Risk Aggregation and Fréchet Problems

Part IV - Uncertainty Bounds for Risk Measures

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Risk measures

A risk measure is a functional $\rho : \mathcal{X} \to [-\infty, \infty]$.

- $\mathcal{X}$ is a convex cone of random variables, $\mathcal{X} \supset L^\infty$.
- $\rho(L^\infty) \subset \mathbb{R}$
- $X \in \mathcal{X}$ represents loss/profit

A law-determined risk measure can be treated as a functional $\rho : \mathcal{D} \to [-\infty, \infty]$.

- $\mathcal{D}$ is the set of distributions of random variables in $\mathcal{X}$.

Risk measures

There are several properties of risk measures discussed in the literature, for instance:

- Monetary risk measure
  - Monotonicity: $\rho(X) \le \rho(Y)$ for $X \le Y,\ X, Y \in \mathcal{X}$
  - Cash-additivity: $\rho(X + c) = \rho(X) + c$ for $X \in \mathcal{X}$ and $c \in \mathbb{R}$
- Coherent risk measure: Monetary + two of the three:
  - Positive homogeneity: $\rho(\lambda X) = \lambda \rho(X)$ for $X \in \mathcal{X}$ and $\lambda \in \mathbb{R}_+$
  - Subadditivity: $\rho(X + Y) \le \rho(X) + \rho(Y)$ for $X, Y \in \mathcal{X}$
  - Convexity: $\rho(\lambda X + (1-\lambda)Y) \le \lambda \rho(X) + (1-\lambda)\rho(Y)$ for $X, Y \in \mathcal{X}$ and $\lambda \in [0, 1]$
- Comonotonic additivity: $\rho(X + Y) = \rho(X) + \rho(Y)$ for $X /\!/ Y,\ X, Y \in \mathcal{X}$

Distortion risk measures

Distortion risk measures

A distortion risk measure is defined as

$$\rho(X) = \int_{\mathbb{R}} x \mathrm{d}h(F_X(x)), \ X \in \mathcal{X}, \ X \sim F_X,$$

where h is an increasing function on $[0,1]$ with $h(0) = 0$ and $h(1) = 1$. h is called a distortion function.

- Yaari (1987):

  distortion risk measure $\Leftrightarrow$ law-determined and comonotonic additive monetary risk measure.

Distortion risk measures

Distortion risk measures

If one of h and F_X^{-1} is continuous, then via a change of variable,

$$\rho(X) = \int_0^1 \mathrm{VaR}_t(X)\mathrm{d}h(t), \ \ X \in \mathcal{X}.$$

- ES and VaR are special cases of distortion risk measures.

Left-tail-ES

Two distortion risk measures, Left-tail-ES (LES) and right-quantiles (VaR^*)[1]:

Left-tail-ES (LES)

$\mathrm{LES}_p : L^0 \to [-\infty, \infty)$,

$$\mathrm{LES}_p(X) = \frac{1}{p}\int_0^p \mathrm{VaR}_q(X)\mathrm{d}q = -\mathrm{ES}_{1-p}(-X), \quad p \in (0,1).$$

Right-quantile (VaR^*)

$\mathrm{VaR}^*_p : L^0 \to (-\infty, \infty)$,

$$\mathrm{VaR}^*_p(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) > p\}, \quad p \in (0,1).$$

[1] We introduce them only for mathematical reasons. LES is not to be implemented in financial regulation and VaR^*_p is indistinguishable from VaR_p in practice.

VaR and ES Bounds

As in Part I: For given $F_1, \dots, F_n \in \mathcal{M}_1$ and $p \in (0,1)$, the four quantities

$$\underline{\mathrm{VaR}}_p(\mathcal{S}_n), \quad \overline{\mathrm{VaR}}_p(\mathcal{S}_n), \quad \underline{\mathrm{ES}}_p(\mathcal{S}_n), \quad \overline{\mathrm{ES}}_p(\mathcal{S}_n)$$

are our primary targets.

