Risk Aggregation and Fréchet Problems

Part II - Preliminaries and Basic Results

Ruodu Wang

http://sas.uwaterloo.ca/~wang

Department of Statistics and Actuarial Science
University of Waterloo, Canada

Minicourse Lectures, University of Milano-Bicocca, Italy
November 9 - 11, 2015

Preliminaries

In the following we briefly give some preliminaries

- copulas
- Fréchet-Hoeffding inequalities
- comonotonicity and counter-monotonicity
- convex order

In this course we will try to avoid copulas as much as possible

Copulas

Lemma 1

For any $X \sim F$. There exists a $\mathrm{U}[0,1]$ random variable U_X such that $X = F^{-1}(U_X)$ a.s.

- Recall that $F^{-1}(t) = \mathrm{VaR}_t(X) = \inf\{x \in \mathbb{R} : F(x) \geq t\}$, $t \in (0,1)$.
- When F is continuous, one can take $U_X = F(X)$ which is a.s. unique.
- When F is not continuous, one can take a distributional transform as in Proposition 1.3 of Rüschendorf (2013).
- $\mathrm{I}_{\{U_X \leq F(x)\}} = \mathrm{I}_{\{F^{-1}(U_X) \leq x\}}$ a.s.

Copulas

Let $\mathcal{C}_n = \mathcal{M}_n(\mathrm{U}[0,1], \ldots, \mathrm{U}[0,1])$.

Definition 2

An n-variate copula is an element in $\mathcal{C}_n$.

Theorem 3 (Sklar's Theorem, Sklar 1959)

For $F_1, \ldots, F_n \in \mathcal{M}_1$, $F \in \mathcal{M}_n(F_1, \ldots, F_n)$ if and only if there exists $C \in \mathcal{C}_n$ such that

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n. \quad (1)$$

C in (1) is called a copula of any random vector $\mathbf{X} \sim F$.

- General reference on copulas: Joe (2014)

Fréchet-Hoeffding inequalities

Theorem 4 (Fréchet-Hoeffding inequalities*)

For any $C \in \mathcal{C}_n$, it holds that

$$\left(\sum_{i=1}^{n} x_i - (n-1)\right)_+ \leq C(x_1, \dots, x_n) \leq \min\{x_1, \dots, x_n\} \qquad (2)$$

for all $(x_1, \dots, x_n) \in \mathbb{R}^n$.

---

*the asterisk always indicates that details are (planned) to be given in the lecture

Classic Fréchet problem

Sharpness*

- $M_n : (x_1, \ldots, x_n) \mapsto \min\{x_1, \ldots, x_n\}$ is a copula for $n \in \mathbb{N}$
- $W_n : (x_1, \ldots, x_n) \mapsto (\sum_{i=1}^n x_i - (n-1))_+$ is a copula only for $n = 1, 2$
- (2) is point-wise sharp for all $n \in \mathbb{N}$

M_n is called the Fréchet upper copula and W_2 is called the Fréchet lower copula.

Classic Fréchet problem

Solution to the classic Fréchet problem*

Given $F_1, F_2 \in \mathcal{M}_1$ and $G \in \mathcal{M}_2$, there exist $F \in \mathcal{M}_2(F_1, F_2)$ such that $F \leq G$ if and only if

$$G(x_1, x_2) \geq F_1(x_1) + F_2(x_2) - 1, \quad \text{for all } (x_1, x_2) \in \mathbb{R}^2.$$

Comonotonicity

Definition 5

A pair of random variables $(X, Y) \in (L^0)^2$ is said to be comonotonic if there exists a random variable Z and two increasing functions f, g such that almost surely $X = f(Z)$ and $Y = g(Z)$.

- X and Y move in the same direction. This is a strongest (and simplest) notion of positive dependence.
- Two risks are not a hedge to each other if they are comonotonic
- We use $X /\!/ Y$ to represent that $(X, Y) \in (L^0)^2$ is comonotonic.

Comonotonicity

Some examples of comonotonic random vectors:

- a constant and any random variable
- X and X
- X and $\mathrm{I}_{\{X \geq 0\}}$
- In the Black-Scholes framework, the time-t price of a stock S and a call option on S

Note: in the definition of comonotonicity, the choice of $\mathbb{P}$ is irrelevant for equivalent probability measures.

