Combining p-values under dependence and optimal transport

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Agenda

This talk

Vladimir Vovk
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Bin Wang
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Matteo Gasparin
(Padova)

Aaditya Ramdas
(Carnegie Mellon)

- Vovk/W.'20, Combining p-values via averaging — Biometrika
- Vovk/Wang/W.'22, Admissible ways of merging p-values under arbitrary dependence — Annals of Statistics
- Gasparin/W./Ramdas'24, Combining exchangeable p-values — arXiv:2404.03484

Also based on 10+ years
of collaboration with
Paul Embrechts (ETH)

Background

Suppose that we are testing the global null hypothesis using $K \geqslant 2$ tests and obtain p-values $p_1, \ldots, p_K$. How can we combine them into a single p-value?

A question of a long history

- Tippett'31, Pearson'33, Fisher'48: assume independence
- We are primarily interested in the case of dependence, in particular, unknown dependence
- The Bonferroni correction: minimum $\times$ correction (K)
- More important is testing multiple hypotheses or selective inference, but for now let us focus on the global null

The value of no assumption

Dependence

- A set of p-values is essentially one vector: difficult to test/verify any dependence model among them
- In modern applications, multiple p-values may come from resampling, data splitting, overlapping experiments, different algorithms, ...
- Efron'10, Large-scale Inference, p50-p51:
  "*independence among the p-values ... usually an unrealistic assumption. ... even PRD* [positive regression dependence] *is unlikely to hold in practice.*"

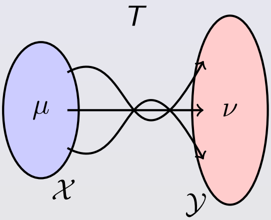
Optimal transport

The Monge (1781) problem: find a transport map $T : \mathcal{X} \to \mathcal{Y}$ that attains

$$\inf \left\{ \int_{\mathcal{X}} c(x, T(x))\, \mathrm{d}\mu(x) \;\middle|\; T_{\#}\mu = \nu \right\}$$

where

- $\mathcal{X}$ and $\mathcal{Y}$ are two Polish spaces (e.g., $\mathbb{R}^d$)
- Cost function $c : \mathcal{X} \times \mathcal{Y} \to [0, \infty]$ or $(-\infty, \infty]$
- Probabilities μ on $\mathcal{X}$ and ν pm $\mathcal{Y}$ are given
- $T_{\#}\mu = \mu \circ T^{-1}$ is the push forward of μ by T
- Such T is an optimal transport map



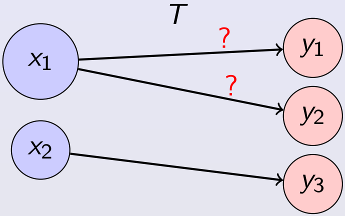
Kantorovich's formulation

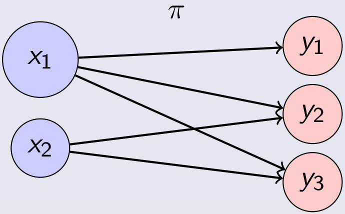
- Monge's formulation may be ill-posed (e.g., point masses)
- Kantorovich's problem: find a probability measure $\pi \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ that attains

$$\inf \left\{ \int_{\mathcal{X}\times\mathcal{Y}} c(x,y)\, \pi(\mathrm{d}x, \mathrm{d}y) \mid \pi \in \Pi(\mu,\nu) \right\},$$

  where $\Pi(\mu,\nu)$ is the set of probability measures on $\mathcal{X} \times \mathcal{Y}$ with marginals μ and ν
- Wasserstein distance is defined this way
- Linear programming

Monge and Kantorovich's formulations





Gaspard Monge
1746-1818

Leonid Kantorovich
1912-1986

Transport duality

If c is non-negative and lower semi-continuous, then duality holds

$$\min_{\pi \in \Pi(\mu,\nu)} \int_{\mathcal{X}\times\mathcal{Y}} c \,\mathrm{d}\pi = \sup \left(\int_{\mathcal{X}} \phi \,\mathrm{d}\mu + \int_{\mathcal{Y}} \psi \,\mathrm{d}\nu \right),$$

where the supremum runs over all pairs of bounded and continuous functions $\phi : \mathcal{X} \to \mathbb{R}$ and $\psi : \mathcal{Y} \to \mathbb{R}$ such that

$$\phi(x) + \psi(y) \leqslant c(x, y) \quad \text{for all } x, y$$

Economic interpretation

- $x \in \mathcal{X}$: the vector of characteristics of a worker
- $y \in \mathcal{Y}$: the vector of characteristics of a firm
- $g(x, y)$ the economic output (production) generated by worker x matched with firm y
- Social economic-output maximization

$$\sup \left\{ \int_{\mathcal{X}\times\mathcal{Y}} g \, \mathrm{d}\pi \mid \pi \in \Pi(\mu, \nu) \right\}$$

- Dual problem $g(x, y) \leqslant \phi(x) + \psi(y)$: social equilibrium
  - ϕ: the equilibrium wage function
  - ψ: the equilibrium profit function

Probabilistic formulation

Let L and R represent random variables on $\mathbb{R}$

- Classic optimal transport

$$\inf_{L\sim\mu, R\sim\nu} \mathbb{E}[c(L,R)]$$

- Martingale optimal transport require: $\mu \preceq_{\mathrm{cx}} \nu$

$$\inf_{L\sim\mu, R\sim\nu} \mathbb{E}[c(L,R)] : \; L = \mathbb{E}[R|L]$$

- Submartingale optimal transport require: $\mu \preceq_{\mathrm{icx}} \nu$

$$\inf_{L\sim\mu, R\sim\nu} \mathbb{E}[c(L,R)] : \; L \leqslant \mathbb{E}[R|L]$$

- Directional optimal transport require: $\mu \preceq_{\mathrm{st}} \nu$

$$\inf_{L\sim\mu, R\sim\nu} \mathbb{E}[c(L,R)] : \; L \leqslant R$$

Martingale: Beiglböck/Henry-Labordère/Penkner'13; Beiglböck/Juillet'16
Submartingale: Nutz/Stebegg'18; Directional: Nutz/W.'22

The multi-marginal formulation

Focus on $\mathbb{R}$ in what follows

- $\mu_1, \ldots, \mu_n$: n probability measures on $\mathbb{R}$
- $\Pi(\mu_1, \ldots, \mu_n)$: the set of probability measures on $\mathbb{R}^n$ with marginals $\mu_1, \ldots, \mu_n$
- $X_1, \ldots, X_n$: random variables ⇐ now probabilistic notation ...

Kantorovich's formulation:

$$\inf \left\{ \int_{\mathbb{R}^n} c(x_1, \ldots, x_n)\, \pi(\mathrm{d}x_1, \ldots, \mathrm{d}x_n) \mid \pi \in \Pi(\mu_1, \ldots, \mu_n) \right\}$$

Probabilistic formulation:

$$\inf \{\mathbb{E}[c(X_1, \ldots, X_n)] \mid X_i \sim \mu_i,\ i \in [n]\}$$

Rüschendorf'13; Pass'15

Dependence uncertainty problems

A random vector $\mathbf{X} = (X_1, \ldots, X_n)$

Assumptions

marginals may be known; dependence is unknown/arbitrary

- Motivation: data scarcity; uncertainty; absent information; lack of models; conservative decisions

Questions:

- properties of $\Psi(\mathbf{X})$ for some $\Psi : \mathbb{R}^n \to \mathbb{R}^d$
- range of $\mathbb{P}(\mathbf{X} \in A)$ for some $A \subseteq \mathbb{R}^n$
- "optimal" dependence structures of $\mathbf{X}$
- statistical decisions based on $\mathbf{X}$

Risk management and dependence uncertainty

Basic setup.

