Risk Aggregation and Fréchet Problems

Part I - Basic concepts, Applications and Examples

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About this minicourse

Instructor: Ruodu Wang

Email: `wang@uwaterloo.ca`

Lectures: 14:30 - 17:30

Nov 9, 10, 11

Notes: blackboard (details)

slides (skeleton)

Website: `http://sas.uwaterloo.ca/~wang`

(slides are available on my website)

Scope of the course

In one sentence:

We study the problem of uncertain dependence in a multivariate model.

Preliminaries

- Knowledge on (undergraduate level) probability theory and mathematical statistics is necessary. Some knowledge on copulas and multivariate models is helpful.
- Some knowledge on (undergraduate level) stochastic processes, finance, and quantitative risk management is helpful but not necessary.

Features of the field

Some features of the field

- easily accessible to graduate students, and even high school students
- practically relevant in risk management
- naturally connected to other fields of finance, statistics, decision making, probability, combinatorics, operations research, numerical calculation, and so on
- a lot of fun

Aim of the course

Aim of the course is to

- understand Fréchet problems, mostly in its particular form of dependence uncertainty in risk aggregation
- understand their relevance in Quantitative Risk Management
- see some nice mathematical results
- see basic techniques in the field, especially some non-standard probabilistic and combinatorial techniques
- enjoy the beauty but not be buried in details
- discuss some open questions in the field

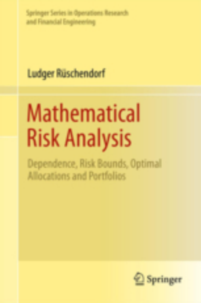
Content

Structure of the course

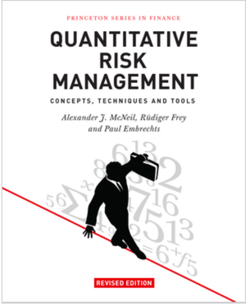
1. Basic concepts, applications and examples
2. Preliminaries and basic results
3. Complete and joint mixability
4. Uncertainty bounds for risk measures

Books relevant to this topic:

Rüschendorf (2013)



McNeil-Frey-Embrechts (2015)



Setup

General setup

- An atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- n is a positive integer
- L^p, $p \in [0, \infty]$: the set of random variables in $(\Omega, \mathcal{F}, \mathbb{P})$, taking values in $\mathbb{R}$, with finite p-th moment
- $\mathcal{X}$: a "suitable" subset of L^0, typically L^∞ or L^1

Notation

Some notation

- $\mathcal{M}_n$: the set of n-variate distributions (cdf)
- $\mathcal{M}_1^p$, $p \in [0, \infty]$: the set of univariate distributions with finite p-th moment
- $X \sim F$ means $X \in L^0$, $F \in \mathcal{M}_1$ and the distribution of X is F
- $X \overset{\mathrm{d}}{=} Y$ means $X, Y \in L^0$ and they have the same distribution
- $X \perp Y$ means $X, Y \in L^0$ and they are independent
- For any monotone (always in the non-strict sense) function $f : \mathbb{R} \to \mathbb{R}$, $f^{-1}(t) := \inf\{x \in \mathbb{R} : f(x) \geq t\}$.
- Convention: $X_i \sim F_i$, $i = 1, \ldots, n$. We frequently use $X_1, \ldots, X_n$ without specifying who they really are

Fréchet problem

What is a Fréchet problem?

For $F_1, \ldots, F_n \in \mathcal{M}_1$, a Fréchet class is defined as

$$\mathcal{M}_n(F_1, \ldots, F_n) = \{F \in \mathcal{M}_n : F \text{ has margins } F_1, \ldots, F_n\}$$

(introduced by Dall'Aglio, 1956).

Classic Fréchet problem

Given $F_1, F_2 \in \mathcal{M}_1$ and $G \in \mathcal{M}_2$, does there exist $F \in \mathcal{M}_2(F_1, F_2)$ such that $F \leq G$?

Answer (we will see this later) was given in Fréchet (1951), and it only works for $n = 2$

Fréchet problem

Pioneer papers: Fréchet (1951), Hoeffding (1940)

Maurice R. Fréchet

(1878 - 1973)

Wassily Hoeffding

(1914 - 1991)

Fréchet problem

(Modern) Fréchet problem

Any questions of the following type: for given $F_1, \dots, F_n \in \mathcal{M}_1$, determine

$$\sup\{\gamma(F) : F \in \mathcal{M}_n(F_1, \dots, F_n)\}$$

where $\gamma : \mathcal{M}_n \to \mathbb{R}$ is some functional, is called a Fréchet problem in this course.

