Diversification quotients: Quantifying diversification via risk measures

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Content

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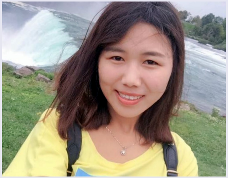

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Agenda

1. Motivation
2. Axiomatic theory of diversification indices
3. Properties of DQ
4. Portfolio optimization
5. Elliptical models
6. Empirical results for financial data

Diversification

How to quantify diversification if you must

Diversification

Portfolio diversification

- Markowitz (mean-variance analysis)
- CAPM (removing idiosyncratic risk)
- ...

As least two approaches to quantify diversification

- Heuristic: number of different investments
- Quantitative: formal reasoning
  - via risk reduction or utility improvement

Diversification indices

Quantitative setup

- $\mathcal{X}$: a convex cone of random variables, e.g., L^1
- One-period portfolio loss/payoff vector: $\mathbf{X} \in \mathcal{X}^n$
- Diversification index: $D : \mathcal{X}^n \to \overline{\mathbb{R}} := [-\infty, \infty]$
  - Convention: smaller D represents better diversification
  - Always write $\mathbf{X} = (X_1, \dots, X_n)$

Examples: diversification ratios (DR) with $0/0 = 0$

$$\mathrm{DR}^{\mathrm{SD}}(\mathbf{X}) = \frac{\mathrm{SD}(\sum_{i=1}^n X_i)}{\sum_{i=1}^n \mathrm{SD}(X_i)} \quad \text{and} \quad \mathrm{DR}^{\mathrm{var}}(\mathbf{X}) = \frac{\mathrm{var}(\sum_{i=1}^n X_i)}{\sum_{i=1}^n \mathrm{var}(X_i)}$$

Diversification indices

$\mathrm{DR}^{\mathrm{SD}}$ and $\mathrm{DR}^{\mathrm{var}}$ satisfy three natural properties

[+] Non-negativity: $D(\mathbf{X}) \geq 0$ for all $\mathbf{X} \in \mathcal{X}^n$

- with $D = 0$ being the most diversified

[LI] Location invariance: $D(\mathbf{X} + \mathbf{c}) = D(\mathbf{X})$ for all $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{X} \in \mathcal{X}^n$

- injecting risk-free payoff to each component does not affect D
- changing initial price of each component does not affect D

[SI] Scale invariance: $D(\lambda\mathbf{X}) = D(\mathbf{X})$ for all $\lambda > 0$ and $\mathbf{X} \in \mathcal{X}^n$

- rescaling of a portfolio does not affect D
- the counting unit or currency (non-random) does not affect D

Generally not convex

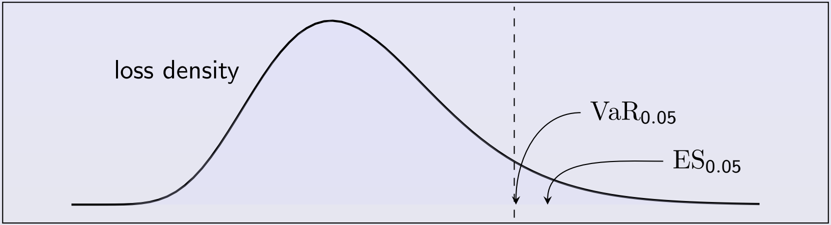
Diversification indices

- SD and var are simple but coarse measures of risk
- By internal needs or regulation, risk should be assessed by risk measures $\phi : \mathcal{X} \to \mathbb{R}$
  - regulatory capital calculation, capital allocation, performance analysis, optimization, ...
  - VaR and ES (CVaR) are popular in banking and insurance regulatory frameworks, such as Basel III/IV and Solvency II
  - monetary/convex/coherent risk measures

Artzner/Delbaen/Eber/Heath'99; Follmer/Schied'16

What is a suitable diversification index based on risk measures?

VaR and ES



Value-at-Risk (VaR), $\alpha \in (0,1)$

$\mathrm{VaR}_\alpha : L^0 \to \mathbb{R}$,

$$\mathrm{VaR}_\alpha(X) = q_\alpha(X)$$
$$= \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) \ge 1-\alpha\}$$

(left-quantile)

Expected Shortfall (ES), $\alpha \in (0,1)$

$\mathrm{ES}_\alpha : L^1 \to \mathbb{R}$,

$$\mathrm{ES}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)\mathrm{d}\beta$$

(also: TVaR/CVaR/AVaR)

Diversification indices based on risk measures

Natural candidate

Tasche'07; McNeil/Frey/Embrechts'15

$$\mathrm{DR}^{\phi}(\mathbf{X}) = \frac{\phi\left(\sum_{i=1}^{n} X_i\right)}{\sum_{i=1}^{n} \phi(X_i)}$$

D	[+]	[LI]	[SI]
DR^{ϕ}	No	No	ϕ pos. hom.

- DR is impossible to interpret if "negative over negative"
- similar problem if we use "difference" instead of "ratio"
- awkward for optimization
- wrong incentives in some simple models ($\Rightarrow$ next slide)
- $\Longrightarrow$ a new index is needed if risk measures are used

Diversification indices based on risk measures

Consider three models (same mean and covariance)

1. iid standard normal: $\mathbf{Z} = (Z_1, \dots, Z_n)$
2. iid shock: $\mathbf{Y} = (\xi_1 Z_1, \dots, \xi_n Z_n)$ where $\xi_1, \dots, \xi_n$ are iid heavy-tailed shocks independent of $\mathbf{Z}$
3. common shock: $\mathbf{Y}' = (\xi Z_1, \dots, \xi Z_n)$ where $\xi \overset{\mathrm{d}}{=} \xi_1$ is a heavy-tailed shock independent of $\mathbf{Z}$

Diversification indices based on risk measures

Consider three models (same mean and covariance)

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Intuitive relation on diversification (smaller $\Rightarrow$ better):

$$\text{Model 1} \le \text{Model 2} < \text{Model 3}$$

- If $\xi^2 \sim \mathrm{ig}(\nu/2, \nu/2)$, then $\mathbf{Y} \sim \mathrm{it}_n(\nu)$ and $\mathbf{Y}' \sim \mathrm{t}(\nu, \mathbf{0}, I_n)$

Diversification indices based on risk measures

Table: DR, where $\alpha = 0.05$ and $n = 10$

D	$\mathrm{DR}^{\mathrm{VaR}_\alpha}$	$\mathrm{DR}^{\mathrm{ES}_\alpha}$	$\mathrm{DR}^{\mathrm{SD}}$	$\mathrm{DR}^{\mathrm{var}}$
$\mathbf{Z} \sim \mathrm{N}(\mathbf{0}, I_n)$	0.3162	0.3162	0.3162	1
$\mathbf{Y} \sim \mathrm{it}_n(3)$	0.3568	0.3058	0.3162	1
$\mathbf{Y}' \sim \mathrm{t}(3, \mathbf{0}, I_n)$	0.3162	0.3162	0.3162	1
Z better than **Y**	Yes	No	No	No
Y better than **Y′**	No	Yes	No	No

$$\mathrm{DR}^{\mathrm{SD}}(\mathbf{Z}) = \mathrm{DR}^{\mathrm{SD}}(\mathbf{Y}') = \mathrm{DR}^{\mathrm{SD}}(\mathbf{Y}) = 1/\sqrt{n}$$

Diversification indices based on risk measures

Question: Can we find a diversification index that is

- based on a specified risk measure (e.g., VaR or ES)
- satisfying the three natural properties [+], [SI] and [LI]
- consistent with common portfolio dependence structures
- natural to interpret
- able to capture heavy tails and common shocks
- convenient to compute and optimize for portfolio selection?