- A vector of risk factors: $\mathbf{X} = (X_1, \ldots, X_n)$
- A financial position $\Psi(\mathbf{X})$
- A risk measure ρ

Key task: Calculate $\rho(\Psi(\mathbf{X}))$

Most practical choices:

- $\Psi(\mathbf{X}) = \sum_{i=1}^{n} X_i$
- $\rho = \mathrm{VaR}_p$ (quantile) or $\rho = \mathrm{ES}_p$ (tail mean)

Challenge: We need a joint model for the random vector $\mathbf{X}$

- Range of $\rho(\Psi(\mathbf{X}))$ under dependence uncertainty

Risk management and dependence uncertainty

Misspecified dependence for credit risks $\Longrightarrow$ 07–09 financial crisis

- In the current Basel FRTB internal model approach, for market risk

$$\text{Capital Charge} = \lambda \mathrm{ES}_p \begin{pmatrix} \text{internal} \\ \text{model} \end{pmatrix} + (1-\lambda)\mathrm{ES}_p \begin{pmatrix} \text{worst-case} \\ \text{dependence} \end{pmatrix}$$

- Similar considerations in other regulatory frameworks, such as Solvency II

General formulation

Minimize or maximize

$$\rho(c(X_1, \ldots, X_n)) \text{ subject to } X_i \sim \mu_i,\ i \in [n]$$

Differences from the classic OT theory:

- ρ may be non-linear, such as VaR, ES and other risk measures
- The optimal value often matters more than the optimizer
- Even the linear case $c(x_1, \ldots, x_n) = \sum_{i=1}^n c_i(x_i)$ is interesting
- Optimizers usually depend on $\mu_1, \ldots, \mu_n$ in a complicated way
- No duality formulas

[many collaborations with P. Embrechts and coauthors]

Merging functions

Let $\mathcal{H}$ be a collection of probability measures ...

Definition (p-variables and merging functions)

(i) A p-variable is a random variable P that satisfies

$$\sup_{\mathbb{P}\in\mathcal{H}} \mathbb{P}(P \leqslant \varepsilon) \leqslant \varepsilon, \quad \varepsilon \in (0,1).$$

(ii) A p-merging function is an increasing Borel function $F : [0,\infty)^K \to [0,\infty)$ such that $F(P_1,\dots,P_K)$ is a p-variable for all p-variables $P_1,\dots,P_K$.

- Controlled type I error under arbitrary dependence

Merging functions

- $\mathcal{U}$: the set of all uniform $[0,1]$ random variables under $\mathbb{P}$

For an increasing Borel $F : [0,\infty)^K \to [0,\infty)$, equivalent are:

- F is a p-merging function w.r.t. some atomless collection $\mathcal{H}$
- F is a p-merging function w.r.t. all collections $\mathcal{H}$
- fixing atomless $\mathbb{P}$, $F(\mathbf{P})$ is a p-variable for all $\mathbf{P} \in \mathcal{U}^K$
- fixing atomless $\mathbb{P}$, for all $\varepsilon \in (0,1)$, $\overline{\mathbb{P}}(F \leqslant \varepsilon) \leqslant \varepsilon$, where

$$\overline{\mathbb{P}}(F \leqslant \varepsilon) = \sup\left\{\mathbb{P}(F(\mathbf{P}) \leqslant \varepsilon) : \mathbf{P} \in \mathcal{U}^K\right\}$$

- It suffices to consider $\mathcal{H} = \{\mathbb{P}\}$ for an atomless $\mathbb{P}$ and $\mathcal{U}^K$

Precise merging functions

- Multi-marginal OT problem:

$$\sup\{\mathbb{E}[\mathbb{1}_{\{F(\mathbf{P})\leqslant\varepsilon\}}] : \mathbf{P} \in \mathcal{U}^K\}$$

- The cost function is not submodular/supermodular
- Dependence assumption is available $\Longrightarrow$ constrained OT

Definition (precise merging functions)

A p-merging function F is precise if, for all $\varepsilon \in (0,1)$, $\overline{\mathbb{P}}(F \leqslant \varepsilon) = \varepsilon$.

Existing methods

- $p_{(1)}, \dots, p_{(K)}$ are the ascending order statistics
- The Bonferroni method

$$F_B(p_1, \dots, p_K) = K p_{(1)}$$

- Order-family (O-family) Rüger'78

$$G_{k,K} = (p_1, \dots, p_K) = \frac{K}{k} p_{(k)}$$

- Hommel–Simes Hommel'83

$$H_K(p_1, \dots, p_K) = \ell_K \bigwedge_{k=1}^{K} \frac{K}{k} p_{(k)}; \qquad \ell_K = \sum_{k=1}^{K} \frac{1}{k}$$

(also appearing in the BH procedure)

Merging p-values via averaging

- Generalized mean — Kolmogorov'30

$$M_{\phi,K}(p_1,\dots,p_K) = \phi^{-1}\left(\frac{\phi(p_1)+\cdots+\phi(p_K)}{K}\right),$$

where $\phi : [0,1] \to [-\infty,\infty]$ is continuous & strictly monotone

- M-family: for $r \in \mathbb{R} \setminus \{0\}$,

$$M_{r,K}(p_1,\dots,p_K) = \left(\frac{p_1^r+\cdots+p_K^r}{K}\right)^{1/r}$$

Limiting cases: geometric, minimum, maximum

- $\phi(x) = \tan((x-\frac{1}{2})\pi)$: Cauchy combination — Liu/Xie'20

Merging p-values via averaging

The arithmetic average $M_{1,K}(p_1, \ldots, p_K) = \frac{1}{K}\sum_{k=1}^{K} p_k$ is not a p-merging function Rüschendorf'82, Meng'93

$$\overline{\mathbb{P}}(M_{1,K} \leqslant \varepsilon) = \min(2\varepsilon, 1).$$

- $\Rightarrow 2M_{1,K}$ is a precise p-merging function

Task. Find $b_{r,K} > 0$ such that (the M-family)

$$F_{r,K} = b_{r,K} M_{r,K} \text{ is precise}$$

- $M_{r,K}$ increases in $r \Longrightarrow b_{r,K}$ should decrease in r.

Translation to a risk aggregation problem

For $\alpha \in (0,1]$ define

$$q_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leqslant x) \geqslant \alpha\};$$
$$\underline{q}_\alpha(F) = \inf\left\{q_\alpha(F(\mathbf{P})) : \mathbf{P} \in \mathcal{U}^K\right\}$$

Lemma 1

For $a > 0$, $r \in [-\infty, \infty]$, and $F = aM_{r,K}$, equivalent are:

(i) *F is a p-merging function, i.e., $\overline{\mathbb{P}}(F \leqslant \varepsilon) \leqslant \varepsilon$ for all $\varepsilon \in (0,1)$;*

(ii) *$\underline{q}_\varepsilon(F) \geqslant \varepsilon$ for all $\varepsilon \in (0,1)$;*

(iii) *$\overline{\mathbb{P}}(F \leqslant \varepsilon) \leqslant \varepsilon$ for some $\varepsilon \in (0,1)$;*

(iv) *$\underline{q}_\varepsilon(F) \geqslant \varepsilon$ for some $\varepsilon \in (0,1)$.*

The same conclusion holds if all $\leqslant$ and $\geqslant$ are replaced by $=$.

Translation to a risk aggregation problem

It boils down to calculate $\underline{q}_\varepsilon(M_{r,K})$, or equivalently:

(i) for $r > 0$, aggregation of Beta risks

$$(\underline{q}_\varepsilon(M_{r,K}))^r = \inf_{U_1,\dots,U_K \in \mathcal{U}} \left\{ q_\varepsilon \left(\frac{1}{K} (U_1^r + \cdots + U_K^r) \right) \right\}$$

(ii) for $r = 0$, aggregation of exponential risks

$$\log(\underline{q}_\varepsilon(M_{r,K})) = \inf_{U_1,\dots,U_K \in \mathcal{U}} \left\{ q_\varepsilon \left(\frac{1}{K} (\log U_1 + \cdots + \log U_K) \right) \right\}$$