- $\gamma(F) = \mathrm{I}_{\{F \le G\}}$ gives the classic Fréchet problem

Handling the Fréchet problem

Many Fréchet problems have the following form: for some $f : \mathbb{R}^n \to \mathbb{R}$, determine

$$\sup \left\{ \int f \mathrm{d}F : F \in \mathcal{M}_n(F_1, \ldots, F_n) \right\}.$$

The brutal way of handling this problem is to

(i) write down its dual (cf. Strassen 1965)

$$\inf \left\{ \sum_{i=1}^n \int f_i \mathrm{d}F_i : f_i \in L^1(F_i),\ i = 1, \ldots, n,\ \ \oplus (f_1, \ldots, f_n) \geq f \right\}$$

where $\oplus(f_1, \ldots, f_n) : (x_1, \ldots, x_n) \mapsto \sum_{i=1}^n f_i(x_i)$

(ii) show that the dual is equal to the primal (typically OK)

(iii) numerically solve the dual (semi-infinite linear programming)

The brutal method

- is typically very difficult or impossible even for modern computational techniques
- cannot answer questions like compatibility
- does not give good visualization
- cannot be easily communicated to students, statisticians or industry

In this course

- we try to avoid linear programming
- we try to work with the primal whenever possible: try to understand the dependence
- we aim for analytical solutions

Our story

Our main object is

$$S_n = \Lambda(X_1, \ldots, X_n)$$

where $\Lambda : \mathbb{R}^n \to \mathbb{R}$ is an aggregation function.

- We mainly look at the case of Λ being the sum.

Risk aggregation

Two aspects of modeling and inference of a multivariate model: **marginal distribution** and **dependence structure**.

"*copula thinking*"

Margins vs Dependence

	data	accuracy	modeling	calculation
margins	rich	good	mature	easy
dependence	limited	poor	limited	heavy

Assumption throughout the course

certain margins, uncertain dependence.

- A common setup in operational risk

Dependence uncertainty

An immediate example: CDO in the subprime crisis

- Between 2003 and 2007, Wall Street issued almost $700 billion in CDOs that included mortgage-backed securities as collateral
- Senior CDO tranches were given high ratings by rating agencies on the grounds that mortgages were diversified by region and so "uncorrelated"
- By October triple-A tranches had started to fall
- CDOs made up over half ($542 billion) of the nearly trillion dollars in losses suffered by financial institutions from 2007 to early 2009

Dependence uncertainty

For example,

$$S_n = X_1 + \cdots + X_n.$$

X_i: individual risks; S_n: risk aggregation

- For a manager, X_i is the loss of a business line i
- For an investor, X_i is the loss of asset i in a portfolio
- For a regulator, X_i is the loss of firm i

Dependence uncertainty

Key question

What are possible distributions of S_n?

- In this course, aggregation always refers to the aggregation of random variables with unspecific dependence

Primary targets

For given $F_1, \dots, F_n$, define the set of aggregate risks

$$\mathcal{S}_n = \mathcal{S}_n(F_1, \dots, F_n) = \{X_1 + \cdots + X_n : X_i \sim F_i,\ i = 1, \dots, n\} \subset L^0.$$

and the set of aggregate distributions

$$\mathcal{D}_n = \mathcal{D}_n(F_1, \dots, F_n) = \{\text{cdf of } S : S \in \mathcal{S}_n(F_1, \dots, F_n)\} \subset \mathcal{M}_1.$$

First things to think about:

- Are $\mathcal{S}_n$ and $\mathcal{D}_n$ properly defined?
- Does $\mathcal{D}_n$ depend on the probability space we choose?
- Is the study of $\mathcal{D}_n$ mathematically meaningful?

We work with $\mathcal{D}_n$ instead of $\mathcal{M}_n$.

Aggregation sets

Some questions to ask:

- **(Compatibility)** For a given F, is $F \in \mathcal{D}_n$?
- **(Mimicking)** What is the best approximation in $\mathcal{D}_n$ to F? That is, find $G \in \mathcal{D}_n$ such that $d(F, G)$ is minimized for some metric d.
- **(Extreme values)** What is $\sup_{S \in \mathcal{S}_n} \rho(S)$ for some functional $\rho : \mathcal{X} \to \mathbb{R}$? $\leftarrow$ measurement of risk aggregation under uncertainty

Aggregation sets

First question to ask: what are the values of

$$\underline{\mathrm{P}}_s(\mathcal{D}_n) = \inf\{F(s) : F \in \mathcal{D}_n\}, \quad s \in \mathbb{R},$$

and

$$\overline{\mathrm{P}}_s(\mathcal{D}_n) = \sup\{F(s) : F \in \mathcal{D}_n\}, \quad s \in \mathbb{R}.$$