Progress

Risk measures

A risk measure $\phi : \mathcal{X} \to \mathbb{R}$ Artzner/Delbaen/Eber/Heath'99

[M] Monotonicity: $\phi(X) \le \phi(Y)$ for all $X, Y \in \mathcal{X}$ with $X \le Y$

[CA] Constant additivity: $\phi(X + c) = \phi(X) + c$ for all $c \in \mathbb{R}$ and $X \in \mathcal{X}$

[PH] Positive homogeneity: $\phi(\lambda X) = \lambda \phi(X)$ for all $\lambda \in (0, \infty)$ and $X \in \mathcal{X}$

[SA] Subadditivity: $\phi(X + Y) \le \phi(X) + \phi(Y)$ for all $X, Y \in \mathcal{X}$

- Coherent risk measures (incl. ES): [M], [CA], [PH] & [SA]
- VaR_α: [M], [CA] & [PH] $\Leftarrow$ call this an MCP risk measure

Axiomatic theory

Setting

- $\mathcal{X} = L^\infty$ and $\phi : \mathcal{X} \to \mathbb{R}$ is a risk measure
- $\mathbf{X} = (X_1, \dots, X_n)$; $\mathbf{Y} = (Y_1, \dots, Y_n)$
- $\mathbf{X} \overset{\mathrm{m}}{\succeq} \mathbf{Y}$: $\phi(X_i) \le \phi(Y_i)$ for each i
- $\mathbf{X} \overset{\mathrm{m}}{\succ} \mathbf{Y}$: $\phi(X_i) < \phi(Y_i)$ for each i
- $\mathbf{X} \overset{\mathrm{m}}{\simeq} \mathbf{Y}$: $\phi(X_i) = \phi(Y_i)$ for each i (marginal equivalence)

Axiom R

$[\mathrm{R}]_\phi$ Rationality under ϕ-marginal equivalence: $D(\mathbf{X}) \le D(\mathbf{Y})$ for $\mathbf{X}, \mathbf{Y} \in \mathcal{X}^n$ satisfying $\mathbf{X} \overset{\mathrm{m}}{\simeq} \mathbf{Y}$ and $\sum_{i=1}^n X_i \le \sum_{i=1}^n Y_i$.

- Given marginal equivalence, preference for less loss

Axiomatic theory

- $\mathbf{0}$ is the n-vector of zeros $\Longrightarrow$ assign $D(\mathbf{0}) = 0$
- A duplicate portfolio: $\mathbf{X}^{\mathrm{du}} = (X, \dots, X) \Longrightarrow D(\mathbf{X}^{\mathrm{du}}) = 1$?
- $\mathbf{0}$ is also duplicate $\Longrightarrow$ assign $D(\mathbf{X}^{\mathrm{du}}) \le 1$
- A worse-than-duplicate portfolio $\mathbf{X}^{\mathrm{wd}} = (X_1, \dots, X_n)$: $\mathbf{X}^{\mathrm{wd}} \overset{\mathrm{m}}{\succ} \mathbf{X}^{\mathrm{du}}$ and $\sum_{i=1}^{n} X_i \ge nX$ for some $\mathbf{X}^{\mathrm{du}} = (X, \dots, X)$
- Assign $D(\mathbf{X}^{\mathrm{wd}}) \ge 1$
  - Diversification disasters e.g., Ibragimov/Jaffee/Walden'11

Axiom N

$[\mathrm{N}]_\phi$ Normalization: $D(\mathbf{0}) = 0$, $D(\mathbf{X}) \le 1$ if $\mathbf{X}$ is duplicate, and $D(\mathbf{X}) \ge 1$ if $\mathbf{X}$ is worse than duplicate.

Axiomatic theory

Axiom C

$[\mathrm{C}]_\phi$ Continuity: For $\{\mathbf{Y}^k\}_{k\in\mathbb{N}} \subseteq \mathcal{X}^n$ and $\mathbf{X} \in \mathcal{X}^n$ satisfying $\mathbf{Y}^k \overset{\mathrm{m}}{\sim} \mathbf{X}$ for each k, if $(\sum_{i=1}^n X_i - \sum_{i=1}^n Y_i^k)_+ \xrightarrow{L^\infty} 0$ as $k \to \infty$, then $(D(\mathbf{X}) - D(\mathbf{Y}^k))_+ \to 0$

- Marginally equivalent portfolios $\mathbf{X}$ and $\mathbf{Y}$
- Total risk of $\mathbf{X}$ is not much worse than $\mathbf{Y}$ in L^∞

  $\implies D(\mathbf{X})$ is not much worse than $D(\mathbf{Y})$
- Robustness with respect to statistical errors

Portfolio convexity

- Simplex: $\Delta_n = \{\mathbf{x} \in [0,1]^n : x_1 + \cdots + x_n = 1\}$
- Portfolio loss vector: $\mathbf{w} \odot \mathbf{X} = (w_1 X_1, \ldots, w_n X_n)$

Axiom PC

[PC] Portfolio convexity: The set $\{\mathbf{w} \in \Delta_n : D(\mathbf{w} \odot \mathbf{X}) \le d)\}$ is convex for each $\mathbf{X} \in \mathcal{X}^n$ and $d \in \overline{\mathbb{R}}$.

- Pooling two well diversified portfolios does not lead to a poorly diversified one

"*Convexity can also be viewed as the formal expression of a basic inclination of economic agents for diversification.*"

Mas-Colell/Whinston/Green'95, Microeconomic Theory, p.44

Portfolio convexity

Remark. For any diversification index D,

- $[\mathrm{PC}]$ is quasi-convexity of $\mathbf{w} \mapsto D(\mathbf{w} \odot \mathbf{X})$
- Convexity or quasi-convexity of $\mathbf{X} \mapsto D(\mathbf{X})$ is not desirable
  - For well-diversified (X, Y) and $Z = (X + Y)/2$, we want $D(Z, Z) > \max\{D(X, Y), D(Y, X)\}$
- Convexity of $\mathbf{w} \mapsto D(\mathbf{w} \odot \mathbf{X})$ is not desirable
  - $D((\mathbf{w}/2 + \mathbf{v}/2) \odot \mathbf{X}) \approx D(\mathbf{w} \odot \mathbf{X})$ if $\mathbf{v}$ has very small scale

Diversification quotients

Our axioms will characterize the following class

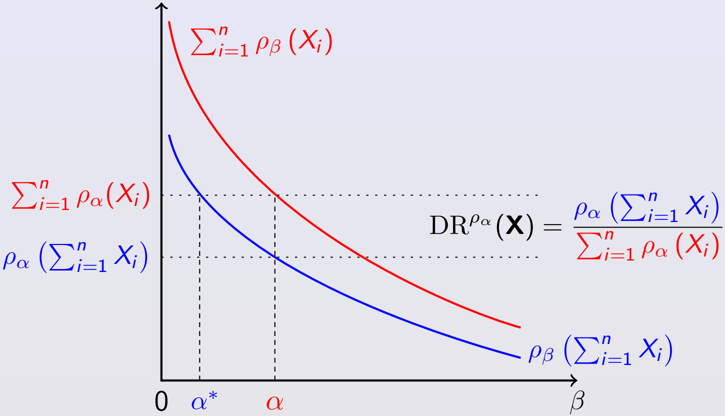
Definition 1 (Diversification quotients)

For $\mathbf{X} \in \mathcal{X}^n$, the diversification quotient based on the decreasing class ρ at level $\alpha \in I$ is defined by $\mathrm{DQ}^{\rho}_{\alpha}(\mathbf{X}) = \alpha^*/\alpha$, where

$$\alpha^* = \inf\left\{\beta \in I : \rho_\beta\left(\sum_{i=1}^n X_i\right) \le \sum_{i=1}^n \rho_\alpha(X_i)\right\}.$$

- Convention: $\inf(\varnothing) = \overline{\alpha}$
- Examples of $(\rho_\alpha)_{\alpha \in I}$: $(\mathrm{VaR}_\alpha)_{\alpha\in(0,1)}$, $(\mathrm{ES}_\alpha)_{\alpha\in(0,1)}$
- DQ can be defined for any decreasing family ρ
- We assume MCP ρ throughout