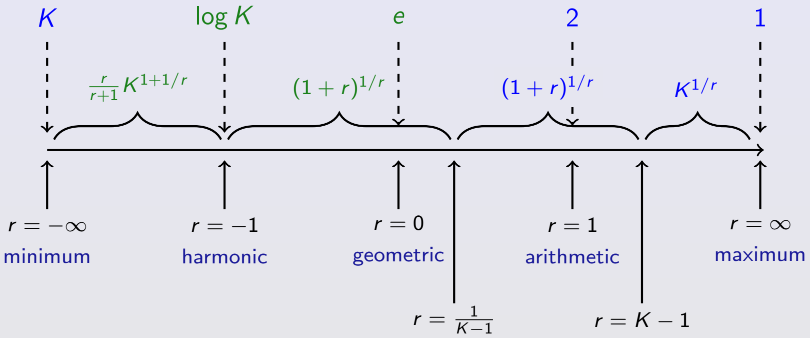
(iii) for $r < 0$, aggregation of Pareto risks

$$(\underline{q}_\varepsilon(M_{r,K}))^r = \sup_{U_1,\dots,U_K \in \mathcal{U}} \left\{ q_{1-\varepsilon} \left(\frac{1}{K} (U_1^r + \cdots + U_K^r) \right) \right\}$$

- A general tool: convolution bound Blanchet/Lam/Liu/W.'24

Summary

Constant multiplier in front of $M_{r,K}$



blue: precise; green: approximate

1. Combining p-values, optimal transport, and risk management
2. P-merging functions
3. Admissibility
4. Combining exchangeable p-values
5. External randomization
6. Simulation and summary

Admissible p-merging functions

For p-merging functions F and G:

- F dominates G if $F \leqslant G$
- F is admissible if it is not dominated by any other one
- F is symmetric if $F(\mathbf{p})$ is invariant under permutation of $\mathbf{p}$
- F is homogeneous if $F(\lambda \mathbf{p}) = \lambda F(\mathbf{p})$ for all $\lambda \in (0, 1]$ and $\mathbf{p}$ with $F(\mathbf{p}) \leqslant 1$

Properties

- Admissible $\Longrightarrow$ precise, lower semicontinuous, grounded
- Any p-merging function is dominated by an admissible one

Rejection regions of admissible p-merging functions

- The rejection region of F at level $\varepsilon \in (0,1)$:

$$R_\varepsilon(F) := \left\{\mathbf{p} \in [0,\infty)^K : F(\mathbf{p}) \leqslant \varepsilon\right\}$$

- An accending chain $\{R_\varepsilon \subseteq [0,\infty)^K : \varepsilon \in (0,1)\}$ of Borel lower sets induces a function $F : [0,\infty)^K \to [0,1]$ via

$$F(\mathbf{p}) = \inf\{\varepsilon \in (0,1) : \mathbf{p} \in R_\varepsilon\} \quad \text{with } \inf \varnothing = 1$$

- F is p-merging $\iff \mathbb{P}(\mathbf{P} \in R_\varepsilon) \leqslant \varepsilon$ for all $\varepsilon \in (0,1)$, $\mathbf{P} \in \mathcal{U}^K$
- F is homogeneous $\implies R_\varepsilon(F) = \varepsilon A$ for some $A \subseteq [0,\infty)^K$.

Rejection regions of admissible p-merging functions

Admissibility $\iff$ rejection region cannot be enlarged

- Precision of p-merging $\iff$ classic OT

$$\text{Compute} \sup_{\mathbf{P}\in\mathcal{U}^K} \mathbb{E}[\mathbb{1}_A(\mathbf{P})]$$

- Admissibility (or optimality) $\iff$ "reverse OT"

$$\text{Given} \sup_{\mathbf{P}\in\mathcal{U}^K} \mathbb{E}[\mathbb{1}_A(\mathbf{P})] \leqslant \varepsilon, \text{ find a maximal } A \subseteq [0,1]^K$$

- Such A needs to be nested
- Techniques in OT can be very helpful

One slide on e-values

Using p-to-e calibrators Vovk/W.'21

- A calibrator is a decreasing function $f : [0, \infty) \to [0, \infty]$ satisfying $f = 0$ on $(1, \infty)$ and $\int_0^1 f(x)\mathrm{d}x \leqslant 1$
- A calibrator f is admissible if it is upper semicontinuous, $f(0) = \infty$, and $\int_0^1 f(x)\mathrm{d}x = 1$

E-variables

An e-variable E for $\mathcal{H}$ is a $[0, \infty]$-valued random variable satisfying $\mathbb{E}^{\mathbb{P}}[E] \leqslant 1$ for $\mathbb{P} \in \mathcal{H}$.

- $f(P/\varepsilon)/\varepsilon$ is an e-variable for $P \in \mathcal{U}$ and $\varepsilon \in (0, 1)$
- $\mathbb{P}(E \geqslant 1/\varepsilon) \leqslant \varepsilon$ for an e-variable E (Markov's inequality)

Representation theorems

Let Δ_K be the standard K-simplex and write $\mathbf{p} = (p_1, \dots, p_K)$.

Theorem 1

For any admissible homogeneous p-merging function F, there exist $(\lambda_1, \dots, \lambda_K) \in \Delta_K$ and admissible calibrators $f_1, \dots, f_K$ such that

$$R_\varepsilon(F) = \left\{ \mathbf{p} \in [0,\infty)^K : \sum_{k=1}^K \lambda_k \frac{1}{\varepsilon} f_k\left(\frac{p_k}{\varepsilon}\right) \geqslant \frac{1}{\varepsilon} \right\}, \ \varepsilon \in (0,1). \quad \text{(H)}$$

Conversely, for any $(\lambda_1, \dots, \lambda_K) \in \Delta_K$ and calibrators $f_1, \dots, f_K$, (H) induces a homogeneous p-merging function.

Connection to e-tests

(H) is equivalent to

$$F(\mathbf{p}) = \inf\left\{\varepsilon \in (0,1) : \sum_{k=1}^{K} \lambda_k f_k\left(\frac{p_k}{\varepsilon}\right) \geqslant 1\right\}$$

- Admissible (homogeneous) ways of merging p-values must be through merging e-values and testing with Markov's inequality
- Proof is based on optimal transport duality

$$\min\left\{\sum_{k=1}^{K}\int_0^1 g_k(x)\mathrm{d}x : \bigoplus_{k=1}^{K} g_k \geqslant \mathbb{1}_{R_\varepsilon(F)}\right\} = \max_{\mathbf{P}\in\mathcal{U}} \mathbb{P}(\mathbf{P} \in R_\varepsilon(F)) = \varepsilon$$

Representation theorems

Theorem 2

For any F that is admissible within the family of homogeneous symmetric p-merging functions, there exists an admissible calibrator f such that

$$R_\varepsilon(F) = \left\{ \mathbf{p} \in [0,\infty)^K : \frac{1}{K}\sum_{k=1}^{K} f\left(\frac{p_k}{\varepsilon}\right) \geqslant 1 \right\}, \ \varepsilon \in (0,1). \quad \text{(SH)}$$

Conversely, for any calibrator f, (SH) induces a homogeneous symmetric p-merging function.

- We say f induces F if (SH) holds
- Converse: not true; calibrator: not unique

Connection to joint mixability

A necessary and sufficient condition for a calibrator f to induce a precise p-merging function via (SH) is

$$\mathbb{P}\left(\frac{1}{K}\sum_{k=1}^{K} f(P_k) = 1\right) = 1 \quad \text{for some } P_1, \dots, P_K \in \mathcal{U}. \qquad \text{(JM)}$$

- $\Longrightarrow$ Joint mixability — Wang/W.'11'16
- Difficult to check for a given f in general
- For a convex f, (JM) holds if and only if $f \leqslant K$ on $(0, 1]$
- Weaker than admissibility

Sufficient conditions for admissibility

Theorem 3

Suppose that an admissible calibrator f is strictly convex or strictly concave on $(0,1]$, $f(0+) \in (K/(K-1), K]$, and $f(1) = 0$. The p-merging function induced by f is admissible.

- Proof based on joint mixability (JM)
- Open question: can strict convexity be reduced to convexity?
- Conditions of this type are not necessary

Hommel's function

Define the Hommel* calibrator f by

$$f : x \mapsto \frac{K \mathbb{1}_{\{\ell_K x \leqslant 1\}}}{\lceil K \ell_K x \rceil}.$$

Theorem 4

The p-merging function $H_K \wedge 1$ is dominated (strictly if $K \geqslant 4$) by the p-merging function H_K^* induced by f via (SH). Moreover, H_K^* is always admissible among symmetric p-merging functions, and it is admissible if K is not a prime number.