- Analytical expression generally unavailable

Risk aggregation

Particular relevant questions in Quantitative Risk Management

- Let $\rho : \mathcal{X} \to \mathbb{R}$ be a risk measure. For some $F_1, \dots, F_n$, $\mathcal{S}_n \subset \mathcal{X}$. Let

$$\overline{\rho}(\mathcal{S}_n) = \sup_{S \in \mathcal{S}_n} \rho(S) \quad \text{and} \quad \underline{\rho}(\mathcal{S}_n) = \inf_{S \in \mathcal{S}_n} \rho(S).$$

- $[\underline{\rho}(\mathcal{S}_n), \overline{\rho}(\mathcal{S}_n)]$ characterizes model uncertainty in the dependence with known marginal distributions.

Risk aggregation

Primary examples: $p \in (0,1)$, $X \sim F$.

Value-at-Risk (VaR)

$\mathrm{VaR}_p : L^0 \to \mathbb{R}$,

$$\mathrm{VaR}_p(X) = F^{-1}(p) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) \geq p\}.$$

Expected Shortfall (ES, or TVaR, CVaR, CTE, AVaR)

$\mathrm{ES}_p : L^0 \to (-\infty, \infty]$,

$$\mathrm{ES}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q(X)\mathrm{d}q \underset{(F \text{ cont.})}{=} \mathbb{E}\left[X | X > \mathrm{VaR}_p(X)\right].$$

Risk aggregation

For given $F_1, \ldots, F_n \in \mathcal{M}_1$ and $p \in (0,1)$, the four quantities

$$\underline{\mathrm{VaR}}_p(\mathcal{S}_n), \quad \overline{\mathrm{VaR}}_p(\mathcal{S}_n), \quad \underline{\mathrm{ES}}_p(\mathcal{S}_n), \quad \overline{\mathrm{ES}}_p(\mathcal{S}_n)$$

are our primary examples.

- $\overline{\mathrm{VaR}}_p(\mathcal{S}_n)$, $\underline{\mathrm{VaR}}_p(\mathcal{S}_n)$ and $\underline{\mathrm{ES}}_p(\mathcal{S}_n)$ are generally analytically unavailable
- $\overline{\mathrm{ES}}_p(\mathcal{S}_n)$ can be analytically calculated

Risk aggregation

The questions of $\underline{\mathrm{P}}_s(\mathcal{D}_n)$ and $\overline{\mathrm{VaR}}_p(\mathcal{S}_n)$:

- One should always keep the problem of finding

$$\overline{\mathrm{VaR}}_p(\mathcal{S}_n) = \sup\{\mathrm{VaR}_p(S) : S \in \mathcal{S}_n\}, \quad p \in (0,1)$$

  and

$$\underline{\mathrm{P}}_s(\mathcal{D}_n) = \inf\{F(s) : F \in \mathcal{D}_n\}, \quad s \in \mathbb{R}$$

  in mind throughout the course.

- The two quantities are inverse to each other; we primarily work with $\overline{\mathrm{VaR}}_p(\mathcal{S}_n)$ for some mathematical elegance

Other applications

Many applications and related problems

- Risk measurement under uncertainty ($\leftarrow$ our main problem)
- Simulation: variance reduction
- Model-independent option pricing
- (Multi-dimensional) Monge-Kantorovich optimal transportation
- Change of measure
- Decision making
- Assembly and scheduling[1]

Many natural questions are not related to statistical uncertainty of a joint model

[1] traditional problem in OR: e.g. Coffman-Yannakakis (1984 MOR)

Assembly and scheduling

Consider the bottleneck of a schedule:

- n steps to produce an equipment
- m workers specialized in each step (mn workers in total)
- produce m equipments simultaneously
- time needed for each worker is recorded in an $m \times n$ matrix
- target: minimize the time T of production of m equipments, $T = \max\{t_1, \ldots, t_m\}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \\ 5 & 5 & 5 \\ 6 & 6 & 6 \\ 7 & 7 & 7 \\ 8 & 8 & 8 \\ 9 & 9 & 9 \end{bmatrix}$$

What is the optimal arrangement of workers for each equipment?

Assembly and scheduling

Simple example: we are allowed to rotate each column.

1	1	1	3
2	2	2	6
3	3	3	9
4	4	4	12
5	5	5	15
6	6	6	18
7	7	7	21
8	8	8	24
9	9	9	27

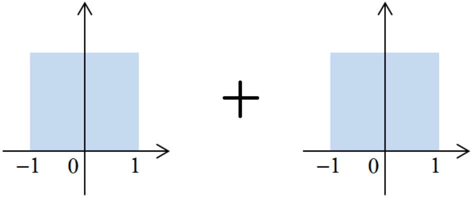
1	7	7	15
2	5	8	15
3	3	9	15
4	9	2	15
5	6	4	15
6	8	1	15
7	2	6	15
8	4	3	15
9	1	5	15

- If $t_1, \dots, t_n$ are identical, then the arrangement is optimal
- When is it possible to have identical $t_1, \dots, t_n$?
- How do we obtain this optimal arrangement?