Diversification quotients



$$\mathrm{DQ}^{\rho}_{\alpha}(\mathbf{X}) = \alpha^*/\alpha$$

Comparing DQ and DR on VaR

- $S = \sum_{i=1}^{n} X_i$
- $s_\alpha = \sum_{i=1}^{n} \mathrm{VaR}_\alpha(X_i)$

Comparing

$$\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = \frac{\mathbb{P}(S > s_\alpha)}{\alpha} \quad \text{and} \quad \mathrm{DR}^{\mathrm{VaR}_\alpha}(\mathbf{X}) = \frac{\mathrm{VaR}_\alpha(S)}{s_\alpha}$$

Duality:

- DQ measures the "probability improvement"
- DR measures the "quantile improvement"

Characterization

Theorem 1

Suppose that ϕ is an MCP risk measure. A diversification index $D : \mathcal{X}^n \to \overline{\mathbb{R}}$ satisfies $[+]$, $[\mathrm{LI}]$, $[\mathrm{SI}]$, $[\mathrm{R}]_\phi$, $[\mathrm{N}]_\phi$ and $[\mathrm{C}]_\phi$ if and only if D is DQ^ρ_α for some α and decreasing class ρ of MCP risk measures with $\rho_\alpha = \phi$.

- First axiomatic characterization of diversification indices

Characterization

Theorem 2

Suppose $n \geq 4$ and ϕ is a non-linear coherent risk measure. A diversification index $D : \mathcal{X}^n \to \overline{\mathbb{R}}$ satisfies $[+]$, $[\mathrm{LI}]$, $[\mathrm{SI}]$, $[\mathrm{R}]_\phi$, $[\mathrm{N}]_\phi$, $[\mathrm{C}]_\phi$ and $[\mathrm{PC}]$ if and only if D is DQ^ρ_α for some α and decreasing class ρ of coherent risk measures with $\rho_\alpha = \phi$.

- DQ based on ES satisfies all axioms

Characterization

Table: Axioms satisfied by DR^{ϕ}, DB^{ϕ} and $\mathrm{DQ}^{\rho}_{\alpha}$ (with $\phi = \rho_\alpha$), where $\mathcal{X}_+$ is the set of non-negative elements in $\mathcal{X}$ and $\alpha \in (0,1)$

Index	Domain	[+]	[LI]	[SI]	$[\mathrm{R}]_\phi$	$[\mathrm{N}]_\phi$	$[\mathrm{C}]_\phi$	[PC]
$\mathrm{DR}^{\mathrm{VaR}_\alpha}/\mathrm{DR}^{\mathrm{ES}_\alpha}$	$\mathcal{X}^n$	×	×	√	×	×	×	×
$\mathrm{DR}^{\mathrm{VaR}_\alpha}$	$\mathcal{X}^n_+$	√	×	√	√	√	√	×
$\mathrm{DR}^{\mathrm{ES}_\alpha}$	$\mathcal{X}^n_+$	√	×	√	√	√	√	√
$\mathrm{DR}^{\mathrm{SD}}$	$\mathcal{X}^n$	√	√	√	×	×	×	√
$\mathrm{DR}^{\mathrm{var}}$	$\mathcal{X}^n$	√	√	√	×	×	×	×
$-\mathrm{DB}^{\mathrm{VaR}_\alpha}$	$\mathcal{X}^n$	×	√	×	√	×	√	×
$-\mathrm{DB}^{\mathrm{ES}_\alpha}$	$\mathcal{X}^n$	×	√	×	√	×	√	√
$\mathrm{DQ}^{\mathrm{VaR}}_\alpha$	$\mathcal{X}^n$	√	√	√	√	√	√	×
$\mathrm{DQ}^{\mathrm{ES}}_\alpha$	$\mathcal{X}^n$	√	√	√	√	√	√	√

Progress

Some properties

- A risk measure is sub-linear if it is convex and PH

Proposition 1

Let $\rho = (\rho_\beta)_{\beta \in I}$ be a decreasing class of sub-linear risk measures and $\alpha \in I$. Then DQ^ρ_α satisfies $[\mathrm{PC}]$. If $n \geq 3$, ρ_α is non-linear and there exists $X \in \mathcal{X}$ such that $\beta \mapsto \rho_\beta(X)$ is strictly decreasing, then $\{\mathrm{DQ}^\rho_\alpha(\mathbf{X}) : \mathbf{X} \in \mathcal{X}^n\} = [0,1]$.

- $\mathrm{DQ}^{\mathrm{ES}}_\alpha$ has the range $[0, 1]$
- For $n \geq 2$ and $\alpha \in (0, 1/n)$, $\mathrm{DQ}^{\mathrm{VaR}}_\alpha$ has the range $[0, n]$

Some properties

As a mapping $D : \bigcup_{n\in\mathbb{N}} \mathcal{X}^n \to \mathbb{R}$, $\mathrm{DQ}^{\rho}_{\alpha}$ with MCP ρ satisfies

[RI] Riskless invariance: $D(\mathbf{X}, c) = D(\mathbf{X})$ for all $\mathbf{X}$ and constant c

- adding a risk-free asset to the portfolio does not affect D

[RC] Replication consistency: $D(\mathbf{X}, \mathbf{X}) = D(\mathbf{X})$ for all $\mathbf{X}$

- replicating the same portfolio composition does not affect D

Remarks on DQ

DQ is connected to

- acceptability indices — Cherny/Madan'09; Rosazza Gianin/Sgarra'13
- PELVE — Li/W.'23
- bPOE — Mafusalov/Uryasev'18

Some properties

- ρ_α satisfies [SA] $\Longrightarrow \mathrm{DQ}^\rho_\alpha$ takes values in $[0,1]$
- Under weak condition: $\mathrm{DQ}^\rho_\alpha(\lambda_1 X, \dots, \lambda_n X) = 1$ for $\lambda_1, \dots, \lambda_n > 0$ and X with no atom

DQ based on VaR and ES

Theorem 3

For $\alpha \in (0,1)$ *and* $\mathbf{X} \in \mathcal{X}^n$, *write* $s_\alpha = \sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)$, $t_\alpha = \sum_{i=1}^n \mathrm{ES}_\alpha(X_i)$ *and* $S = \sum_{i=1}^n X_i$. *We have*

$$\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = \frac{1}{\alpha}\mathbb{P}(S > s_\alpha).$$

If $\mathbb{P}(S > t_\alpha) > 0$, *then*

$$\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X}) = \frac{1}{\alpha} \min_{r\in(0,\infty)} \mathbb{E}\left[(r(S - t_\alpha) + 1)_+\right],$$

and otherwise $\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{X}) = 0$.

- Inverting the ES curve

bPOE, Mafusalov/Uryasev'18

DQ based on VaR and ES

Proof of the last statement.

$$
\begin{aligned}
\mathrm{DQ}^{\mathrm{ES}}_{\alpha}(\mathbf{X}) &= \frac{1}{\alpha}\inf\{\beta\in(0,1):\mathrm{ES}_{\beta}(S)-t_{\alpha}\le 0\} \\
&= \frac{1}{\alpha}\inf\{\beta\in(0,1):\mathrm{ES}_{\beta}(S-t_{\alpha})\le 0\} \\
(*) &= \frac{1}{\alpha}\inf\left\{\beta\in(0,1):\min_{t\in\mathbb{R}}\left\{t+\frac{1}{\beta}\mathbb{E}\left[(S-t_{\alpha}-t)_{+}\right]\right\}\le 0\right\} \\
&= \frac{1}{\alpha}\inf\left\{\beta\in(0,1):\exists t\in\mathbb{R}\text{ s.t. }\frac{1}{\beta}\mathbb{E}\left[(S-t_{\alpha}-t)_{+}\right]\le -t\right\} \\
&= \frac{1}{\alpha}\inf\left\{\beta\in(0,1):\exists r>0\text{ s.t. }\mathbb{E}\left[(r(S-t_{\alpha})+1)_{+}\right]\le\beta\right\} \\
&= \frac{1}{\alpha}\inf_{r>0}\mathbb{E}\left[(r(S-t_{\alpha})+1)_{+}\right]
\end{aligned}
$$

(*): Rockafellar/Uryasev'02

Asymptotic behaviour of DQ as $\alpha \downarrow 0$

Proposition 2

Suppose that $X_1, \dots, X_n$ are iid random variables. If $X_1 \in \mathrm{RV}_\gamma$ has positive density over its support, then $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) \to n^{1-\gamma}$ as $\alpha \downarrow 0$.