- $\overline{\mathrm{ES}}_p(\mathcal{S}_n) = \sum_{i=1}^n \mathrm{ES}_p(X_i)$
- $\underline{\mathrm{LES}}_p(\mathcal{S}_n) = \sum_{i=1}^n \mathrm{LES}_p(X_i)$
- The others are generally open
- $\underline{\mathrm{LES}}_p$ and $\overline{\mathrm{LES}}_p$ are symmetric to ES

We assume the marginal distributions $F_1, \dots, F_n$ have finite means.

Basic ideas

Observation: ES_p preserves convex order.

Finding $\underline{\mathrm{ES}}_p(S_n)$

Search for a smallest element in $\mathcal{S}_n$ with respect to convex order, if it exists.

- If $(F_1, \ldots, F_n)$ is JM, then such an element is a constant.

VaR does not respect convex order: more tricky.

For $i = 1, \dots, n$ and $U \sim \mathrm{U}[0,1]$, let $F_i^{[p,1]}$ be the distribution of $F_i^{-1}(p + (1-p)U)$, and $F_i^{[0,p]}$ be the distribution of $F_i^{-1}(pU)$.

Lemma 1 (VaR bounds*)

For $p \in (0,1)$ and $F_1, \dots, F_n \in \mathcal{M}_1$,

$$\overline{\mathrm{VaR}}_p^*(\mathcal{S}_n) = \sup\{\text{ess-inf}\, S : S \in \mathcal{S}_n(F_1^{[p,1]}, \dots, F_n^{[p,1]})\},$$

and

$$\underline{\mathrm{VaR}}_p(\mathcal{S}_n) = \inf\{\text{ess-sup} S : S \in \mathcal{S}_n(F_1^{[0,p]}, \dots, F_n^{[0,p]})\}.$$

Basic ideas

Corollary 2 (VaR bounds*)

Suppose that T is the smallest element in $\mathcal{S}_n(F_1^{[p,1]}, \ldots, F^{[p,1]})$ with respect to convex order. Then $\overline{\mathrm{VaR}}_p^*(\mathcal{S}_n) = \text{ess-inf}\, T$.

Finding $\overline{\mathrm{VaR}}_p(\mathcal{S}_n)$

Search for a smallest element in $\mathcal{S}_n(F_1^{[p,1]}, \ldots, F^{[p,1]})$ with respect to convex order.

- $\overline{\mathrm{VaR}}_p^*(\mathcal{S}_n) = \overline{\mathrm{VaR}}_p(\mathcal{S}_n)$ if $F_1^{-1}, \ldots, F_n^{-1}$ are continuous at p.

Smallest element wrt convex order

Question

For general $F_1, \ldots, F_n \in \mathcal{M}_1$, does there always exist a smallest element wrt $\prec_{\mathrm{cx}}$ in $\mathcal{S}_n$ (or $\mathcal{D}_n$)?

- If $(F_1, \ldots, F_n)$ is JM, then there exists one
- The largest element wrt $\prec_{\mathrm{cx}}$ is always the comonotonic sum for any marginal distributions

Smallest element wrt convex order

Example*

Let $F_1 = D\{0, 3, 8\}$ $F_2 = D\{0, 6, 16\}$ and $F_3 = D\{0, 7, 13\}$.