- Primality appears in the proof due to factoring the set $[K]$
- H_K^* is not admissible for $K = 2, 3$ (we guess also 5)

Hommel's function and the O-family

- The Bonferroni method is admissible
- Members $G_{k,K}$ of the O-family are admissible after truncation at 1 except for $k = K$
- Members $F_{r,K}$ of the M-family are not admissible except for $r = -\infty$
- $F_{r,K}$ can be strictly improved to $F^*_{r,K}$
- $F^*_{-1,K}$ is similar to H^*_K

Combining exchangeable p-values

Definition 1 (Exchangeability)

The random vector $(P_1, \dots, P_K)$ is exchangeable if

$$(P_1, \dots, P_K) \overset{\mathrm{d}}{=} (P_{\sigma(1)}, \dots, P_{\sigma(K)})$$

for all K-permutations σ.

Examples

- Split conformal prediction
- Median-of-means methods
- Overlapping sample

Exchangeable/uniformly-randomized Markov inequalities

Proposition 1 (Ramdas/Manole'24 STS)

Let $X_1, X_2, \ldots$ form an exchangeable sequence of non-negative random variables. Then, for any $a > 0$,

$$\mathbb{P}\left(\exists k \geqslant 1 : \frac{1}{k}\sum_{i=1}^{k} X_i \geqslant \frac{1}{a}\right) \leqslant a\mathbb{E}[X_1].$$

Proof.

- Let $S_k = \sum_{i=1}^{k} X_i$
- $\mathbb{E}[S_k|S_{k+1}] = kS_{k+1}/(k+1) \implies (S_k/k)_{k\in[K]}$ is a backward martingale
- Ville's inequality gives the desired statement

Combining exchangeable p-values

Theorem 5

Let f be a calibrator, and $\mathbf{P} = (P_1, \dots, P_K) \in \mathcal{U}^K$ be exchangeable. For each $\alpha \in (0,1)$, we have

$$\mathbb{P}\left(\exists k \leqslant K : \frac{1}{k}\sum_{i=1}^{k} f\left(\frac{P_i}{\alpha}\right) \geqslant 1\right) \leqslant \alpha.$$

Combining exchangeable p-values

- For a calibrator f and $\alpha \in (0,1)$, let

$$R_\alpha = \left\{ \mathbf{p} \in [0,1]^K : \frac{1}{k}\sum_{i=1}^{k} f\left(\frac{p_i}{\alpha}\right) \geqslant 1 \text{ holds for some } k \leqslant K \right\}$$

and

$$\begin{aligned} F(\mathbf{p}) &= \inf\{\alpha \in (0,1) : \mathbf{p} \in R_\alpha\} \\ &= \inf \left\{ \alpha \in (0,1) : \bigvee_{k=1}^{K} \left(\frac{1}{k}\sum_{i=1}^{k} f\left(\frac{p_i}{\alpha}\right)\right) \geqslant 1 \right\}, \qquad \text{(Ex)} \end{aligned}$$

with the convention $\inf \varnothing = 1$

- (Ex) is always smaller or equal than the deterministic version

Combining exchangeable p-values

Definition 2

An ex-p-merging function is an increasing Borel function $F : [0,1]^K \to [0,1]$ such that $\mathbb{P}(F(\mathbf{P}) \leqslant \alpha) \leqslant \alpha$ for all $\alpha \in (0,1)$ and $\mathbf{P} \in \mathcal{U}^K$ that is exchangeable.

Theorem 6

If f is a calibrator and $\mathbf{P} \in \mathcal{U}^K$ is exchangeable, then F in (Ex) is a homogeneous ex-p-merging function, and dominating (H).

Combining exchangeable p-values

Proposition 2

A symmetric ex-p-merging function is necessarily a p-merging function.

- Hence, for an ex-p-merging function to strictly dominate an admissible p-merging function, it cannot be symmetric

Proposition 3

Suppose that f is a convex admissible calibrator with $f(0+) \leqslant K$ and $f(1) = 0$, and F is in (Ex). For $\alpha \in (0,1)$, there exists an exchangeable $\mathbf{P} \in \mathcal{U}^K$ such that $\mathbb{P}(F(\mathbf{P}) \leqslant \alpha) = \alpha$.

- Again relying on (JM)

Combining exchangeable p-values

Theorem 7

Let $P_1, P_2, \ldots \in \mathcal{U}^\infty$ be an infinite exchangeable sequence and let F be the function defined in (Ex). Then,

$$\mathbb{P}(\exists k \geqslant 1 : F(\mathbf{P}_k) \leqslant \alpha) \leqslant \alpha,$$

where $\mathbf{P}_k = (P_1, \ldots, P_k)$.

External randomization

Proposition 4

Let X be a non-negative random variable independent of $U \in \mathcal{U}$. Then, for any $a > 0$,

$$\mathbb{P}(X \geqslant U/a) \leqslant a\mathbb{E}[X].$$

External randomization

$$\mathcal{U}^K \otimes \mathcal{U} = \{(\mathbf{P}, U) : \mathbf{P} \in \mathcal{U}^K \text{ and } U \in \mathcal{U} \text{ are independent}\}$$

Theorem 8

Let $f_1, \ldots, f_K$ be calibrators and $(P_1, \ldots, P_K, U) \in \mathcal{U}^K \otimes \mathcal{U}$. For each $\alpha \in (0,1)$ and $(\lambda_1, \ldots, \lambda_K) \in \Delta_K$, we have

$$\mathbb{P}\left(\sum_{k=1}^K \lambda_k f_k\left(\frac{P_k}{\alpha}\right) \geqslant U\right) \leqslant \alpha.$$

If $f_1, \ldots, f_K$ are admissible and $\mathbb{P}(\sum_{k=1}^K \lambda_k f_k(P_k/\alpha) \leqslant 1) = 1$, then

$$\mathbb{P}\left(\sum_{k=1}^K \lambda_k f_k\left(\frac{P_k}{\beta}\right) \geqslant U\right) = \beta \quad \text{for all } \beta \in (0, \alpha].$$

External randomization

- Let $f_1, \ldots, f_K$ be calibrators and $(\lambda_1, \ldots, \lambda_K) \in \Delta_K$
- For $\alpha \in (0,1)$, let

$$R_\alpha = \left\{ (\mathbf{p}, u) \in [0,1]^{K+1} : \sum_{k=1}^{K} \lambda_k f_k \left(\frac{p_k}{\alpha} \right) \geqslant u \right\}$$

where we set $f_k(p_k/u) = 0$ if $u = 0$, and

$$\begin{aligned} F(\mathbf{p}, u) &= \inf\{\alpha \in (0,1) : (\mathbf{p}, u) \in R_\alpha\} \\ &= \inf \left\{ \alpha \in (0,1) : \sum_{k=1}^{K} \lambda_k f_k \left(\frac{p_k}{\alpha} \right) \geqslant u \right\}, \qquad \text{(U)} \end{aligned}$$

with the convention $\inf \varnothing = 1$ and $0 \times \infty = \infty$

External randomization

Definition 3

A randomized p-merging function is an increasing Borel function $F : [0,1]^{K+1} \to [0,1]$ such that $\mathbb{P}(F(\mathbf{P}, U) \leqslant \alpha) \leqslant \alpha$ for all $\alpha \in (0,1)$ and $(\mathbf{P}, U) \in \mathcal{U}^K \otimes \mathcal{U}$.

Theorem 9

If $f_1, \dots, f_K$ are calibrators and $(\lambda_1, \dots, \lambda_K) \in \Delta_K$, then F in (U) is a homogenous randomized p-merging function. Moreover, F is lower semicontinuous.