- $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) \approx n$ for ultra heavy-tailed iid model ($\gamma \downarrow 0$)
- $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = n$ for some complicated model with both positive and negative dependence

Laws of large numbers for DQ

Theorem 4

Let $X_1, X_2, \ldots$ be a sequence of uncorrelated random variables in L^2. Assume $\sup_{i\in\mathbb{N}} \mathrm{var}(X_i) < \infty$ and $\inf_{i\in\mathbb{N}}\{\rho_\alpha(X_i) - \mathbb{E}[X_i]\} > 0$. For $\alpha \in (0,1)$, and ρ being VaR or ES,

$$\lim_{n\to\infty} \mathrm{DQ}^{\rho}_{\alpha}(X_1, \ldots, X_n) = 0.$$

- If $X_1, X_2, \ldots$ are iid, then $\lim_{n\to\infty} \mathrm{DQ}^{\rho}_{\alpha}(X_1, \ldots, X_n) = 0$ for ρ being VaR or ES with $\rho_\alpha(X_1) > \mathbb{E}[X_1]$

Laws of large numbers for DQ

Proposition 3

Let $X_1, X_2, \ldots$ be a sequence of exchangeable random variables in L^2. Denote by $\mu = \mathbb{E}[X_1]$, $\sigma^2 = \mathrm{var}(X_1)$ and $r = \mathrm{corr}(X_1, X_2)$. For $\alpha \in (0,1)$ and ρ being VaR or ES, if $\rho_\alpha(X_1) > \mu$, then

$$\lim_{n\to\infty} \mathrm{DQ}_\alpha^\rho(X_1, \ldots, X_n) \le \frac{1}{\alpha} \frac{r\sigma^2}{r\sigma^2 + (\rho_\alpha(X_1) - \mu)^2}.$$

- The limit of DQ exists under exchangeability
- Proof: bounds on VaR/ES e.g., Li-Shao-W.-Yang'18

$$\mathrm{VaR}_\beta(X) \le \mathrm{ES}_\beta(X) \le \mathbb{E}[X] + \mathrm{SD}(X)\sqrt{(1-\beta)/\beta}$$

- $r \to 0 \implies \lim_{n\to\infty} \mathrm{DQ}_\alpha^\rho(X_1, \ldots, X_n) \to 0$
- $r = 1 \implies \mathrm{DQ}_\alpha^\rho(X_1, \ldots, X_n) = 1$ under mild conditions (sharp)

Progress

Optimal portfolio diversification

Optimal one-period diversification problem

$$\min_{\mathbf{w}\in\Delta_n} \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{w}\odot\mathbf{X}) \quad \text{and} \quad \min_{\mathbf{w}\in\Delta_n} \mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{w}\odot\mathbf{X}) \qquad \text{(OD)}$$

Optimal portfolio diversification

- Write $\mathbf{x}_\alpha^\rho = (\rho_\alpha(X_1), \dots, \rho_\alpha(X_n))$

Proposition 4

For $\rho = \mathrm{VaR}$, *if each component of* $\mathbf{X}$ *is non-constant, then* (OD) *is solved by*

$$\min_{\mathbf{w}\in\Delta_n} \mathbb{P}\left(\mathbf{w}^\top\left(\mathbf{X} - \mathbf{x}_\alpha^{\mathrm{VaR}}\right) > 0\right).$$

For $\rho = \mathrm{ES}$, *if* $\mathbb{P}(\mathbf{w}^\top(\mathbf{X} - \mathbf{x}_\alpha^{\mathrm{ES}}) = 0) = 0$ *for all* $\mathbf{w} \in \Delta_n$, *then* (OD) *is solved by*

$$\min_{\mathbf{v}\in\mathbb{R}_+^n} \mathbb{E}\left[\left(\mathbf{v}^\top\left(\mathbf{X} - \mathbf{x}_\alpha^{\mathrm{ES}}\right) + 1\right)_+\right],$$

and the optimal $\mathbf{w}$ *is given by* $\mathbf{v}/\|\mathbf{v}\|_1$.

Portfolio optimization of DQ for a data sample

$\min_{\mathbf{w}\in\Delta_n} \mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{w}\odot\mathbf{X})$ for data sample $\mathbf{X}^{(1)},\dots,\mathbf{X}^{(N)}$

$\Longrightarrow$ convex programming

$$\text{minimize} \quad \sum_{j=1}^{N}\left(\mathbf{v}^\top\left(\mathbf{X}^{(j)}-\widehat{\mathbf{x}}_\alpha^{\mathrm{ES}}\right)+1\right)_+ \quad \text{over } \mathbf{v}\in\mathbb{R}_+,$$

where $\widehat{\mathbf{x}}_\alpha^{\mathrm{ES}}$ is the empirical version of $\mathbf{x}_\alpha^{\mathrm{ES}}$ based on the sample

- Practically use $\|\mathbf{v}\|_1 \le M$ for a large M, e.g., $M=100$
- Apply a tie-breaking rule if needed

Portfolio optimization of DQ for a data sample

$\min_{\mathbf{w}\in\Delta_n} \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{w}\odot\mathbf{X})$ for data sample $\mathbf{X}^{(1)},\dots,\mathbf{X}^{(N)}$

$\Longrightarrow$ linear integer programming

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^N z_j \quad \left(=\textstyle\sum_{j=1}^N \mathbb{1}_{\{\mathbf{w}^\top \mathbf{y}^{(j)}>0\}}\right) \qquad \text{(LIP)} \\ \text{subject to} \quad & \mathbf{w}^\top \mathbf{y}^{(j)} - M z_j \le 0, \quad \textstyle\sum_{j=1}^n w_j = 1 \\ & z_j \in \{0,1\}, \quad w_j \ge 0 \ \text{ for all } j\in\{1,\dots,N\}, \end{aligned}$$

where

- $\mathbf{y}^{(j)} = \mathbf{X}^{(j)} - \widehat{\mathbf{x}}_\alpha^{\mathrm{VaR}}$
- $\widehat{\mathbf{x}}_\alpha^{\mathrm{VaR}}$ is the empirical version of $\mathbf{x}_\alpha^{\mathrm{VaR}}$ based on the sample
- $M>0$: $z_j = 1 \Longrightarrow \mathbf{w}^\top \mathbf{y}^{(j)} - M z_j \le 0$ (the Big M method)

Portfolio optimization of DQ for a data sample

Tie-breaking

- the objective function of (LIP) takes integer values
- let m^* be the optimal value of (LIP)
- pick the closest one (in L^1-norm $\|\cdot\|_1$ on $\mathbb{R}^n$) to a given benchmark $\mathbf{w}_0$ among tied optimizers

$$\begin{aligned} \text{minimize} \quad & \|\mathbf{w}-\mathbf{w}_0\|_1 \\ \text{subject to} \quad & \sum_{j=1}^{N} \mathbb{1}_{\{\mathbf{w}^\top \mathbf{y}^{(j)}>0\}} \le m^* \\ & \mathbf{w} \in \Delta_n \end{aligned}$$

Outline

Elliptical models

- Elliptical distributions are popular in QRM
- Two examples: normal and t distributions
- Fundamental theorem of QRM: for elliptical models, any PH risk measures are "equivalent" (Embrechts'19 keynote at IME)
- Explicit formulas for $\mathrm{DQ}^{\mathrm{VaR}}_{\alpha}$ and $\mathrm{DQ}^{\mathrm{ES}}_{\alpha}$ are available
- Asymptotic results for $n \to \infty$ and $\alpha \downarrow 0$ are available