Dependence (a)

$$\begin{pmatrix} X_1(\omega_1) & X_2(\omega_1) & X_3(\omega_1) \\ X_1(\omega_2) & X_2(\omega_2) & X_3(\omega_2) \\ X_1(\omega_3) & X_2(\omega_3) & X_3(\omega_3) \end{pmatrix} = \begin{pmatrix} 3 & 16 & 0 \\ 0 & 6 & 13 \\ 8 & 0 & 7 \end{pmatrix}$$

Dependence (b)

$$\begin{pmatrix} X_1(\omega_1) & X_2(\omega_1) & X_3(\omega_1) \\ X_1(\omega_2) & X_2(\omega_2) & X_3(\omega_2) \\ X_1(\omega_3) & X_2(\omega_3) & X_3(\omega_3) \end{pmatrix} = \begin{pmatrix} 0 & 16 & 0 \\ 3 & 0 & 13 \\ 8 & 6 & 7 \end{pmatrix}$$

Example 3.1 of Bernard-Jiang-W. (2014); this example was provided by Bin Wang

Smallest element wrt convex order

Some conclusions:

- $\mathcal{S}_n$ does not always admit a smallest element wrt $\prec_{\mathrm{cx}}$
- To search for a smallest element wrt $\prec_{\mathrm{cx}}$ might not be a viable solution

Homogeneous model with a decreasing density

Suppose that $F_1 = \cdots = F_n := F \in \mathcal{M}_1^1$ which has a decreasing density on its support. Define three quantities:

$$H(x) = (n-1)F^{-1}((n-1)x) + F^{-1}(1-x), \quad x \in \left[0, \frac{1}{n}\right],$$

$$a = \min\left\{c \in \left[0, \frac{1}{n}\right] : \int_c^{\frac{1}{n}} H(t)\mathrm{d}t \geq \left(\frac{1-nc}{n}\right) H(c)\right\},$$

and

$$D = \frac{n}{1-na}\int_a^{\frac{1}{n}} H(x)\mathrm{d}x = n\frac{\int_{(n-1)a}^{1-a} F^{-1}(y)\mathrm{d}y}{1-na}.$$

Note that if $a > 0$ then $D = H(a)$. Finally, let

$$T(u) = H(u/n)\mathrm{I}_{\{u \leq na\}} + D\mathrm{I}_{\{u > na\}}, \quad u \in [0,1].$$

Homogeneous model with a decreasing density

Theorem 3 (Homogeneous model with a decreasing density*)

Suppose that $F_1 = \cdots = F_n := F \in \mathcal{M}_1^1$ *which has a decreasing density on its support. Then*

(i) $T(U) \in \mathcal{S}_n$ *for some* $U \sim \mathrm{U}[0,1]$*;*

(ii) $T(U) \prec_{\mathrm{cx}} S$ *for all* $S \in \mathcal{S}_n$*.*

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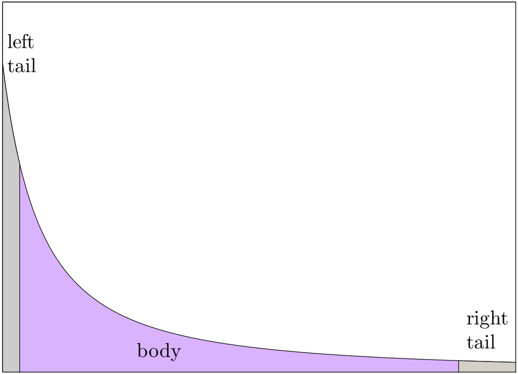
*This is a weaker version of Theorem 3.1 of Bernard-Jiang-W. (2014) which was essentially shown in Wang-W. (2011)

Homogeneous model with a decreasing density



The corresponding dependence structure:

- On $\{U \leq na\}$: almost mutual exclusivity
- On $\{U > na\}$: a joint mix

Homogeneous model with a decreasing density

General model with decreasing densities

Suppose that each of $F_1, \dots, F_n$ has a decreasing density. Then there exists an element T in $\mathcal{S}_n$ such that $T \prec_{\rm cx} S$ for all $S \in \mathcal{S}_n$.