External randomization

Remarks:

- If P_1 is independent of $(P_2, \dots, P_K)$, then we can use $U = P_1$ and merge $(P_2, \dots, P_K)$
- Merging asymptotic p-values: if $(\mathbf{P}_n)_{n\in\mathbb{N}} \xrightarrow{\text{law}} \mathbf{P}$, then

$$\limsup_{n\to\infty} \mathbb{P}(\mathbf{P}_n \in R_\alpha) \leqslant \mathbb{P}(\mathbf{P} \in R_\alpha) \leqslant \alpha$$

because the rejection sets in (H), (Ex), and (U) are closed

Summary

Rule	Arbitrary dependence	Exchangeable	Randomized
Rüger's	$\frac{K}{k} p_{(k)}$	$\frac{K}{k} \bigwedge_{m=1}^{K} p^m_{(\lambda_m)}$	$\frac{K}{k} p_{(\lceil uk \rceil)}$
Arithmetic	$2A(\mathbf{p})$	$2 \bigwedge_{m=1}^{K} A(\mathbf{p}_m)$	$\frac{2}{2-u} A(\mathbf{p})$
Geometric	$eG(\mathbf{p})$	$e \bigwedge_{m=1}^{K} G(\mathbf{p}_m)$	$e^u G(\mathbf{p})$
Harmonic	$(T_K + 1)H(\mathbf{p})$	$(T_K + 1) \bigwedge_{m=1}^{K} H(\mathbf{p}_m)$	$(T_K u + 1)H(\mathbf{p})$

- $\mathbf{p} = (p_1, \dots, p_K)$
- $\mathbf{p}_m$: the vector containing the first m values of $\mathbf{p}$
- $p_{(k)}$ is the k-th smallest value of $\mathbf{p}$
- $p^m_{(\lambda_m)}$ is the $\lambda_m = \lceil m \frac{k}{K} \rceil$ ordered value of $\mathbf{p}_m$
- A, G and H: the arithmetic, the geometric, and the harmonic mean
- $T_K = \log K + \log\log K + 1,\ K \geqslant 2.$

Simulation results

- One-side z-test of $\mu = 0$ against the alternative $\mu > 0$
- $Z, Z_1, \ldots, Z_K \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$
- P-values are given by

$$X_k = \rho Z + \sqrt{1-\rho^2} Z_k - \mu, \quad P_k = \Phi(X_k),$$

  where Φ is standard normal cdf
- $\mu \geqslant 0$ and $\rho \in [0,1]$ are constants
- $P_1, \ldots, P_K$ are exchangeable
- $\alpha = 0.05$
- $B = 10,000$ replications

Simulation results

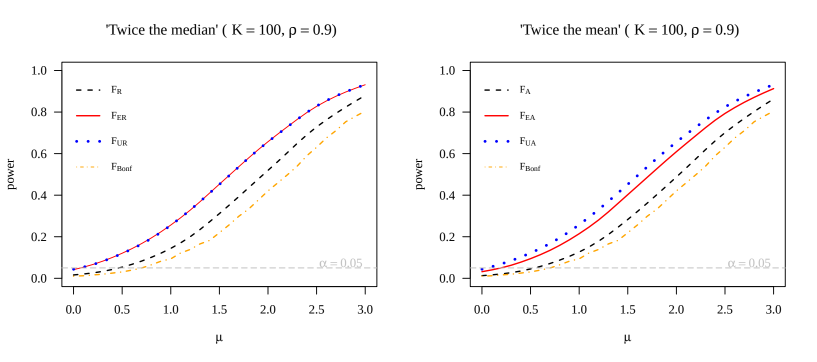


Figure: Empirical power of different combination rules, $\rho = 0.9$

Simulation results

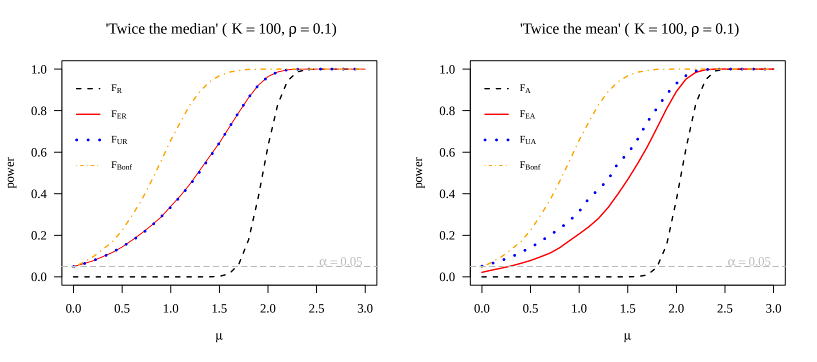


Figure: Empirical power of different combination rules, $\rho = 0.1$

Simulation results

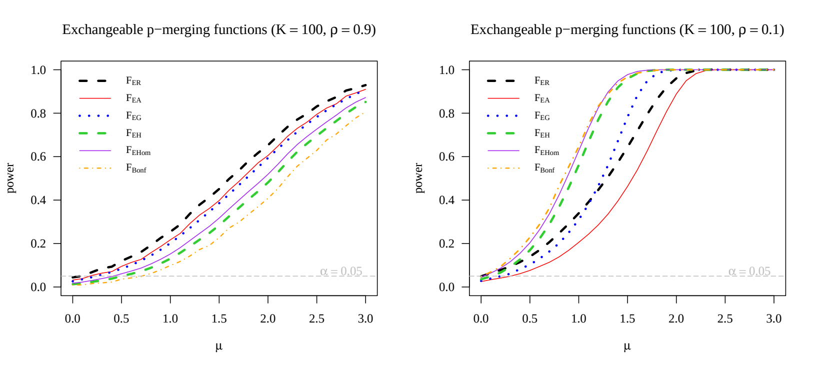


Figure: Combination of p-values using different exchangeable p-merging functions. The performance of the different exchangeable p-merging functions is almost reversed in the two situations.

Simulation results

- $X_{ij} \overset{\text{iid}}{\sim} \mathcal{N}(\mu, 1)$ for $i = 0, 1, \dots, K$ and $j = 1, \dots, n_i$
- X_{0j}, $j = 1, \dots, n_0$: common sample
- Sample average

$$\bar{X}_i = \frac{1}{\sqrt{n_i}} \sum_{j=1}^{n_i} X_{ij} \sim \mathcal{N}(\sqrt{n_i}\mu, 1), \quad i = 0, 1, \dots, K$$

- Test statistics and p-values, exchangeable under the null

$$T_i = \frac{\bar{X}_i + \bar{X}_0}{\sqrt{2}} \text{ and } P_i = 2 \min \{\Phi(T_i), \Phi(-T_i)\}, \quad i = 1, \dots, K$$

- Order $P_1, \dots, P_K$ by $n_1, \dots, n_K$

Simulation results

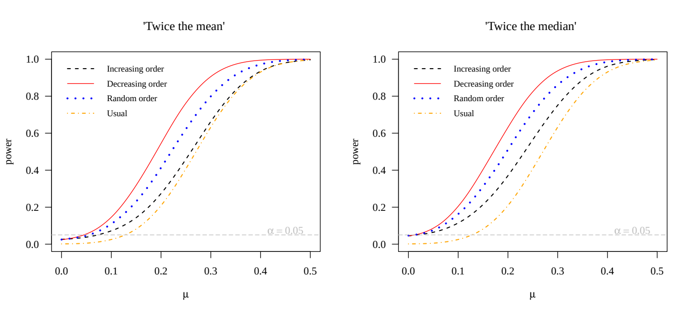


Figure: Combination of p-values using different ex-p-merging functions and different ordering based on the number of observations, where $K = 10$ and $n_i = 10 \times i$. The ex-p-merging rules are more powerful if p-values are ordered in decreasing order with respect to the number of observations.