- We also say "X and Y are comonotonic" when there is no confusion
- comonotonicity can be generalized to n-vectors

Comonotonicity

Theorem 6

For $X \sim F$, $Y \sim G$, the following are equivalent:

(i) $X /\!/ Y$;

(ii) *For some strictly increasing functions* f, g, $f(X) /\!/ g(Y)$;

(iii) $\mathbb{P}(X \le x, Y \le y) = \min\{F(x), G(y)\}$ *for all* $(x, y) \in \mathbb{R}^2$;

(iv) $(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \ge 0$ *for a.s.* $(\omega, \omega') \in \Omega \times \Omega$.

(v) *There exists* $U \sim \mathrm{U}[0,1]$ *such that* $X = F^{-1}(U)$ *and* $Y = G^{-1}(U)$ *almost surely.*

(vi) *A copula of* (X, Y) *is the Fréchet upper copula.*

In the following, the four random variables $X, Y, X', Y' \in L^2$ satisfy $X \stackrel{\mathrm{d}}{=} X'$ and $Y \stackrel{\mathrm{d}}{=} Y'$.

Proposition 7

Suppose $X /\!/ Y$. The following hold:

(i) $\mathbb{P}(X \le x, Y \le y) \ge \mathbb{P}(X' \le x, Y' \le y)$ *for all* $(x, y) \in \mathbb{R}^2$;

(ii) $\mathbb{E}[XY] \ge \mathbb{E}[X'Y']$;

(iii) $\mathrm{Corr}(X, Y) \ge \mathrm{Corr}(X', Y')$.

Comonotonicity

Let $F \oplus G$ be the distribution of $F^{-1}(U) + G^{-1}(U)$ for some $U \in \mathrm{U}[0,1]$.

Proposition 8

Suppose $X /\!/ Y$, $X \sim F$ and $Y \sim G$. Let H be the distribution of $X + Y$. Then

(i) $H = F \oplus G$;

(ii) $H^{-1} = F^{-1} + G^{-1}$;

(iii) $\mathrm{VaR}_p(X+Y) = \mathrm{VaR}_p(X) + \mathrm{VaR}_p(Y)$, $p \in (0,1)$;

(iv) $\mathrm{ES}_p(X+Y) = \mathrm{ES}_p(X) + \mathrm{ES}_p(Y)$, $p \in (0,1)$.

- VaR_p and ES_p are comonotonic additive.

Counter-monotonicity

Definition 9

A pair of random variables $(X, Y) \in (L^0)^2$ is said to be counter-monotonic if $(X, -Y)$ is comonotonic.

- We use $X \rightleftharpoons Y$ to represent that $(X, Y) \in (L^0)^2$ is counter-monotonic.
- Counter-monotonicity is not easy to generalize to n-vectors for $n \geq 3$.

Counter-monotonicity

Theorem 10

For $X \sim F$, $Y \sim G$, the following are equivalent:

(i) $X \rightleftharpoons Y$;

(ii) *For some strictly increasing functions f, g, $f(X) \rightleftharpoons g(Y)$;*

(iii) $\mathbb{P}(X \le x, Y \le y) = (F(x) + G(y) - 1)_+$ *for all* $(x, y) \in \mathbb{R}^2$;

(iv) $(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \le 0$ *for a.s.* $(\omega, \omega') \in \Omega \times \Omega$.

(v) *There exists $U \sim \mathrm{U}[0,1]$ such that $X = F^{-1}(U)$ and $Y = G^{-1}(1-U)$ almost surely.*

(vi) *A copula of (X, Y) is the Fréchet lower copula.*

Convex order

Definition 11 (Convex order)

For $X, Y \in L^1$, X is smaller than Y in (resp. increasing) convex order, denoted as $X \prec_{\mathrm{cx}} Y$ (resp. $X \prec_{\mathrm{icx}} Y$), if $\mathbb{E}[f(X)] \le \mathbb{E}[f(Y)]$ for all (resp. increasing) convex functions f such that the expectations exist.