A simple example

One simple example: $n = 2$, $F_1 = F_2 = \mathrm{U}[-1, 1]$.

What is a possible distribution of $S_2 = X_1 + X_2$?

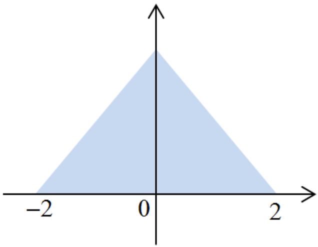


Obvious constraints

- $\mathbb{E}[S_2] = 0$
- range of S_2 in $[-2, 2]$
- $\mathrm{Var}(S_2) \leq 4/3$

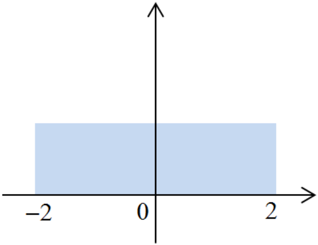
Uniform example I

Is the following distribution possible for S_2?



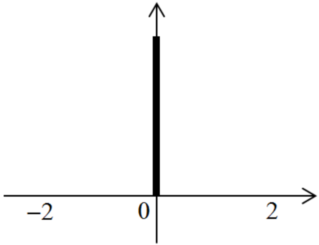
Uniform example II

Is the following distribution possible for S_2?

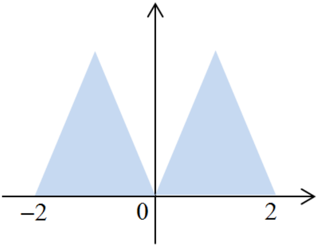


Uniform example III

Is the following distribution possible for S_2?

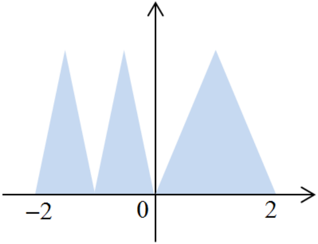


Is the following distribution possible for S_2?



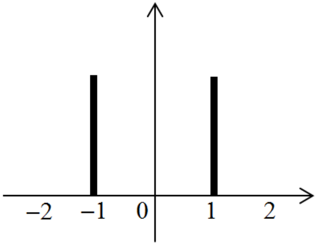
Uniform example V

Is the following distribution possible for S_2?



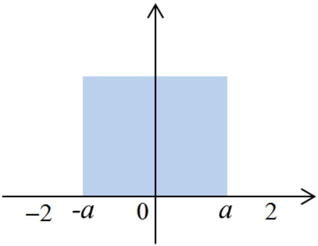
Uniform example VI

Is the following distribution possible for S_2?



Uniform example VII

Is the following distribution possible for S_2?

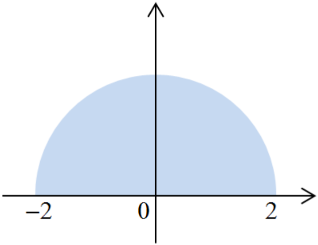


This it not trivial any more[2].

[2]the case $[-1, 1]$ obtained in Rüschendorf (1982); general case $[-a, a]$ obtained in Wang-W. (2015+)

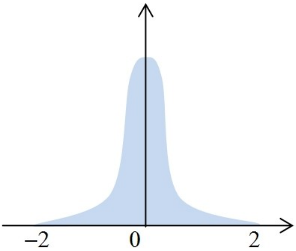
Uniform example VIII

Is the following distribution possible for S_2?



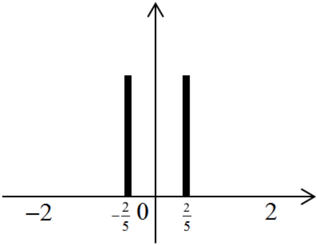
Uniform example IX

Is the following distribution possible for S_2?



Uniform example X

Is the following distribution possible for S_2?

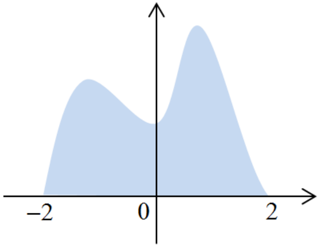


We will come back to this example later[3].

[3] This is essentially Example 3.3 of Mao and W. (2015)

Uniform example XI

Is the following distribution possible for S_2?



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