Summary

Take home these

$$e \bigwedge_{m=1}^{K} G(\mathbf{p}_m); \qquad (\log K + \log\log K + 2) \bigwedge_{m=1}^{K} H(\mathbf{p}_m)$$

$$\frac{K}{k} p_{(\lceil uk \rceil)}; \qquad e^u G(\mathbf{p})$$

- Merging p-values under unknown dependence $\Longrightarrow$ merging e-values
- Exchangeability allows for processing p-values sequentially
- Randomization improves power
- Dependence problems can be solved with OT theory
- Many open questions!
- Write me if you have a problem whose challenge lies in dependence

Thank you

Thank you for your kind attention

Vladimir Vovk

Bin Wang

Matteo Gasparin

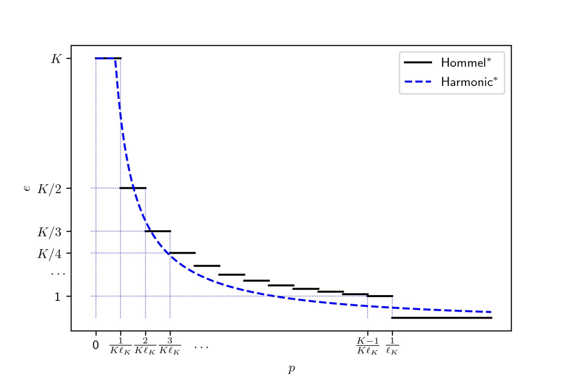
Aaditya Ramdas

Paul Embrechts

Hommel's function and the O-family

Define the Hommel* calibrator f by

$$f : x \mapsto \frac{K \mathbb{1}_{\{\ell_K x \leqslant 1\}}}{\lceil K \ell_K x \rceil}.$$



Hommel's function and the O-family

Theorem 10

The p-merging function $H_K \wedge 1$ is dominated (strictly if $K \geqslant 4$) by the p-merging function H_K^* induced by the Hommel calibrator f,*

$$R_\varepsilon(H_K^*) = \varepsilon \left\{ \mathbf{p} \in [0,\infty)^K : \frac{1}{K} \sum_{k=1}^{K} f(p_k) \geqslant 1 \right\}, \quad \varepsilon \in (0,1).$$

Moreover, H_K^* is always admissible among symmetric p-merging functions, and it is admissible if K is not a prime number.

Hommel's function and the O-family

Proof sketch.

- Recall: $H_K(\mathbf{p}) = \ell_K \bigwedge_{k=1}^K \frac{K}{k} p_{(k)}$ where $\ell_K = \sum_{i=1}^K \frac{1}{k}$.
- Induced by the calibrator $f \implies H_K^*$ is a p-merging function.
- Verify $H_K \geqslant H_K^*$: $H_K(\mathbf{p}) \leqslant \varepsilon \implies$ there exists m such that $K\ell_K p_{(m)} \leqslant \varepsilon$ $\implies \#\{k : K\ell_K p_k/m \leqslant \varepsilon\} \geqslant m \implies$

$$\sum_{k=1}^K \frac{\mathbb{1}_{\{\ell_K p_k \leqslant \varepsilon\}}}{\lceil K\ell_K p_i/\varepsilon \rceil} \geqslant \sum_{k=1}^K \frac{1}{m} \mathbb{1}_{\{K\ell_K p_k/\varepsilon \leqslant m\}} = \frac{1}{m} \#\{k : K\ell_K p_k/m \leqslant \varepsilon\} \geqslant 1$$

$\implies R_\varepsilon(H_K) \subseteq R_\varepsilon(H_K^*) \implies H_K \geqslant H_K^*$.

- Check $H_K = H_K^*$ if and only if $K \leqslant 3$.

Hommel's function and the O-family

Proof sketch (continued).

- Suppose H_K^* is not admissible among symmetric p-merging functions.
- There exists a calibrator g satisfying

$$\left\{\mathbf{p}\in[0,\infty)^K : \frac{1}{K}\sum_{k=1}^K f(p_k)\geqslant 1\right\} \subsetneq \left\{\mathbf{p}\in[0,\infty)^K : \frac{1}{K}\sum_{k=1}^K g(p_k)\geqslant 1\right\}.$$

- Denote by $\tau := 1/(K\ell_K)$. For $x\in(0,K\tau]$, set $p_1=\cdots=p_m=x$ and $p_{m+1}=\cdots=p_K>1$, where $m:=\lceil \tau x\rceil$.
- $f(x)=K/m \implies \sum_{k=1}^K f(p_k)=K \implies K\leqslant\sum_{k=1}^K g(p_k)=mg(x)\implies g(x)\geqslant K/m=f(x)$.
- $\int_0^{K\tau} g(x)\mathrm{d}x \geqslant \int_0^{K\tau} f(x)\mathrm{d}x = 1 \implies g=f$ almost everywhere on $[0,1]$.
- f is left-continuous $\implies g\leqslant f$, a contradiction.
- The admissibility statement for non-prime K is much more complicated.

Hommel's function and the O-family

- $S_K \leqslant H_K^* \leqslant H_K \implies 1/\ell_K \leqslant H_K^*/H_K \leqslant 1$
- H_K^* may not be admissible for a prime K

Example 1

In case $K = 2$, $H_2^* = H_2 : (p_1, p_2) \mapsto (3p_{(1)}) \wedge (\frac{3}{2}p_{(2)})$ is strictly dominated by $F : (p_1, p_2) \mapsto (3p_1) \wedge (\frac{3}{2}p_2)$, which is a (non-symmetric) p-merging function because

$$\mathbb{P}(F(P_1, P_2) \leqslant \alpha) \leqslant \mathbb{P}\left(P_1 \leqslant \frac{1}{3}\alpha\right) + \mathbb{P}\left(P_2 \leqslant \frac{2}{3}\alpha\right) \leqslant \alpha.$$

Hommel's function and the O-family

Theorem 11

The p-merging function $\mathbf{p} \mapsto G_{k,K}(\mathbf{p}) \wedge \mathbb{1}_{\{\min(\mathbf{p})>0\}}$ *is admissible for* $k = 1, \ldots, K-1$*, and it is admissible among symmetric p-merging functions for* $k = K$*.*

The M-family

- $F_{r,K} = (b_{r,K} M_{r,K}) \wedge 1$
- For $r \neq \{-1, 0\}$ and $r < 1/(K-1)$, denote by c_r the unique number $c \in (0, 1/K)$ solving the equation

$$\frac{(K-1)(1-(K-1)c)^r + c^r}{K} = \frac{(1-(K-1)c)^{r+1} - c^{r+1}}{(r+1)(1-Kc)}$$

- c_{-1} and c_0 are limits of c_r
- Set $c_r := 0$ for $r \geqslant 1/(K-1)$
- Write $d_r := 1 - (K-1)c_r$

The M-family

Proposition 5

For $K \geqslant 3$ and $r \in (-\infty, \frac{1}{K-1})$,

$$b_{r,K} = 1/M_{r,K}(c_r, d_r, \ldots, d_r).$$

- If $r < s$ and $rs > 0$, then

$$K^{1/s-1/r} b_{r,K} \leqslant b_{s,K} \leqslant b_{r,K}$$

The M-family

For $r < 0$:

- Rejection region

$$R_\varepsilon(F_{r,K}) = \varepsilon \left\{ \mathbf{p} \in [0,\infty)^K : \frac{\sum_{k=1}^K p_k^r}{c_r^r + (K-1)d_r^r} \geqslant 1 \right\}$$

$$= \varepsilon \left\{ \mathbf{p} \in [0,\infty)^K : \sum_{k=1}^K \frac{p_k^r - d_r^r}{c_r^r - d_r^r} \geqslant 1 \right\}.$$

- Define a calibrator

$$f_r : x \mapsto K \left(\frac{x^r - d_r^r}{c_r^r - d_r^r} \wedge 1 \right)_+ .$$

- f_r is strictly convex on $[c_r, d_r]$.

The M-family

Let $F^*_{r,K}$ be the p-merging function induced by f_r, i.e.,

$$R_\varepsilon(F^*_{r,K}) = \varepsilon \left\{ \mathbf{p} \in [0,\infty)^K : \sum_{k=1}^K \left(\frac{p_k^r - d_r^r}{c_r^r - d_r^r} \right)_+ \geqslant 1 \right\}, \ \varepsilon \in (0,1).$$

- $R_\varepsilon(F_{r,K}) \subset R_\varepsilon(F^*_{r,K})$
- $F^*_{r,K}$ is admissible

The M-family

Theorem 12

For $K \geqslant 3$ and $r \in (-\infty, K-1)$, $F_{r,K}$ is strictly dominated by the p-merging function $F^*_{r,K}$ defined via, for $\mathbf{p} \in (0,\infty)^K$ and $\varepsilon \in (0,1)$,

$$F^*_{r,K}(\mathbf{p}) \leqslant \varepsilon \iff F_{r,K}(\mathbf{p} \wedge (\varepsilon d_r \mathbf{1})) \leqslant \varepsilon \quad \text{or} \quad \min(\mathbf{p}) = 0.$$

Moreover, $F^*_{r,K}$ is admissible unless $r = 1$.