- The distribution of T consists of a point-mass part and a continuous part, both of which can be calculated via a set of functional equations.
- The structure is very similar to the homogeneous model: an almost mutually exclusive part and a part of joint mix.

obtained in Jakobsons-Han-W. (2015+)

Summary of existing results

- Homogeneous model ($F_1 = \cdots = F_n$)
  - $\underline{\mathrm{ES}}_p(\mathcal{S}_n)$ solved analytically for decreasing densities, e.g. Pareto, Exponential
  - $\overline{\mathrm{VaR}}_p(\mathcal{S}_n)$ solved analytically for tail-decreasing densities, e.g. Pareto, Gamma, Log-normal
- Inhomogeneous model
  - Semi-analytical results are available for decreasing densities
- Numerical method: Rearrangement Algorithm (RA)[2]
- Real data analysis: DNB[3]

[2] Embrechts-Puccetti-Rüschendorf (2013)

[3] Aas-Puccetti (2014)

VaR bounds - homogeneous model

Theorem 4 (Sharp VaR bounds for homogeneous model*)

Suppose that $F_1 = \cdots = F_n := F \in \mathcal{M}_1^1$ *which has a decreasing density on* $[b, \infty)$ *for some* $b \in \mathbb{R}$. *Then, for* $p \in [F(b), 1)$ *and* $X \sim F$,

$$\overline{\mathrm{VaR}}_p(S_n) = n\mathbb{E}[X | X \in [F^{-1}(p + (n-1)c), F^{-1}(1-c)]],$$

where c *is the smallest number in* $[0, \frac{1}{n}(1-p)]$ *such that*

$$\int_{p+(n-1)c}^{1-c} F^{-1}(t)\mathrm{d}t \geq \frac{1-p-nc}{n}((n-1)F^{-1}(p+(n-1)c) + F^{-1}(1-c)).$$

- $c = 0$: $\overline{\mathrm{VaR}}_p(S_n) = \overline{\mathrm{ES}}_p(S_n)$.

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obtained in W.-Peng-Yang (2013)

VaR bounds - homogeneous model

Theorem 5 (Sharp VaR bounds for homogeneous model II*)

Suppose that $F_1 = \cdots = F_n := F \in \mathcal{M}_1^1$ *which has a decreasing density on its support. Then for* $p \in (0,1)$ *and* $X \sim F$,

$$\underline{\mathrm{VaR}}_p(S_n) = \max\{(n-1)F^{-1}(0) + F^{-1}(p), n\mathbb{E}[X|X \le F^{-1}(p)]\}.$$

obtained in Bernard-Jiang-W. (2014)

ES bounds - homogeneous model

Theorem 6 (Sharp ES bounds for homogeneous model*)

Suppose that $F_1 = \cdots = F_n := F \in \mathcal{M}_1^1$ *which has a decreasing density on its support. Then for* $p \in (1 - na, 1)$, $q = (1-p)/n$ *and* $X \sim F$,

$$\underline{\mathrm{ES}}_p(S_d) = \frac{1}{q}\int_0^q \left((n-1)F^{-1}((n-1)t) + F^{-1}(1-t)\right) \mathrm{d}t,$$

$$= (n-1)^2 \mathrm{LES}_{(n-1)q}(X) + \mathrm{ES}_{1-q}(X).$$

- One large outcome is coupled with $d-1$ small outcomes.

Rearrangement Algorithm (RA)[4]

- A fast numerical procedure
- Discretization of relevant quantile regions
- The idea is to approximate a $\prec_{\text{cx}}$-smallest element assuming one exists
- n possibly large
- Applicable to $\overline{\text{VaR}}_p$, $\underline{\text{VaR}}_p$ and $\underline{\text{ES}}_p$

[4] Puccetti-Rüschendorf (2012) and Embrecths-Puccetti-Rüschendorf (2013)