- For (increasing) convex order, the choice of $\mathbb{P}$ is relevant.
- If $X \prec_{\mathrm{cx}} Y$ then $\mathbb{E}[X] = \mathbb{E}[Y]$.
- Increasing convex order is also called second-order stochastic dominance or stop-loss order
- We abuse the notation here: for $F, G \in \mathcal{M}_1^1$ and $X \in L^1$, we sometimes write $X \prec_{\mathrm{cx}} F$ and $G \prec_{\mathrm{cx}} F$

Convex order

- Increasing convex order describes a preference among risks for risk-averse investors
- a risk-averse investor prefers a risk with less variability (uncertainty) against one with larger variability, and she prefers a risk with a certainly smaller loss against a risk with a larger loss
- convex order and increasing convex order are based on the law of random variables

Some examples and properties (all random variables are in L^1):

- $X \prec_{\rm cx} Y$ implies $X \prec_{\rm icx} Y$.
- $X \le Y$ a.s. implies $X \prec_{\rm icx} Y$.
- If $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, and $X = aY$, $a > 1$, then $Y \prec_{\rm cx} X$.
- If $X \prec_{\rm icx} Y$, $Y \prec_{\rm icx} Z$, then $X \prec_{\rm icx} Z$.
- If $X \prec_{\rm icx} Y$, then $f(X) \prec_{\rm icx} f(Y)$ for any increasing function f.
- $X \prec_{\rm cx} Y$ if and only if $X \prec_{\rm icx} Y$ and $\mathbb{E}[X] = \mathbb{E}[Y]$.

Reference: Shaked-Shanthikumar (2007)

Convex order

Theorem 12 (Martingale Theorem for convex order)

For $X, Y \in L^1$, $X \prec_{\mathrm{cx}} Y$ if and only if there exists $Z \stackrel{\mathrm{d}}{=} X$ such that $Z = \mathbb{E}[Y|Z]$ almost surely.

- $\mathbb{E}[Y|\mathcal{G}] \prec_{\mathrm{cx}} Y$ for any σ-field $\mathcal{G}$. In particular, $\mathbb{E}[Y] \prec_{\mathrm{cx}} Y$.

Theorem 13 (Separation Theorem)

For $X, Y \in L^1$, $X \prec_{\mathrm{icx}} Y$ if and only if there exists $Z \in L^0$ such that

$$X \leq Z \prec_{\mathrm{cx}} Y \quad \text{almost surely.}$$

Proposition 14

For $X, Y \in L^1$, the following are equivalent:

(i) $X \prec_{\mathrm{icx}} Y$;

(ii) $\mathrm{ES}_p(X) \le \mathrm{ES}_p(Y)$ *for all* $p \in (0,1)$;

(iii) $\mathbb{E}[(X-t)_+] \le \mathbb{E}[(Y-t)_+]$ *for all* $t \in \mathbb{R}$.

Convex order and comonotonicity

Theorem 15

Suppose that $X \overset{\mathrm{d}}{=} X' \in L^1$, $Y \overset{\mathrm{d}}{=} Y' \in L^1$.

(i) *If* $X \mathbin{/\!/} Y$, *then* $X' + Y' \prec_{\mathrm{cx}} X + Y$.

(ii) *If* $X \leftrightharpoons Y$, *then* $X + Y \prec_{\mathrm{cx}} X' + Y'$.

- The case of $n \geq 3$ is still true for (i) but for (ii) it becomes unclear
- A general version of the above theorem dates back to Lorentz (1951)
- More information: Puccetti-W. (2015)

Aggregation sets

For $F \in \mathcal{M}_1^1$, let $\mathcal{M}^*(F) = \{G \in \mathcal{M}_1 : G \prec_{\rm cx} F\}$ and $\mathcal{X}^*(F) = \{X \in L^0 : X \prec_{\rm cx} F\}$.

Proposition 16 (Basic properties*)

For $F_1, \ldots, F_n \in \mathcal{M}_1^1$, the following hold:

(i) $\mathcal{S}_n \subset \mathcal{X}^* \left(\oplus_{i=1}^n F_i\right)$;

(ii) $\mathcal{D}_n \subset \mathcal{M}^* \left(\oplus_{i=1}^n F_i\right)$;

(iii) *Both the sets $\mathcal{D}_n$ and $\mathcal{M}^* \left(\oplus_{i=1}^n F_i\right)$ are convex and closed with respect to convergence in distribution.*

Aggregation sets

Uniform example*

For $F_1 = F_2 = \mathrm{U}[-1,1]$, $\mathcal{D}_2 \subsetneq \mathcal{M}^*\left(\mathrm{U}[-2,2]\right)$.

- see Example X.