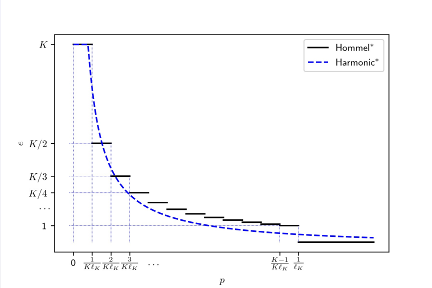
The M-family

Recall

$$f_{-1} : x \mapsto K \left(\frac{x^{-1} - d_{-1}^{-1}}{c_{-1}^{-1} - d_{-1}^{-1}} \wedge 1 \right)_+ ;$$

the Hommel* calibrator

$$f : x \mapsto \frac{K \mathbb{1}_{\{\ell_K x \leqslant 1\}}}{\lceil K \ell_K x \rceil}.$$



- When taking values in $(0, K)$:

$$f_{-1}(x) = a/x - b \quad \text{vs} \quad f(y) = a'/\lceil b'y \rceil$$

The M-family

Proposition 6

For $K \geqslant 3$ *and* $\mathbf{p} \in [0,\infty)^K$, *we have, if* $r \in (-\infty, 1/(K-1))$,

$$F^*_{r,K}(\mathbf{p}) = \left(\bigwedge_{m=1}^{K} \frac{M_{r,m}(p_{(1)},\dots,p_{(m)})}{M_{r,m}(c_r, d_r, \dots, d_r)} \right) \wedge 1,$$

and, if $r \in [1/(K-1), K-1)$, *with the convention* $\cdot/0 = \infty$,

$$F^*_{r,K}(\mathbf{p}) = \left(\bigwedge_{m=1}^{K} \frac{M_{r,m}(p_{(1)},\dots,p_{(m)})}{\left(1 - \frac{rK}{(r+1)m}\right)_+} \right) \wedge \mathbb{1}_{\{p_{(1)}>0\}}.$$

The M-family

Proposition 7

For $r < s$, $K \geqslant 2$ and $a, b > 0$, the following statements hold.

(i) $aM_{r,K} \leqslant bM_{s,K}$ *if and only if* $a \leqslant b$.

(ii) $bM_{s,K} \leqslant aM_{r,K}$ *if and only if* $rs > 0$ *and* $aK^{-1/r} \geqslant bK^{-1/s}$.

Proposition 8

Suppose $r \neq s$. *If* $K = 2$, $F_{r,K} \geqslant F_{s,K}$ *if and only if* $1 \leqslant r < s$ *or* $s < r \leqslant 1$. *If* $K \geqslant 3$, $F_{r,K} \geqslant F_{s,K}$ *if and only if* $K - 1 \leqslant r < s$.

Magnitude of improvement

Proposition 9

For $K \geqslant 3$, we have

$$\inf_{\mathbf{p}>\mathbf{0}} \frac{F^*_{1,K}(\mathbf{p})}{F_{1,K}(\mathbf{p})} = \inf_{\mathbf{p}>\mathbf{0}} \frac{F^*_{0,K}(\mathbf{p})}{F_{0,K}(\mathbf{p})} = 0, \quad \inf_{\mathbf{p}>\mathbf{0}} \frac{F^*_{-1,K}(\mathbf{p})}{F_{-1,K}(\mathbf{p})} = 1-(K-1)c_{-1},$$

and

$$\min_{\mathbf{p}>\mathbf{0}} \frac{H^*_K(\mathbf{p})}{H_K(\mathbf{p})} = \min\left\{ t>0 : \sum_{k=1}^{K} \frac{\mathbb{1}_{\{t \geqslant k/K\}}}{\lceil k/t \rceil} \geqslant 1 \right\} =: \gamma_K.$$

Moreover, $c_{-1} \sim 1/(K \log K)$ and $\gamma_K \sim 1/\log K$ as $K \to \infty$.

Magnitude of improvement

- $F^*_{-1,K}$ improves $F_{-1,K}$ only by a factor $1 - 1/\log K \sim 1$
- H^*_K can improve H_K by a significant factor of $1/\log K$
- $H^*_K(\mathbf{p})/H_K(\mathbf{p}) = \gamma_K$ is attained by $\mathbf{p} = (\alpha, 2\alpha, \cdots, K\alpha)$ for $\alpha \in (0, 1/K\ell_K]$.
- Since $H_K = \ell_K S_K$ and

$$\gamma_K \sim 1/\log K \sim 1/\ell_K,$$

H^*_K performs similarly to the Simes function S_K for some values of $\mathbf{p}$ above

Simes function

Proof sketch.

- Take any symmetric p-merging function F and $\mathbf{p} = (p_1, \dots, p_K)$
- Let $\alpha := S_K(\mathbf{p})/K \implies p_{(k)} \geqslant k\alpha$ for each k
- Symmetry and monotonicity of $F \implies$

$$F(\mathbf{p}) = F(p_{(1)}, \dots, p_{(K)}) \geqslant F(\alpha, 2\alpha, \dots, K\alpha) =: \beta$$

- Let Π be the set of all permutations of $(\alpha, 2\alpha, \dots, K\alpha)$, and $\mu = \mathrm{U}(\Pi)$
- Take $(P_1, \dots, P_K) \sim K\alpha\mu + (1 - K\alpha)\delta_{(1,\dots,1)}$
- For each k, $P_k \sim \sum_{k=1}^K \alpha\delta_{k\alpha} + (1 - K\alpha)\delta_1 \implies P_k$ is a p-variable
- F is a p-merging function $\implies$

$$\beta \geqslant \mathbb{P}(F(P_1, \dots, P_K) \leqslant \beta) \geqslant \mathbb{P}((P_1, \dots, P_K) \in \Pi) = K\alpha$$

- $F(\mathbf{p}) \geqslant K\alpha = S_K(\mathbf{p}) \implies S_K$ dominates all symmetric p-merging functions
- $S_K = \bigwedge_{k=1}^K G_{k,K}$

Representation theorems

Proof sketch.

- For decreasing functions $g_1, \dots, g_K$ on $[0, \infty)$, denote by

$$\left(\bigoplus_{k=1}^{K} g_k\right)(x_1, \dots, x_K) := \sum_{k=1}^{K} g_k(x_k)$$

- Classic duality ($R_\varepsilon(F)$ is closed and F is precise)

$$\min\left\{\sum_{k=1}^{K}\int_0^1 g_k(x)\mathrm{d}x : \bigoplus_{k=1}^{K} g_k \geqslant \mathbb{1}_{R_\varepsilon(F)}\right\} = \max_{\mathbf{P}\in\mathcal{U}} \mathbb{P}(\mathbf{P} \in R_\varepsilon(F)) = \varepsilon$$

- Take $(g_1^\varepsilon, \dots, g_K^\varepsilon)$ such that $\bigoplus_{k=1}^K g_k^\varepsilon \geqslant \mathbb{1}_{R_\varepsilon(F)}$ and $\sum_{k=1}^K \int_0^1 g_k^\varepsilon(x)\mathrm{d}x = \varepsilon$
- Choose g_k^ε to be non-negative and left-continuous
- Monotonicity

$$\max_{\mathbf{P}\in\mathcal{U}^K} \mathbb{P}(\mathbf{P} \in R_\varepsilon(F)) = \varepsilon \implies \max_{\mathbf{P}\in\mathcal{U}^K} \mathbb{P}(\varepsilon\mathbf{P} \in R_\varepsilon(F)) = 1$$

Representation theorems

Proof sketch (continued).