Numerical calculation

Example of RA borrowed from Marius Hofert:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 5 & 4 \\ 4 & 7 & 8 \end{pmatrix} \underset{\Sigma_{-1}=\begin{pmatrix} 2 \\ 5 \\ 9 \\ 15 \end{pmatrix}}{\Longrightarrow} \begin{pmatrix} 4 & 1 & 1 \\ 3 & 3 & 2 \\ 2 & 5 & 4 \\ 1 & 7 & 8 \end{pmatrix} \underset{\Sigma_{-2}=\begin{pmatrix} 5 \\ 5 \\ 6 \\ 9 \end{pmatrix}}{\Longrightarrow} \begin{pmatrix} 4 & 7 & 1 \\ 3 & 5 & 2 \\ 2 & 3 & 4 \\ 1 & 1 & 8 \end{pmatrix} \underset{\Sigma_{-3}=\begin{pmatrix} 11 \\ 8 \\ 5 \\ 2 \end{pmatrix}}{\Longrightarrow}$$

$$\begin{pmatrix} 4 & 7 & 1 \\ 3 & 5 & 2 \\ 2 & 3 & 4 \\ 1 & 1 & 8 \end{pmatrix} \underset{\Sigma_{-1}=\begin{pmatrix} 8 \\ 7 \\ 7 \\ 9 \end{pmatrix}}{\Longrightarrow} \begin{pmatrix} 2 & 7 & 1 \\ 4 & 5 & 2 \\ 3 & 3 & 4 \\ 1 & 1 & 8 \end{pmatrix} \underset{\Sigma_{-2}=\begin{pmatrix} 3 \\ 6 \\ 7 \\ 9 \end{pmatrix}}{\Longrightarrow} \begin{pmatrix} 2 & 7 & 1 \\ 4 & 5 & 2 \\ 3 & 3 & 4 \\ 1 & 1 & 8 \end{pmatrix} \underset{\Sigma_{-3}=\begin{pmatrix} 9 \\ 9 \\ 6 \\ 2 \end{pmatrix}}{\Longrightarrow}$$

$$\begin{pmatrix} 2 & 7 & 2 \\ 4 & 5 & 1 \\ 3 & 3 & 4 \\ 1 & 1 & 8 \end{pmatrix} \checkmark \underset{\Sigma=\begin{pmatrix} 11 \\ 10 \\ 10 \\ 10 \end{pmatrix}}{\Longrightarrow} \widehat{\mathrm{VaR}}_\alpha(L^+) \approx 10$$

Numerical calculation

Example of RA not working:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} \underset{\sum_{-1}=\left(\begin{smallmatrix}2\\4\\6\end{smallmatrix}\right)}{\Longrightarrow} \begin{pmatrix} 3 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix} \underset{\sum_{-2}=\left(\begin{smallmatrix}4\\4\\4\end{smallmatrix}\right)}{\Longrightarrow} \begin{pmatrix} 3 & 3 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 3 \end{pmatrix} \quad \checkmark$$

$$\underset{\sum=\left(\begin{smallmatrix}7\\6\\5\end{smallmatrix}\right)}{\Longrightarrow} \widehat{\mathrm{VaR}}_\alpha(L^+) \approx 5 < 6 \quad \underset{\sum=\left(\begin{smallmatrix}5\\6\\7\end{smallmatrix}\right)}{\Longrightarrow} \text{ for } \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

Asymptotic equivalence

Consider the case $n \to \infty$. What would happen to $\overline{\mathrm{VaR}}_p(\mathcal{S}_n)$?

- Clearly always $\overline{\mathrm{VaR}}_p(\mathcal{S}_n) \le \overline{\mathrm{ES}}_p(\mathcal{S}_n)$.
- Recall that $\overline{\mathrm{VaR}}_p(\mathcal{S}_n)$ has an ES-type part.

Under some weak conditions,

$$\lim_{n\to\infty} \frac{\overline{\mathrm{ES}}_p(\mathcal{S}_n)}{\overline{\mathrm{VaR}}_p(\mathcal{S}_n)} = 1.$$

- When arbitrary dependence is allowed, the worst-case VaR_p of a portfolio behaves like the worst-case ES_p

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This was shown first for homogeneous models and then extended to general inhomogeneous models. The first result is in Puccetti-Rüschendorf (2014).