Numerical examples: normal and t-distributions

Consider two dispersion matrices, parametrized by $r \in [0,1]$ and $n \in \mathbb{N}$,

- Equicorrelation

$$\Sigma_1 = (\sigma_{ij})_{n\times n}, \quad \text{where } \sigma_{ii} = 1 \text{ and } \sigma_{ij} = r \text{ for } i \neq j,$$

- Autoregressive AR(1)

$$\Sigma_2 = (\sigma_{ij})_{n\times n}, \quad \text{where } \sigma_{ii} = 1 \text{ and } \sigma_{ij} = r^{|j-i|} \text{ for } i \neq j.$$

Let $\mathbf{X}_i \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma_i)$ and $\mathbf{Y}_i \sim \mathrm{t}(\nu, \boldsymbol{\mu}, \Sigma_i)$, $i = 1, 2$

DQ for varying α

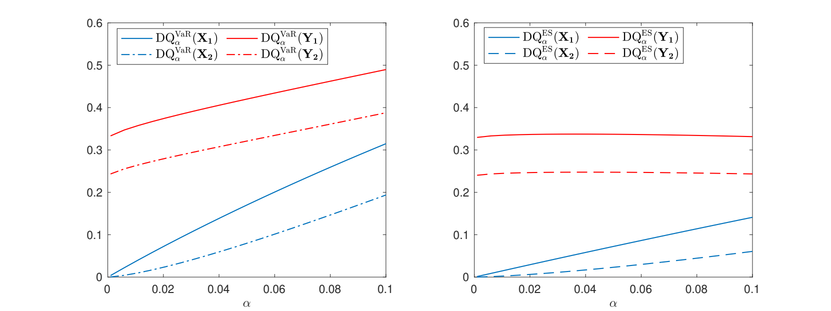


Figure: DQ based on VaR and ES for $\alpha \in (0, 0.1)$ with fixed $\nu = 3$, $r = 0.3$ and $n = 4$

DQ for varying correlation coefficient

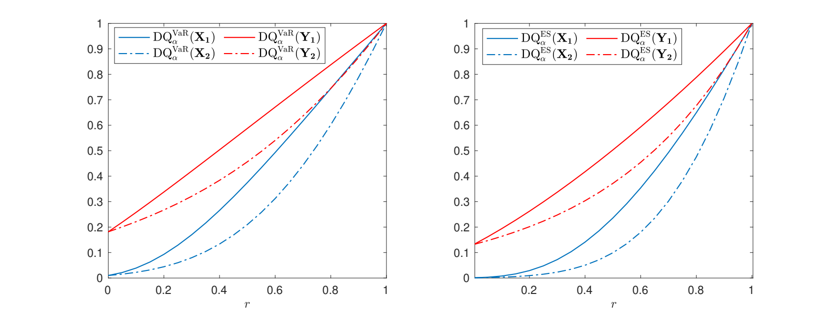


Figure: DQ based on VaR and ES for $r \in [0, 1]$ with fixed $\alpha = 0.05$, $\nu = 3$, and $n = 4$

DQ for t-models with varying tail parameter

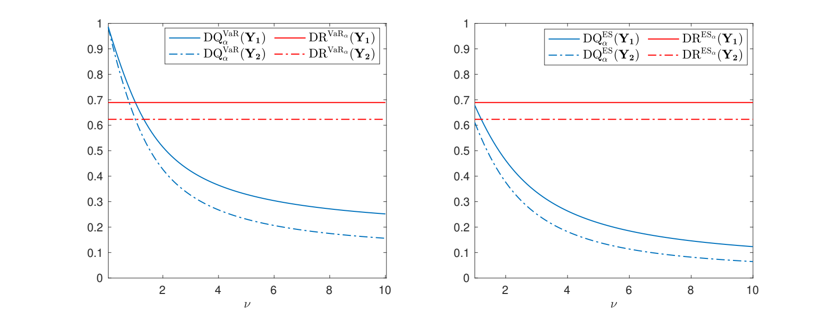


Figure: DQ based on VaR for $\nu \in (0, 10]$ and ES for $\nu \in (1, 10]$ with fixed $\alpha = 0.05$, $r = 0.3$ and $n = 4$

DQ for elliptical models as the dimension n varies

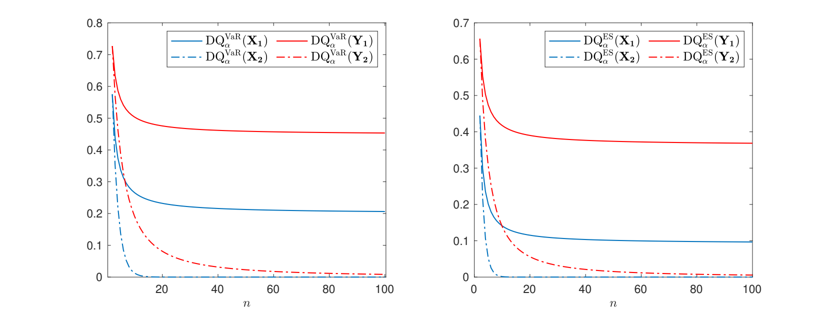


Figure: DQ based on VaR and ES for $n \in [2, 100]$ with fixed $\alpha = 0.05$, $r = 0.5$ and $\nu = 3$

The joint t-model has a common shock

Table: DQ/DR based on VaR, ES and SD, where $\alpha = 0.05$ and $n = 10$

D	$\mathrm{DQ}_\alpha^{\mathrm{VaR}}$	$\mathrm{DQ}_\alpha^{\mathrm{ES}}$	$\mathrm{DR}^{\mathrm{VaR}_\alpha}$	$\mathrm{DR}^{\mathrm{ES}_\alpha}$	$\mathrm{DR}^{\mathrm{SD}}$
$\mathbf{Z} \sim \mathrm{N}(\mathbf{0}, I_n)$	$\sim 10^{-6}$	$\sim 10^{-9}$	0.3162	0.3162	0.3162
$\mathbf{Y} \sim \mathrm{it}_n(3)$	0.0231	0.0144	0.3568	0.3058	0.3162
$\mathbf{Y}' \sim \mathrm{t}(3, \mathbf{0}, I_n)$	0.0502	0.0340	0.3162	0.3162	0.3162
$D(\mathbf{Z}) < D(\mathbf{Y})$	Yes	Yes	Yes	No	No
$D(\mathbf{Y}) < D(\mathbf{Y}')$	Yes	Yes	No	Yes	No

- DQ: iid normal < iid t < joint t
- DR: iid normal = joint t $\overset{?}{\sim}$ iid t
- DQ captures tail heaviness/common shock which DR ignores

DQ for t-models with varying tail parameter

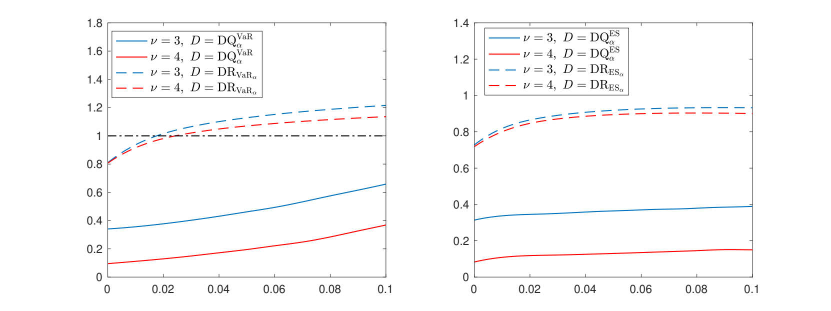


Figure: $D(\mathbf{Y}')/D(\mathbf{Y})$ based on VaR and ES for $\alpha \in (0, 0.1]$ with fixed $n = 10$

