Risk Aggregation and Fréchet Problems

Part III - Complete and Joint Mixability

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1 Definitions and examples

2 Properties

3 Necessary conditions

4 Sufficient conditions

5 An unexpected application

6 References

Aggregation sets

Observe that

$$S = X_1 + \cdots + X_n \Leftrightarrow X_1 + \cdots + X_n - S = 0$$

Hence,

$$F_S \in \mathcal{D}_n(F_1, \ldots, F_n) \Leftrightarrow \delta_0 \in \mathcal{D}_{n+1}(F_1, \ldots, F_n, F_{-S}).$$

To answer

is a distribution in $\mathcal{D}_n$, $n \geq 2$?

We study

is a point-mass in $\mathcal{D}_{n+1}$, $n \geq 2$?

Joint mixability

Joint mix

A random vector $(X_1, \ldots, X_n)$ is a joint mix if $X_1 + \cdots + X_n$ is a constant.

- Example: a multinomial random vector

Joint mixability

Definition 1 (Joint mixability)

An n-tuple of univariate distributions $(F_1, \dots, F_n)$ is jointly mixable (JM) if there exists a joint mix with marginal distributions $(F_1, \dots, F_n)$.

- The property concerns whether the n-tuple is able to support a joint mix.

definition given in W.-Peng-Yang (2013)

Remark 1 (Equivalent definitions)

An n-tuple of univariate distributions $(F_1, \dots, F_n)$ is JM if either

(i) there exists $F \in \mathcal{M}_n(F_1, \dots, F_n)$ supported in a hyperplane $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = K\}$ for some $K \in \mathbb{R}$, or

(ii) $\mathcal{D}_n(F_1, \dots, F_n)$ contains a point-mass.

- The above K is called a center of $(F_1, \dots, F_n)$.
- We write $\mathcal{J}_n(K)$, $K \in \mathbb{R}$ as the set of jointly mixable tuples with center K, and let $\mathcal{J}_n = \bigcup_{K \in \mathbb{R}} \mathcal{J}_n(K)$.

Joint mixability

Proposition 2 (Center of JM*)

Suppose that $F_1, \dots, F_n$ have finite means $\mu_1, \dots, \mu_n$ respectively, and $(F_1, \dots, F_n)$ is JM, then the center of $(F_1, \dots, F_n)$ is unique and it is $\sum_{i=1}^n \mu_i$.

Question

Is the center always unique? That is, are the sets $\mathcal{J}_n(K)$ disjoint for $K \in \mathbb{R}$?

Motivation

Reasons to study JM

- To understand and characterize $\mathcal{D}_n$
- A notion of extremal negative dependence
  - The safest dependence structure for random variables in $\mathcal{S}_n$; this leads to at least $\underline{\mathrm{ES}}_p(\mathcal{S}_n)$ and later we will see it also serves as a building block for $\overline{\mathrm{VaR}}_p(\mathcal{S}_n)$ and $\underline{\mathrm{VaR}}_p(\mathcal{S}_n)$
  - All the applications in Part I

History

Who first came with the idea of a constant sum[1]?

- Gaffke-Rüschendorf (1981) and Rüschendorf (1982)
  - the target was to study $\underline{\mathrm{P}}_n(\mathcal{D}_n)$
  - obtained analytical results for several $\mathrm{U}[0,1]$ distributions
- Knott-Smith (2006) - first version 1998
  - the target was variance reduction
  - obtained results for three radially symmetric distributions
- Rüschendorf-Uckelmann (2002)
  - the target was variance reduction
  - obtained analytical results for unimodal-symmetric distributions
- Müller-Stoyan (2002) book
  - the target was the safest dependence structure for risks
  - provided several examples

[1] the knowledge of W. is very limited

Definition 3 (Complete mixability)

We say a univariate distribution F is n-completely mixabe (n-CM) if exists an n-dimensional joint mix with identical marginal distributions F.

- Equivalently, $(F, \ldots, F) \in \mathcal{J}_n(n\mu)$ for some $\mu \in \mathbb{R}$.
- μ is called the center of F (uniqueness?). If the mean of F is finite, then it is equal to μ.
- We write $\mathcal{I}_n(\mu)$, $\mu \in \mathbb{R}$ as the set of completely mixable distributions with center μ, and let $\mathcal{I}_n = \bigcup_{\mu \in \mathbb{R}} \mathcal{I}_n(\mu)$.

definition given in Wang-W. (2011)

Examples

Examples:

- F is 1-CM if and only if F is the distribution of a constant.
- F is 2-CM if and only if F is symmetric, i.e. $X \sim F$ and $a - X \sim F$ for some constant a.
- An discrete uniform distribution on n points is n-CM.
- Suppose that $r = \frac{p}{q}$ is rational, $p, q \in \mathbb{N}$. The Bernoulli distribution $\mathrm{Bern}(r)$ is q-CM.