- Using duality again

$$\min\left\{\sum_{k=1}^K \frac{1}{\varepsilon}\int_0^\varepsilon g_k(x)\mathrm{d}x : \bigoplus_{k=1}^K g_k \geqslant \mathbb{1}_{R_\varepsilon(F)}\right\} = 1$$

$\Longrightarrow \sum_{k=1}^K \int_0^\varepsilon g_k^\varepsilon(x)\mathrm{d}x \geqslant \varepsilon \Longrightarrow g_k^\varepsilon(x) = 0$ for $x > \varepsilon$

- Define the set $A_\varepsilon := \{\mathbf{p} \in [0,\infty)^K : \sum_{k=1}^K g_k^\varepsilon(p_k) \geqslant 1\}$
- $\bigoplus_{k=1}^K g_k^\varepsilon \geqslant \mathbb{1}_{R_\varepsilon(F)} \Longrightarrow R_\varepsilon(F) \subseteq A_\varepsilon$
- By Markov's inequality,

$$\sup_{\mathbf{P}\in\mathcal{U}^K} \mathbb{P}\left(\bigoplus_{k=1}^K g_k^\varepsilon(\mathbf{P}) \geqslant 1\right) \leqslant \sup_{P\in\mathcal{U}} \sum_{k=1}^K \mathbb{E}[g_k^\varepsilon(P)] = \varepsilon$$

- Define a function F' with rejection region $R_\delta(F') = \delta\varepsilon^{-1}A_\varepsilon$ for $\delta \in (0,1)$
- F' is a valid homogeneous p-merging function and $F' \leqslant F$

Representation theorems

Proof sketch (continued).

- Admissibility of $F \implies F = F'$, thus

$$R_\varepsilon(F) = A_\varepsilon = \varepsilon\left\{\mathbf{p} \in [0,\infty)^K : \sum_{k=1}^{K} g_k^\varepsilon(\varepsilon p_k) \geqslant 1\right\} \quad \text{for each } \varepsilon \in (0,1)$$

- $\varepsilon^{-1} R_\varepsilon(F) = \varepsilon^{-1} A_\varepsilon$ does not depend on $\varepsilon \in (0,1)$
- For a fixed $\varepsilon \in (0,1)$ and each k, let $\lambda_k := \varepsilon^{-1} \int_0^\varepsilon g^\varepsilon(x)\mathrm{d}x$ and $f_k : (0,\infty) \to \mathbb{R}$, $x \mapsto g_k^\varepsilon(\varepsilon x)/\lambda_k$ (if $\lambda_k = 0$, then let $f_k := 1$), and further set $f_k(0) = \infty$
- For each k with $\lambda_k \neq 0$,

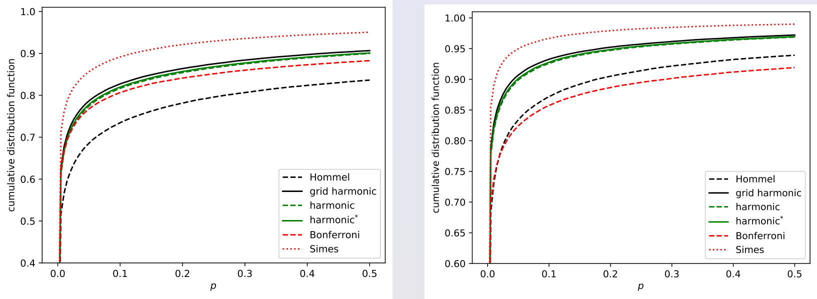
$$\int_0^1 f_k(x)\mathrm{d}x = \frac{\int_0^1 \varepsilon g_k^\varepsilon(\varepsilon x)\mathrm{d}x}{\int_0^1 g_k^\varepsilon(x)\mathrm{d}x} = \frac{\int_0^\varepsilon g_k^\varepsilon(x)\mathrm{d}x}{\int_0^1 g_k^\varepsilon(x)\mathrm{d}x} = 1$$

  $\implies f_k$ is an admissible calibrator
- Converse statement: Markov's inequality

Simulation results

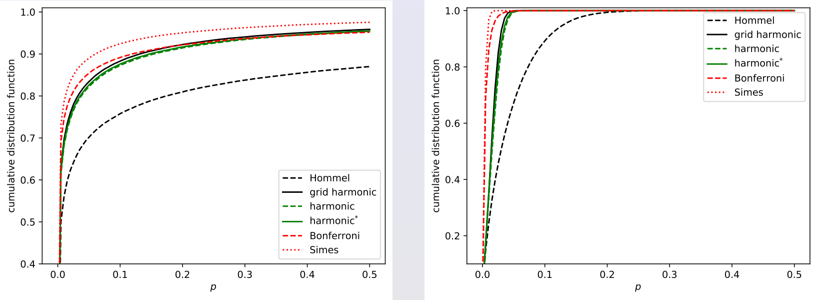
- Correlated z-tests
- $K = 10^6$ observations from $\mathrm{N}(\mu, 1)$
- Pairwise correlation: 0.9
- Last observation: -0.9 correlation with all others
- $\mu = 0$ for null and $\mu = -5$ for alternative
- K_1 observations are drawn from the alternative; the rest from the null
- P-values are $\Phi(x)$
- $F_{-\infty,K}$ (Bonferroni); H_K (Hommel); $F_{-1,K}$ (harmonic); $F^*_{-1,K}$ (harmonic*'); H^*_K (grid harmonic'); S_K (Simes),

Simulation results



$K_1 = 10^3$ (left panel) and $K_1 = 10^4$ (right panel)

Simulation results



$K_1 = 10^5$ with correlation 0.5 (left panel) and 0 (right panel) in place of 0.9

Simulation results

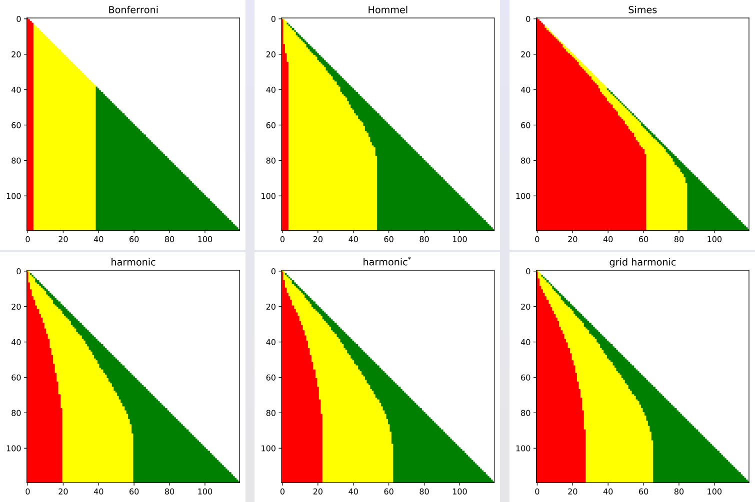
- GWGS discovery matrix Genovese/Wasserman'04; Goeman/Solari'11
- $\mathrm{DM}_{i,j}$: a p-value for testing "there are less than j true discoveries among the i rejected hypotheses"
- $\mathcal{N} \subseteq [K]$: nulls
- Jointly validity: for each $\alpha \in (0,1)$,

$$\mathbb{P}\left(\exists (i,j) \in D_\alpha : \#(R_i \setminus \mathcal{N}) < j\right) \leqslant \alpha$$

where R_i is the set of i hypotheses with smallest p-values and

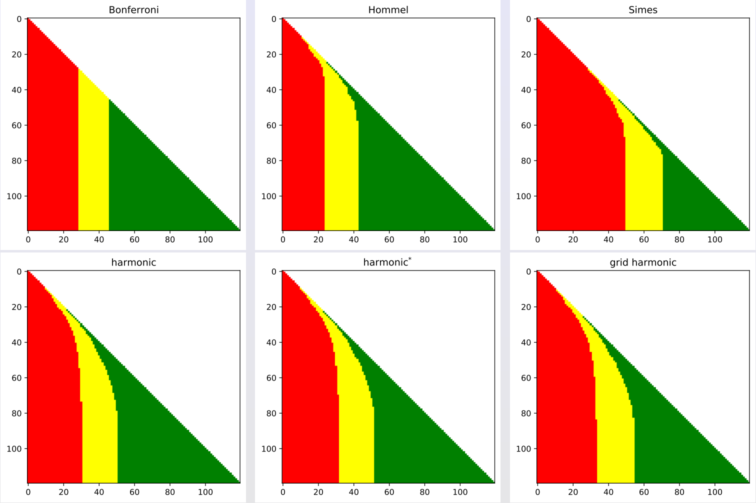
$$D_\alpha = \{(i,j) : \mathrm{DM}_{i,j} \leqslant \alpha\}$$

Simulation results



GWGS discovery matrices with correlation 0.9 and significance levels 1% and 5%

Simulation results



GWGS discovery matrices with correlation 0.5 and significance levels 1% and 5%