Optimization for the elliptical models

- Optimal diversification for DQ ($\boldsymbol{\sigma}$ is the diagonal of Σ)

$$\underset{\mathbf{w}\in\Delta_n}{\arg\min}\, \mathrm{DQ}_{\alpha}^{\mathrm{VaR}}(\mathbf{w}\odot\mathbf{X}) = \underset{\mathbf{w}\in\Delta_n}{\arg\min}\, \frac{\sqrt{\mathbf{w}^\top\Sigma\mathbf{w}}}{\mathbf{w}^\top\boldsymbol{\sigma}}$$

- Optimal diversification for DR

$$\underset{\mathbf{w}\in\Delta_n}{\arg\min}\, \mathrm{DR}^{\mathrm{VaR}_{\alpha}}(\mathbf{w}\odot\mathbf{X}) = \underset{\mathbf{w}\in\Delta_n}{\arg\min}\, \frac{\mathbf{w}^\top\boldsymbol{\mu} + y_\alpha\sqrt{\mathbf{w}^\top\Sigma\mathbf{w}}}{\mathbf{w}^\top\boldsymbol{\mu} + y_\alpha\mathbf{w}^\top\boldsymbol{\sigma}}$$

where $y_\alpha = \mathrm{VaR}_\alpha(Y)$ and $Y \sim \mathrm{E}_1(0,1,\phi)$

- The two have the same optimizers if $\boldsymbol{\mu} = \mathbf{0}$ and $y_\alpha \neq 0$
- In case $\Sigma = I_n$: $\mathbf{w}^* = (\frac{1}{n}, \dots, \frac{1}{n})$

Outline

DQ for different portfolios

Data: daily losses from S&P 500 constituents

- Period: January 3, 2012 to December 31, 2021
- 2518 daily losses; moving window of 500 days

Portfolios with stock compositions:

(A) 2 largest stocks from each of 10 different sectors of S&P 500

(B) 1 largest stock from each of 5 different sectors of S&P 500

- XOM (ENR), AAPL (IT), BRK/B (FINL), WMT (CONS), GE (INDU)

(C) 5 largest stocks from the Information Technology (IT) sector

(D) 5 largest stocks from the Financials (FINL) sector

DQ for different portfolios

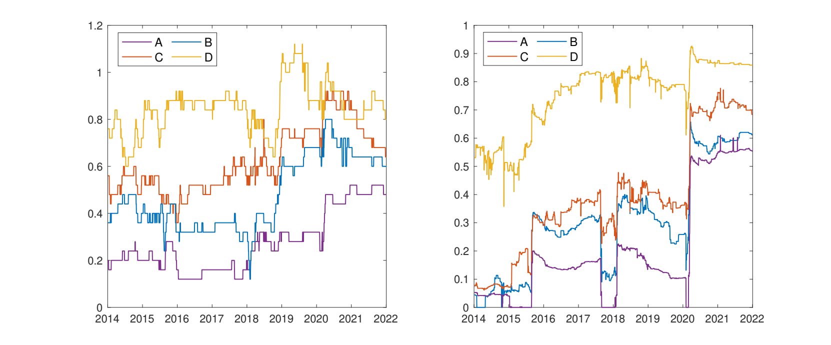


Figure: DQ based on VaR (left) and ES (right) with $\alpha = 0.05$

- Observation: A (20) $<$ B (5) $<$ C (5 IT) $<$ D (5 FINL)
- Large jump for DQ based on ES at the COVID outbreak
- DQ based on VaR can be larger than 1

Optimal diversified portfolios

Portfolios (monthly rebalancing) with 4 largest stocks from each of the 10 sectors of S&P 500 in 2012 (40 in total)

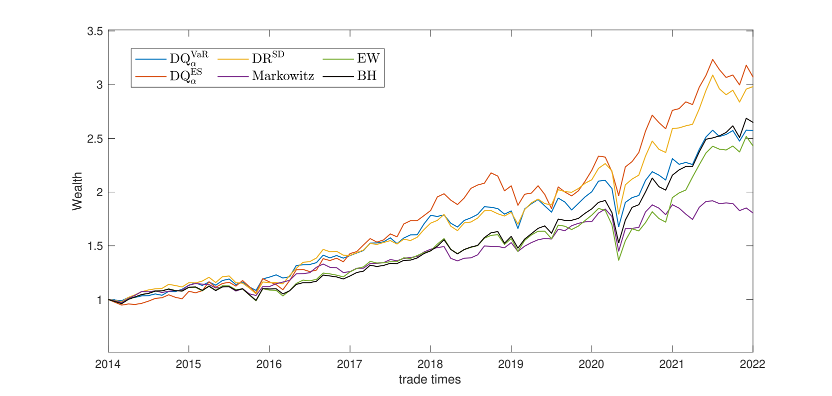


Figure: Wealth processes

Optimal diversified portfolios

%	$\mathrm{DQ}_\alpha^{\mathrm{VaR}}$	$\mathrm{DQ}_\alpha^{\mathrm{ES}}$	$\mathrm{DR}^{\mathrm{SD}}$	Markowitz	EW	BH
AR	12.562	14.695	14.364	7.929	11.906	12.883
AV	14.643	15.818	14.994	12.976	15.918	14.343
SR	66.397	74.942	76.854	39.222	56.955	70.023

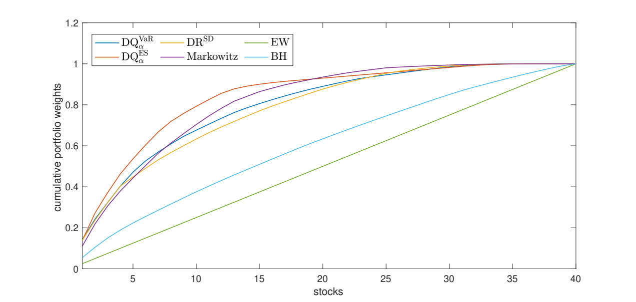
Table: Annualized return (AR), annualized volatility (AV) and Sharpe ratio (SR) for different portfolio strategies from Jan 2014 to Dec 2021

- Risk-free rate: 2.84% (= 10-y US treasury yield, Jan 2014)
- $\alpha = 0.1$
- EW = equally weighted; BH = buy and hold
- The target AR for the Markowitz portfolio is set to 10%

Optimal diversified portfolios

%	$\mathrm{DQ}_\alpha^{\mathrm{VaR}}$	$\mathrm{DQ}_\alpha^{\mathrm{ES}}$	$\mathrm{DR}^{\mathrm{SD}}$	Markowitz	EW	BH
ATP	19.29	14.75	15.61	18.79	4.43	0

Table: Average trading proportion (ATP) from Jan 2014 to Dec 2021



Average cumulative portfolio weights

Optimal diversified portfolios

Portfolios (monthly rebalancing) with 4 largest stocks from each of the 10 sectors of S&P 500 in 2002 (40 in total)

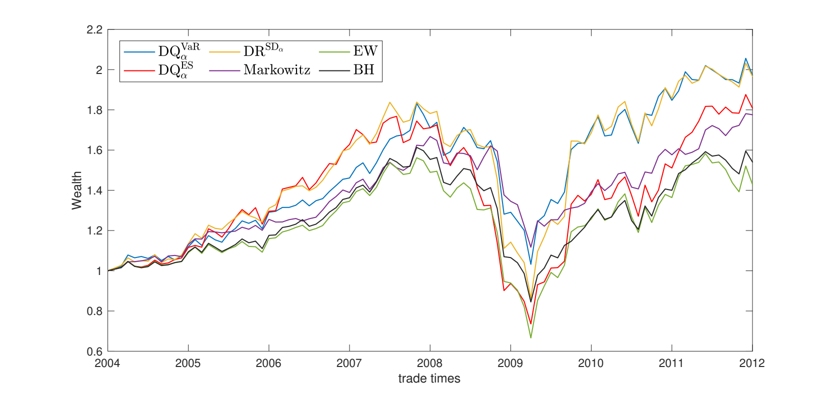


Figure: Wealth processes

Optimal diversified portfolios

%	$\mathrm{DQ}^{\mathrm{VaR}}_{\alpha}$	$\mathrm{DQ}^{\mathrm{ES}}_{\alpha}$	$\mathrm{DR}^{\mathrm{SD}}$	Markowitz	EW	BH
AR	9.456	8.129	9.103	7.980	5.300	6.235
AV	16.653	21.452	20.915	11.976	20.154	15.530
SR	30.478	17.474	22.582	30.064	4.566	11.944