We say F is discrete uniform on $(a_1, \ldots, a_n) \in \mathbb{R}^n$ if

$$F(x) = \frac{1}{n} \sum_{i=1}^{n} \mathrm{I}_{\{a_i \le x\}}, \; x \in \mathbb{R}.$$

We write $F = D\{a_1, \ldots, a_n\}$.

Duality

Dual of mixability

$(F_1, \ldots, F_n) \in \mathcal{J}_n(K)$ if and only if for all measurable functions $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, n$ such that $\sum_{i=1}^n f_i(x_i) \geq \mathrm{I}_{\{x_1+\cdots+x_n=K\}}$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\sum_{i=1}^n \int f_i \mathrm{d}F_i \geq 1,$$

whenever the left-hand side of the above equation is finite.

- In this course we will not work with the dual.

Mixability

An open research area:

what distributions are CM/JM?

The research in this area is very much marginal-dependent - copula techniques do not help much!

recent summary paper: Puccetti-W. (2015)

1 Definitions and examples

2 Properties

3 Necessary conditions

4 Sufficient conditions

5 An unexpected application

6 References

Properties

- We focus on theoretical properties of CM; these for JM can be analogously formulated.
- In the following proposition F_X stands for the distribution of $X \in L^0$.

Properties

Proposition 4 (Basic properties*)

Take any $n \in \mathbb{N}$ and $\mu \in \mathbb{R}$.

(i) *For $a, b \in \mathbb{R}$, $F_X \in \mathcal{I}_n(\mu) \Rightarrow F_{aX+b} \in \mathcal{I}_n(a\mu + b)$.*

(ii) *$\mathcal{I}_n(\mu)$ is a convex set.*

(iii) *For any $k \in \mathbb{N}$, $\frac{n}{n+k}\mathcal{I}_n + \frac{k}{n+k}\mathcal{I}_k \subset \mathcal{I}_{n+k}$. In particular, $\mathcal{I}_n \subset \mathcal{I}_{nk}$.*

(iv) *Suppose $X \perp Y$ and $F_X, F_Y \in \mathcal{I}_n$. Then $F_{X+Y} \in \mathcal{I}_n$.*

(v) *$\mathcal{I}_n(\mu)$ and $\mathcal{I}_n$ are both closed under convergence in distribution.*

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mostly given in Wang-W. (2011)

similar properties hold for $\mathcal{D}_n$; see Remark 2.2 of Bernard-Jiang-W. (2014).

Example:

- Suppose that $r = \frac{p}{q}$ is rational, $p, q \in \mathbb{N}$. The binomial distribution $\mathrm{Bin}(n, r)$ is q-CM.

Decomposition

Theorem 5 (Decomposition Theorem*)

For $\mu \in \mathbb{R}$, a discrete distribution $F \in \mathcal{I}_n(\mu)$ if and only if it has a decomposition:

$$F = \sum_{i=1}^{\infty} b_i F_i,$$

where $\sum_{i=1}^{\infty} b_i = 1,\ b_i \geq 0,\ i \in \mathbb{N}$ and $F_i,\ i \in \mathbb{N}$ are n-discrete uniform distributions with mean μ.

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given in Puccetti-Wang-W. (2012); a stronger version is given in W. (2015)

1 Definitions and examples

2 Properties

3 Necessary conditions

4 Sufficient conditions

5 An unexpected application

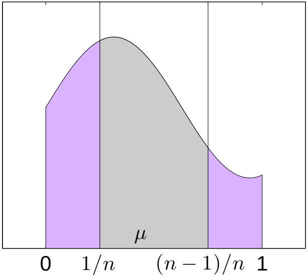
6 References

Mean condition

Proposition 6 (Mean condition for CM*)

Suppose that $F \in \mathcal{I}_n(\mu)$ and the essential support of F is $[a, b]$, $a, b \in \mathbb{R}$. Then

$$a + \frac{b-a}{n} \leq \mu \leq b - \frac{b-a}{n}. \tag{1}$$

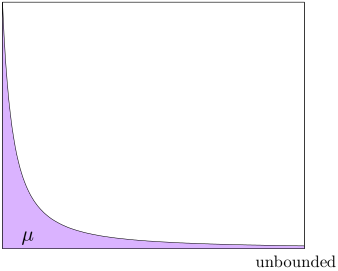


this condition was given in Wang-W. (2011)

Mean condition

Remark 2 (One-side unbounded distributions*)

If $b = \infty$ and $a > -\infty$, F cannot be n-CM.



Mean condition

For $i = 1, \ldots, n$, let μ_i, a_i, b_i be respectively the mean, essential infimum, and essential supremum of $X_i \sim F_i$, and $\ell = \max_{i=1,\ldots,n}\{b_i - a_i\}$.