Bernoulli example*

For $F_1 = F_2 = \mathrm{Bern}(p)$, $p \in [0,1]$, we have $\mathcal{D}_2 = \mathcal{M}^*\left(\mathrm{Bern}(p) \oplus \mathrm{Bern}(p)\right) \cap R$ where R is the set of distributions supported in $\{0,1,2\}$.

Aggregation sets

Question

Does the equality $\mathcal{D}_n = \mathcal{M}^* \left(\oplus_{i=1}^{n} F_i \right)$ hold for some non-degenerate distributions?

Expected Shortfall bounds

Proposition 17 (Expected Shortfall naive bounds*)

For $F_1, \dots, F_n \in \mathcal{M}_1$, $X_i \sim F_i$, $i = 1, \dots, n$ and $p \in (0,1)$, the following hold:

(i) $\overline{\mathrm{ES}}_p(\mathcal{S}_n) = \sum_{i=1}^n \mathrm{ES}_p(X_i)$;

(ii) $\underline{\mathrm{ES}}_p(\mathcal{S}_n) \geq \sum_{i=1}^n \mathbb{E}[X_i]$.

- ES_p is comonotonic additive and preserves convex order

Value-at-Risk bounds

Proposition 18 (Value-at-Risk naive bounds*)

For $F_1, \dots, F_n \in \mathcal{M}_1$, $X_i \sim F_i$, $i = 1, \dots, n$ and $p \in (0,1)$, the following hold:

(i) $\sum_{i=1}^n \mathrm{VaR}_p(X_i) \le \overline{\mathrm{VaR}}_p(\mathcal{S}_n) \le \sum_{i=1}^n \mathrm{ES}_p(X_i)$;

(ii) $\sum_{i=1}^n \mathrm{VaR}_p(X_i) \ge \underline{\mathrm{VaR}}_p(\mathcal{S}_n) \ge -\sum_{i=1}^n \mathrm{ES}_{1-p}(-X_i)$.

- VaR_p is comonotonic additive but it does not preserve convex order

The problem of $\overline{\mathrm{VaR}}_p$ for $n = 2$

Theorem 19 ($\overline{\mathrm{VaR}}_p(\mathcal{S}_2)$ and $\underline{\mathrm{VaR}}_p(\mathcal{S}_2)$*)

For any $p \in (0,1)$ and $F_1, F_2 \in \mathcal{M}_1$ with F_1^{-1}, F_2^{-1} being continuous,

$$\overline{\mathrm{VaR}}_p(\mathcal{S}_2) = \inf_{x \in [0,1-p]} \{F_1^{-1}(p+x) + F_2^{-1}(1-x)\},$$

and

$$\underline{\mathrm{VaR}}_p(\mathcal{S}_2) = \sup_{x \in [0,p]} \{F_1^{-1}(x) + F_2^{-1}(p-x)\}.$$

- The dependence structure: a combination of comonotonicity and counter-monotonicity

---

* The result dates back to Makarov (1981) and Rüschendorf (1982); both studied $\underline{\mathrm{P}}_s(\mathcal{S}_2)$, the former based on construction and the latter based on duality.

The problem of $\overline{\mathrm{VaR}}_p$ for $n = 2$

Example:

- For $F_1 = F_2 = \mathrm{U}[0,1]$,

$$\overline{\mathrm{VaR}}_p(\mathcal{S}_2) = \overline{\mathrm{ES}}_p(\mathcal{S}_2) = 1 + p.$$

- For a concave distribution function $F_1 = F_2$ (decreasing density),

$$\overline{\mathrm{VaR}}_p(\mathcal{S}_2) = 2\mathrm{VaR}_{\frac{1+p}{2}}(X_1).$$

References I

- Joe, H. (2014). *Dependence Modeling with Copulas*. London: Chapman & Hall.
- Lorentz, G. G. (1953). An inequality for rearrangements. *American Mathematical Monthly*, **60**, 176–179.
- Makarov, G.D. (1981). Estimates for the distribution function of the sum of two random variables with given marginal distributions. *Theory of Probability and its Applications*, **26**, 803–806.
- Puccetti, G. and Wang R. (2015). Extremal dependence concepts. *Statistical Science*, forthcoming.
- Rüschendorf, L. (1982). Random variables with maximum sums. *Advances in Applied Probability*, **14**(3), 623–632.
- Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic orders*. Springer Series in Statistics.