Table: Annualized return (AR), annualized volatility (AV) and Sharpe ratio (SR) for different portfolio strategies from Jan 2004 to Dec 2011

- Risk-free rate: 4.38% (= 10-y US treasury yield, Jan 2004)
- $\alpha = 0.1$
- EW = equally weighted; BH = buy and hold
- The target AR for the Markowitz portfolio is set to 5%

Thank you

Thank you for your kind attention

- Axiomatic theory paper: https://arxiv.org/abs/2206.13679
- Particular models paper: https://arxiv.org/abs/2301.03517
- Working papers series: Risk management with risk measures `http://sas.uwaterloo.ca/~wang/pages/WPS5.html`

Some literature on measuring diversification

- Markowitz'52 JF: Mean-variance theory
- Diversification ratio
  - Tasche'07; Choueifaty/Coignard'08 JPM; Bürgi/Dacorogna/Iles'08; Embrechts/Wang/W.'15 FS
- Number of unique investments or naive diversification
  - Rudin/Morgan'06 JPM; DeMiguel/Garlappi/Uppal'09 RFS; Pflug/Pichler/Wozabal'12 JBF
- Diversification benefit in multivariate regular variation models
  - Mainik/Rüchendorf'10 FS; Mainik/Embrechts'13 AAS
- Koumou/Dionne'22: Axioms for correlation diversification measures

Some recent work on VaR and ES

- Axiomatic characterizations
  - VaR: Kou/Peng' 16 OR; He/Peng' 18 OR; Liu/W.'21 MOR
  - ES: W./Zitikis'21 MS; Embrechts/Mao/Wang/W.'21 MF
- Risk sharing
  - Embrechts/Liu/W.'18 OR; Embrechts/Liu/Mao/W.'20 MP
- Robustness in optimization
  - Emberchts/Schied/W.'22 OR
- Calibrating levels between VaR and ES
  - Li/W.'23 JE
- Forecasting and backtesting
  - Fissler/Ziegel'16 AOS; Nolde/Ziegel'17 AOAS; Du/Escanciano'17 MS

Connecting DQ and DR

Proposition 5

For a given $\phi : \mathcal{X} \to \mathbb{R}_+$*, we have* $\mathrm{DQ}^{\rho}_{\alpha} = \mathrm{DR}^{\phi}$ *where* $\rho = (\phi/\alpha)_{\alpha \in (0,\infty)}$*. The same holds if* $\rho = (b\mathbb{E} + c\phi/\alpha)_{\alpha \in (0,\infty)}$ *for some* $b \in \mathbb{R}$ *and* $c > 0$ *and* $\mathcal{X} = L^1$*.*

- $\mathrm{DR}^{\mathrm{var}}$ and $\mathrm{DR}^{\mathrm{SD}}$ are special cases of DQ
- If ϕ satisfies $[\mathrm{CA}]_0$, then $\rho_\alpha = b\mathbb{E} + c\phi/\alpha$ satisfies $[\mathrm{CA}]_b$
- $b\mathbb{E} + c\phi/\alpha$ includes mean-standard deviation, mean-variance, and mean-Gini (Denneberg'90)

Elliptical models

- A random vector $\mathbf{X}$ is elliptically distributed if it has a characteristic function

$$\psi(\mathbf{t}) = \mathbb{E}\left[\exp\left(\mathtt{i}\mathbf{t}^\top \mathbf{X}\right)\right] = \exp\left(\mathtt{i}\mathbf{t}^\top \boldsymbol{\mu}\right)\phi\left(\mathbf{t}^\top \Sigma \mathbf{t}\right),$$

for some $\boldsymbol{\mu} \in \mathbb{R}^n$, positive semi-definite matrix $\Sigma \in \mathbb{R}^{n\times n}$, and $\phi : \mathbb{R}_+ \to \mathbb{R}$ (the characteristic generator)

- This distribution is denoted by by $\mathrm{E}_n(\boldsymbol{\mu}, \Sigma, \phi)$
- Write $\Sigma = (\sigma_{ij})_{n\times n}$, $\sigma_i^2 = \sigma_{ii}$, $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$, and

$$k_\Sigma = \frac{\sum_{i=1}^n \sigma_i}{\left(\sum_{i,j}^n \sigma_{ij}\right)^{1/2}} \in [1, \infty)$$

DQ for elliptical models

Proposition 6

Suppose that $\mathbf{X} \sim \mathrm{E}_n(\boldsymbol{\mu}, \Sigma, \phi)$. *We have*

$$\mathrm{DQ}^{\mathrm{VaR}}_{\alpha}(\mathbf{X}) = \frac{1 - F(k_{\Sigma}\mathrm{VaR}_{\alpha}(Y))}{\alpha}; \ \mathrm{DQ}^{\mathrm{ES}}_{\alpha}(\mathbf{X}) = \frac{1 - \widetilde{F}(k_{\Sigma}\mathrm{ES}_{\alpha}(Y))}{\alpha},$$

for $\alpha \in (0,1)$, *where* $Y \sim \mathrm{E}_1(0,1,\phi)$ *with distribution function* F, *and* $\widetilde{F}$ *is the superquantile transform of* F. *Moreover,*

(i) $\alpha \mapsto \mathrm{DQ}^{\mathrm{VaR}}_{\alpha}(\mathbf{X})$ *takes value in* $[0,1]$ *on* $(0,1/2]$ *and it takes value in* $[1,2]$ *on* $(1/2,1)$;

(ii) $k_{\Sigma} \mapsto \mathrm{DQ}^{\mathrm{VaR}}_{\alpha}(\mathbf{X})$ *is decreasing for* $\alpha \in (0,1/2]$ *and increasing for* $\alpha \in (1/2,1)$;

(iii) $k_{\Sigma} \mapsto \mathrm{DQ}^{\mathrm{ES}}_{\alpha}(\mathbf{X})$ *is decreasing for* $\alpha \in (0,1)$.

Asymptotic behaviour of DQ

Proposition 7

Suppose that $\mathbf{X} \sim \mathrm{E}_n(\boldsymbol{\mu}, \Sigma, \phi)$.

(i) *Let* $Y \sim \mathrm{E}_1(0, 1, \phi)$ *and* f *be the density function of* Y. *We have*

$$\lim_{\alpha \downarrow 0} \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = \lim_{x \to \infty} k_\Sigma \frac{f(k_\Sigma x)}{f(x)}$$

if $\mathrm{VaR}_0(Y) = \infty$ *and the limit exists, and* $\lim_{\alpha \downarrow 0} \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = 0$ *if* $\mathrm{VaR}_0(Y) < \infty$.

(ii) *Let* $\mathrm{AC}_\Sigma = 1/k_\Sigma^2$. *If* $\lim_{n \to \infty} \mathrm{AC}_\Sigma = 0$, *then*

$$\lim_{n \to \infty} \mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{X}) = \lim_{n \to \infty} \mathrm{DQ}_\beta^{\mathrm{ES}}(\mathbf{X}) = 0$$

for $\alpha \in (0, 1/2)$ *and* $\beta \in (0, 1)$.