Proposition 7 (Mean condition for JM)

If $(F_1, \ldots, F_n) \in \mathcal{J}_n$ *and* $\mu_i, a_i, b_i \in \mathbb{R}$ *for* $i = 1, \ldots, n$, *then*

$$\sum_{i=1}^{n} a_i + \ell \le \sum_{i=1}^{n} \mu_i \le \sum_{i=1}^{n} b_i - \ell \qquad (2)$$

- We can always scale and shift the distributions such that $\sum_{i=1}^n a_i = 0$ and $\sum_{i=1}^n b_i = 1$. In that case, (2) becomes

$$\ell \le \sum_{i=1}^{n} \mu_i \le 1 - \ell.$$

Norm inequality

Definition 8 (Pseudo-norm)

A pseudo-norm $||\cdot||$ is a map from L^0 to $[0,\infty]$, such that

(i) $||aX|| = |a|\cdot||X||$ for $a\in\mathbb{R}$ and $X\in L^0$;

(ii) $||X+Y||\le||X||+||Y||$ for $X,Y\in L^0$;

(iii) $||X||=0$ implies $X=0$ a.s.;

(iv) $||X||=||Y||$ if $X\overset{\mathrm{d}}{=}Y$, $X,Y\in L^0$.

- The L^p-norms, $p\in[1,\infty)$, and the L^∞-norm,

$$||\cdot||_p : L^0\to[0,\infty],\ X\mapsto(\mathbb{E}[|X|^p])^{1/p}$$

and

$$||\cdot||_\infty : L^0\to[0,\infty],\ X\mapsto\text{ess-sup}(|X|)$$

are pseudo-norms.

Norm inequality

Proposition 9 (Norm inequality*)

If $(F_1, \ldots, F_n) \in \mathcal{J}_n$ *and* $\mu_1, \ldots, \mu_n \in \mathbb{R}$, *then*

$$\sum_{i=1}^{n} ||X_i - \mu_i|| \geq 2 \max_{i=1,\ldots,n} ||X_i - \mu_i||,$$

where $X_i \sim F_i$, $i = 1, \ldots, n$ *and* $|| \cdot ||$ *is any pseudo-norm on* L^0.

- A polygon inequality

a stronger version is given in W. (2015)

Norm inequality

A special case of the norm inequality,

Variance condition

If $(F_1, \dots, F_n)$ is JM with finite variance $\sigma_1^2, \dots, \sigma_n^2$, then

$$\max_{i=1,\dots,n} \sigma_i \leq \frac{1}{2} \sum_{i=1}^{n} \sigma_i. \tag{3}$$

this trivial condition was first given in W.-Peng-Yang (2013)

1 Definitions and examples

2 Properties

3 Necessary conditions

4 Sufficient conditions

5 An unexpected application

6 References

Monotone densities

Theorem 10 (CM for monotone densities*)

Suppose that F admits a monotone density on its bounded essential support. Then F is n-CM if and only if the mean condition (1) *is satisfied.*

- In general, the mean condition is not sufficient
- The mean condition is weaker as n grows

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*given in Wang-W. (2011)

Monotone densities

Corollary 11 (CM for uniform distributions)

For any $a, b \in \mathbb{R}$, $a < b$, $\mathrm{U}[a, b]$ *is* n*-CM for* $n \geq 2$.

Example:

- The Beta distribution $\mathrm{Beta}(\alpha, \beta)$ with parameters $\alpha, \beta > 0$ where $(\alpha - 1)(\beta - 1) \leq 0$ has a monotone density. Thus it is n-CM for $\frac{1}{n} \leq \frac{\alpha}{\alpha+\beta} \leq \frac{n-1}{n}$.

the corollary is already given in Rüschendorf (1982)

Application: VaR bounds

Corollary 12 (VaR bounds for uniform distributions*)

Suppose $F_1 = \cdots = F_n = \mathrm{U}[0, a]$. *Then*

$$\overline{\mathrm{VaR}}_p(\mathcal{S}_n) = \overline{\mathrm{ES}}_p(\mathcal{S}_n) = \frac{na}{2}(1 + p).$$

- Again, a combination of comonotonicity and extremal negative dependence (cf Theorem 19. Part I); a coincidence, maybe?

Unimodal-symmetric densities

Theorem 13 (CM for unimodal-symmetric densities)

Suppose that F admits a unimodal-symmetric density. Then F is n-CM for $n \geq 2$.

Example:

- The normal distribution and the Cauchy distribution are n-CM for $n \geq 2$.

result given in Rüschendorf and Uckelmann (2002)

Concave densities

Theorem 14 (CM for concave densities)

Suppose that F admits a concave density on its essential support. Then F is n-CM for $n \geq 3$.