Asymptotic equivalence - homogeneous model

Theorem 7 ($(\mathrm{VaR}_p, \mathrm{ES}_p)$-equivalence for homogeneous model*)

In the homogeneous model, $F_1 = F_2 = \cdots = F$, for $p \in (0,1)$ and $X \sim F$,

$$\lim_{n\to\infty} \frac{1}{n}\overline{\mathrm{VaR}}_p(\mathcal{S}_n) = \mathrm{ES}_p(X).$$

Corollary 3.7 in Wang-W. (2015)

Asymptotic equivalence - worst-cases

Theorem 8 (($\mathrm{VaR}_p, \mathrm{ES}_p$)-equivalence)

Suppose the continuous distributions F_i, $i \in \mathbb{N}$ satisfy that for $X_i \sim F_i$ and some $p \in (0,1)$,

(i) $\mathbb{E}[|X_i - \mathbb{E}[X_i]|^k]$ *is uniformly bounded for some* $k > 1$;

(ii) $\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathrm{ES}_p(X_i) > 0$.

Then as $n \to \infty$,

$$\frac{\overline{\mathrm{ES}}_p(\mathcal{S}_n)}{\overline{\mathrm{VaR}}_p(\mathcal{S}_n)} = 1 + O(n^{1/k-1}).$$

- $k = 1$ is not ok

Theorem 3.3 of Embrechts-Wang-W. (2015)

Asymptotic equivalence - best-cases

Similar results holds for $\underline{\mathrm{VaR}}_p$ and $\underline{\mathrm{ES}}_p$: assume (i) and

(iii) $\liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathrm{LES}_p(X_i) > 0$,

then

$$\lim_{n\to\infty} \frac{\underline{\mathrm{VaR}}_p(\mathcal{S}_n)}{\underline{\mathrm{LES}}_p(\mathcal{S}_n)} = 1,$$

$$\lim_{n\to\infty} \frac{\underline{\mathrm{ES}}_p(\mathcal{S}_n)}{\sum_{i=1}^n \mathbb{E}[X_i]} = 1,$$

and

$$\frac{\underline{\mathrm{VaR}}_p(\mathcal{S}_n)}{\underline{\mathrm{ES}}_p(\mathcal{S}_n)} \approx \frac{\sum_{i=1}^n \mathrm{LES}_p(X_i)}{\sum_{i=1}^n \mathbb{E}[X_i]} \le 1, \quad n \to \infty.$$

Example: Pareto(2) risks

Bounds on VaR and ES for the sum of n Pareto(2) distributed rvs for $p = 0.999$; VaR_p^+ corresponds to the comonotonic case.

	$n = 8$	$n = 56$
$\underline{\mathrm{VaR}}_p$	31	53
$\underline{\mathrm{ES}}_p$	178	472
VaR_p^+	245	1715
$\overline{\mathrm{VaR}}_p$	465	3454
$\overline{\mathrm{ES}}_p$	498	3486
$\overline{\mathrm{VaR}}_p/\mathrm{VaR}_p^+$	1.898	2.014
$\overline{\mathrm{ES}}_p/\overline{\mathrm{VaR}}_p$	1.071	1.009

Example: Pareto(θ) risks

Bounds on the VaR and ES for the sum of $n = 8$ Pareto(θ)-distributed rvs for $p = 0.999$.

	$\theta = 1.5$	$\theta = 2$	$\theta = 3$	$\theta = 5$	$\theta = 10$
$\overline{\mathrm{VaR}}_p$	1897	465	110	31.65	9.72
$\overline{\mathrm{ES}}_p$	2392	498	112	31.81	9.73
$\overline{\mathrm{ES}}_p/\overline{\mathrm{VaR}}_p$	1.261	1.071	1.018	1.005	1.001

General risk measures

Let $\mathcal{D}_n(F) = \mathcal{D}_n(F, \dots, F)$ (homogeneous model).