Cross-comparison between DQ based on VaR and ES

Associating VaR and ES levels by PELVE (Li/W.'22)

$$\mathrm{ES}_{c\alpha}(X) = \mathrm{VaR}_{\alpha}(X)$$

Table: Values of DQ based on VaR at level $\alpha = 0.01$ and ES at level $c\alpha$, where $n = 4$ and $r = 0.3$

	c	$c\alpha$	$\mathrm{DQ}_{\alpha}^{\mathrm{VaR}}$	$\mathrm{DQ}_{c\alpha}^{\mathrm{ES}}$
$\mathbf{X}_1 \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma_1)$	2.58	0.0258	0.0369	0.0377
$\mathbf{X}_2 \sim \mathrm{N}(\boldsymbol{\mu}, \Sigma_2)$	2.58	0.0258	0.0024	0.0025
$\mathbf{Y}_1 \sim \mathrm{t}(3, \boldsymbol{\mu}, \Sigma_1)$	3.31	0.0331	0.3558	0.3373
$\mathbf{Y}_2 \sim \mathrm{t}(3, \boldsymbol{\mu}, \Sigma_2)$	3.31	0.0331	0.2094	0.1961

Dependence and portfolio risks

For a random variable X and $\alpha \in (0,1)$

(i) A tail event of X is an event $A \in \mathcal{F}$ with $0 < \mathbb{P}(A) < 1$ such that $X(\omega) \geq X(\omega')$ holds for a.s. all $\omega \in A$ and $\omega' \in A^c$

(ii) A random vector $(X_1, ..., X_n)$ is α-concentrated if its component share a common tail event of probability α (W./Zitikis'21)

For $\mathbf{X} \in \mathcal{X}^n$ and $\alpha \in (0, 1/n)$, an α-CE model satisfies

- $\mathbb{P}(X_i > \mathrm{VaR}_\alpha(X_i)) = \alpha$
- $\mathbb{P}(X_i \geq \mathrm{VaR}_\alpha(X_i)) \geq n\alpha$
- $(X_1, \ldots, X_n)$ are $(n\alpha)$-concentrated
- $\{X_i > \mathrm{VaR}_\alpha(X_i)\}$, $i = 1, \ldots, n$, are mutually exclusive

Dependence and portfolio risks

Theorem 5

Let $\alpha \in (0,1)$ *and* $n \geq 2$ *satisfy* $n \leq 1/\alpha$.

(i) $\mathrm{DQ}^{\mathrm{VaR}}_{\alpha}$ *has a range* $[0, n]$ *and* $\mathrm{DQ}^{\mathrm{ES}}_{\alpha}$ *has a range* $[0, 1]$.

(ii) *If* $\sum_{i=1}^{n} X_i$ *is a constant, then* $\mathrm{DQ}^{\mathrm{VaR}}_{\alpha}(\mathbf{X}) = \mathrm{DQ}^{\mathrm{ES}}_{\alpha}(\mathbf{X}) = 0$.

(iii) *For* ρ *being* VaR *or* ES, *if* $\mathbf{X}$ *is* α*-concentrated, then* $\mathrm{DQ}^{\rho}_{\alpha}(\mathbf{X}) \leq 1$. *If, in addition,* ρ *is continuous and non-flat from the left at* $(\alpha, \sum_{i=1}^{n} X_i)$, *then* $\mathrm{DQ}^{\rho}_{\alpha}(\mathbf{X}) = 1$.

(iv) *If* $\mathbf{X}$ *has an* α*-CE model, then* $\mathrm{DQ}^{\mathrm{VaR}}_{\alpha}(\mathbf{X}) = n$ *and* $\mathrm{DQ}^{\mathrm{ES}}_{n\alpha}(\mathbf{X}) = 1$.

Numerical example

Assume that $\mathbf{X} \sim \mathrm{t}(\nu, \boldsymbol{\mu}, \Sigma)$ where $\nu = 3$ and the dispersion matrix is given by

$$\Sigma = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 2 \end{pmatrix}$$

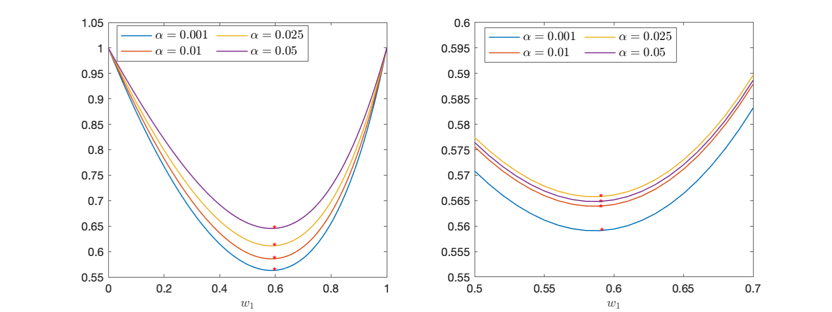


Figure: Values of $\mathrm{DQ}_\alpha^{\mathrm{VaR}}(\mathbf{w} \odot \mathbf{X})$ and $\mathrm{DQ}_\alpha^{\mathrm{ES}}(\mathbf{w} \odot \mathbf{X})$ for $w_1 \in [0, 1]$

Comparison of DQ and DR

- Portfolio: 1 largest stock from each of 5 sectors (2012 market cap)
  - XOM (ENR), AAPL (IT), BRK/B (FINL), WMT (CONS), GE (INDU)
- Period: January 3, 2012 to December 31, 2021
- 2518 daily losses; moving window of 500 days

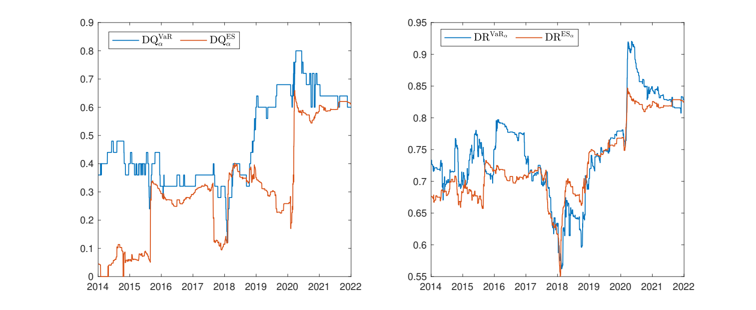


Figure: DQ (left) and DR (right) on VaR/ES with $\alpha = 0.05$

Optimal diversified portfolio

Portfolios (monthly rebalancing) with 2 largest stocks from each of the 10 sectors of S&P 500 in 2012 (20 in total)

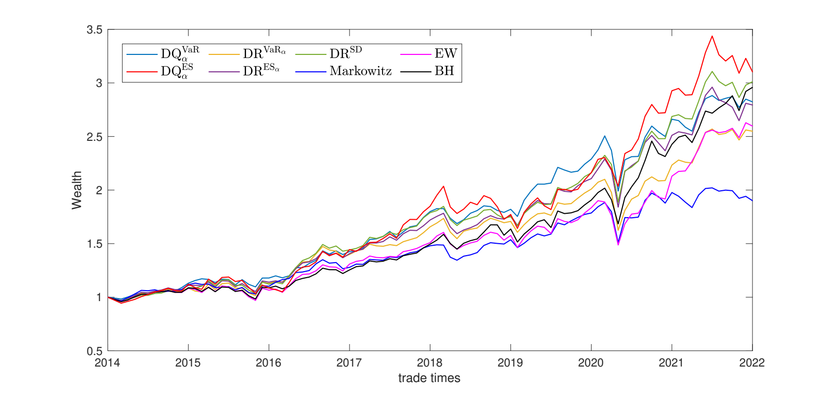


Figure: Wealth processes

Optimal diversified portfolio

%	DQ^{VaR}_{α}	DQ^{ES}_{α}	$DR^{VaR_{\alpha}}$	$DR^{ES_{\alpha}}$
AR	13.5449	14.4763	12.7657	13.8492
AV	13.4340	15.7689	14.4079	14.5265
SR	79.6853	73.7905	68.8908	75.7867
%	DR^{SD}	Markowitz	EW	BH
AR	14.3663	8.5884	12.7359	14.2236
AV	14.2887	12.7355	14.6834	13.9614
SR	80.6671	45.1371	67.3952	81.5362

Table: Annualized return (AR), annualized volatility (AV) and Sharpe ratio (SR) for different portfolio strategies from Jan 2014 to Dec 2021