q(E=0)>0$ and $Q(E=\infty)>0$, but there are finite examples

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- $\Psi^Q(E)=-\infty$ for $Q(E=0)>0$ may be sensible

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- $\Psi^Q(E)=-\infty$ for $Q(E=0)>0$ may be sensible, ... but $-\infty$ also for $E=\exp(m-X)$ with $X\overset{\mathrm{d}}{\sim}\mathrm{Pareto}(1)$ and $m\in\mathbb{R}$?

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- $\Psi^Q(E)=-\infty$ for $Q(E=0)>0$ may be sensible, ... but $-\infty$ also for $E=\exp(m-X)$ with $X \overset{\mathrm{d}}{\sim} \mathrm{Pareto}(1)$ and $m\in\mathbb{R}$?
- $\Psi^Q(E_1E_2)=\Psi^Q(E_1)+\Psi^Q(E_2)$ regardless of E_1 and E_2

E-power

Let us be formal

- Fix $(\Omega, \mathcal{F})$ and a probability Q
- Let $\mathcal{X}$ be a set of bounded nonnegative measurable functions (e-variables for some $\mathbb{P}$)
- For now we exclude unbounded random variables
- A candidate e-power function $\Pi : \mathcal{X} \to [-\infty, \infty]$

E-power

What are some good properties?

P1 Law-invariance: $\Pi(E)$ is determined by the distribution of E under Q

E-power

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P2 Strict monotonicity: $\Pi(E_1) \le \Pi(E_2)$ if $E_1 \le E_2$, and $\Pi(E_1) < \Pi(E_2)$ if $Q(E_1 < E_2) = 1$

E-power

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- P5 Symmetry: $\Pi(E^{-1}) = -\Pi(E)$ if $E^{-1} \in \mathcal{X}$

Characterization

Theorem 8

A function $\Pi : \mathcal{X} \to [-\infty, \infty]$ satisfies P1-P5 if and only if there exists a strictly increasing and symmetric function f such that

$$\Pi(E) = f(\mathbb{E}^Q[\log E]) \quad \text{for all } E \in \mathcal{X}.$$

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- Proof based on a recent result Mu/Pomatto/Strack/Tamuz'24 ECMA
- Justified Ψ^Q as an e-power function
- Can be extended beyond $\mathcal{X}$ as long as $\Psi^Q(E)$ is well-defined
- Did not address the problem of undefinedness of $\Psi^Q(E)$

Impossible to get rid of the undesirable properties if we wish to keep the desirable ones

E-power

Write

$$L_t(E) = \frac{1}{t}\log \mathbb{E}^Q[E^t] \text{ for } t \in \mathbb{R}\setminus\{0\}; \quad L_0(E) = \mathbb{E}^Q[\log E]$$

$$L_{-\infty}(E) = \operatorname{ess\text{-}inf}_Q \log E; \quad L_\infty(E) = \operatorname{ess\text{-}sup}_Q \log E;$$

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$$\Pi(E) = f\left(\int_{-\infty}^{\infty} L_t(E) \mathrm{d}\mu(t)\right) \quad \text{for some } \mu \in \mathcal{M}(\mathbb{R}) \text{ and } f \in \mathcal{I}$$

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P1-P4 $\Longleftrightarrow$

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Take-away messages

- A factor of 2 does not hurt in many situations
- Taking a supremum does not hurt in some situations
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  ... but with e-removable (?) drawbacks

Thank you

Thank you for your kind attention

Based on joint work with

Christopher Blier-Wong
(Watrerloo)