- The mean condition is precisely satisfied by the concavity.

Examples:

- The Beta distribution $\mathrm{Beta}(\alpha, \beta)$ with $1 \leq \alpha, \beta \leq 2$ is a typical distribution with a concave density. Thus it is n-CM for $n \geq 3$.
- Any triangular distribution has a concave density and hence it is n-CM for $n \geq 3$.

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result given in Puccett-Wang-W. (2012)

Positive densities

Theorem 15 (CM for positive densities)

A distribution on $[0,1]$ with density $p(x) \geq 3/n$, $x \in [0,1]$ is n-CM.

- $3/n$ cannot be lowered to $2/n$.

Corollary 16

A distribution on a finite interval with density $p(x) > \epsilon > 0$ is n-CM for sufficiently large n.

Question

Can we remove the condition $p(x) > \epsilon > 0$? ($p(x) > 0$ or $p(x) \geq 0$?)

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result given in Puccett-Wang-W. (2013)

Joint mixability

Theorem 17 (JM for monotone densities)

The mean condition (2) is sufficient for a tuple of distributions with increasing (decreasing) densities and bounded supports to be JM.

- This of course includes the previous result on CM for monotone densities, but the proof is much more complicated
- $(\mathrm{U}[0, a], \mathrm{U}[0, b], \mathrm{U}[0, c])$ is jointly mixable if and only if $\frac{1}{2}(a + b + c) \geq \max\{a, b, c\}$.

result given in Wang-W. (2015+)

Joint mixability

Theorem 18 (JM for symmetric distributions*)

The variance condition (3) is sufficient for the joint mixability of

(i) *a tuple of uniform distributions,*

(ii) *a tuple of marginal distributions of a multivariate elliptical distribution,*

(iii) *a tuple of distributions with unimodal-symmetric densities in the same location-scale family.*

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result given in Wang-W. (2015+)

Joint mixability

Theorem 19 (Sum of two uniform distributions*)

Suppose that F has a unimodal-symmetric density. For $a > 0$, $(\mathrm{U}[0,a], \mathrm{U}[0,a], F)$ is JM if and only if F is supported in an interval of length at most $2a$.

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result given in Wang-W. (2015+)

Joint mixability

Some remarks:

- Determination of JM is still open
- 12 open questions on mixability: W. (2015)
- Determination of JM in discrete setting is NP-complete[2].

[2] see Haus (2015)

1 Definitions and examples

2 Properties

3 Necessary conditions

4 Sufficient conditions

5 An unexpected application

6 References

An irrelevant question

Question

Can we use integer-valued decreasing densities to approximate an arbitrary decreasing density?

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this question was raised during collaborative research with J. Shen (Waterloo) and Y. Shen (Waterloo)

Density question

For $T \in (0,1)$, denote

$$E_T^M = \left\{ f : [0, T] \to \mathbb{N}_0 : f \text{ is decreasing and } \int_0^T f(x)\mathrm{d}x \le 1 \right\},$$

$$I_T^M = \overline{\mathrm{cx}(E_T^M)},$$

that is, (weak-) closed convex hull of E_T^M, and

$$A_T^M = \left\{ f : [0, T] \to \mathbb{R}_+ : f \text{ is decreasing and } \int_0^T f(x)\mathrm{d}x \le 1 \right\}.$$

Obviously $E_T^M \subset I_T^M \subset A_T^M$.

- When we take f in E_T^M, I_T^M or A_T^M, we treat f as a function on $\mathbb{R}$ taking value 0 on $\mathbb{R} \setminus [0, T]$.

Density question

The question is

- Is it $I_T^M = A_T^M$?
- If the above is not true, for $f \in A_T^M$, how can we determine whether f is in I_T^M? That is, to characterize I_T^M.

This question is purely analysis. It has barely anything to do with probability.

Density question

Proposition 20 (*)

For any $f \in A_T^M$, *let* $N = \lceil f(0) \rceil$, *and define the distribution functions*

$$F_i : \mathbb{R} \to [0,1], \quad x \mapsto \min\{(i - f(x))_+, 1\}\mathrm{I}_{\{x \geq 0\}}, \quad i = 1, \dots, N.$$

Then $f \in I_T^M$ *if* $(F_1, \dots, F_N)$ *is jointly mixable.*

Density question

Proposition 21 (*)

Suppose that $f \in A_T^M$ *is convex on* $[0, T]$ *and*

$$\sum_{i=0}^{N} f^{-1}(i) \leq \int_0^T f(x)\mathrm{d}x + f^{-1}(1).$$

Then $f \in I_T^M$.

- Non-trivial results in joint mixability!

Density question

Proposition 22 (*)

Suppose that $f \in A_T^M$ is linear on its essential support $[0, b]$ and $f(b) = 0$. Then $f \in I_T^M$.

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