For a law-determined risk measure ρ, define

$$\Gamma_\rho(X) = \lim_{n\to\infty} \frac{1}{n} \sup \{\rho(S) : F_S \in \mathcal{D}_n(F_X)\} .$$

Γ_ρ is also a law-determined risk measure.

- $\Gamma_\rho \geq \rho$.
- If ρ is subadditive then $\Gamma_\rho = \rho$.

Aggregation of risk measures

Take $\mathcal{X} = L^\infty$.

Theorem 9 ((ρ_h, ρ_{h^*})-equivalence for homogeneous model)

We have

$$\Gamma_{\rho_h}(X) = \rho_{h^*}(X), \quad X \in \mathcal{X},$$

where h^* is the largest convex distortion function dominated by h.

Theorem 3.2 of W.-Bignozzi-Tsanakas (2015)

Aggregation of risk measures

For distortion risk measures

- $\Gamma_{\mathrm{VaR}_p} = \mathrm{ES}_p$
- ρ_h is coherent if and only if $h^* = h$

For law-determined convex risk measures.

- Γ_ρ is the smallest coherent risk measure dominating ρ
- If ρ is a convex shortfall risk measure, then Γ_ρ is a coherent expectile

Theorem 10 (Uncertainty spread)

Take $1 > q \geq p > 0$. Under weak regularity conditions, for inhomogeneous models,

$$\liminf_{n\to\infty} \frac{\overline{\mathrm{VaR}}_q(\mathcal{S}_n) - \underline{\mathrm{VaR}}_q(\mathcal{S}_n)}{\overline{\mathrm{ES}}_p(\mathcal{S}_n) - \underline{\mathrm{ES}}_p(\mathcal{S}_n)} \geq 1.$$

- The uncertainty-spread of VaR is generally bigger than that of ES.
- In recent Consultative Documents of the Basel Committee, $\mathrm{VaR}_{0.99}$ is compared with $\mathrm{ES}_{0.975}$: $p = 0.975$ and $q = 0.99$.

Theorem 4.1 of Embrechts-Wang-W. (2015)

Dependence-uncertainty spread

ES and VaR of $S_n = X_1 + \cdots + X_n$, where

- $X_i \sim \mathrm{Pareto}(2 + 0.1i),\ i = 1, \ldots, 5;$
- $X_i \sim \mathrm{Exp}(i - 5),\ i = 6, \ldots, 10;$
- $X_i \sim \mathrm{Log\text{–}Normal}(0, (0.1(i - 10))^2),\ i = 11, \ldots, 20.$

	$n = 5$			$n = 20$		
	best	worst	spread	best	worst	spread
$\mathrm{ES}_{0.975}$	22.48	44.88	22.40	29.15	102.35	73.20
$\mathrm{VaR}_{0.975}$	9.79	41.46	31.67	21.44	100.65	79.21
$\mathrm{VaR}_{0.99}$	12.96	62.01	49.05	22.29	136.30	114.01
$\frac{\overline{\mathrm{ES}}_{0.975}}{\overline{\mathrm{VaR}}_{0.975}}$		1.08			1.02	

Dependence-uncertainty spread

Features/Risk measure	VaR	Tail-VaR
Frequency captured?	Yes	Yes
Severity captured?	No	Yes
Sub-additive?	Not always	Always
Diversification captured?	Issues	Yes
Back-testing?	Straight-forward	Issues
Estimation?	Feasible	Issues with data limitation
Model uncertainty?	Sensitive to aggregation	Sensitive to tail modelling
Robustness I (with respect to "Lévy metric[33]")?	Almost, only minor issues	No
Robustness II (with respect to "Wasserstein metric[34]")?	Yes	Yes

From the International Association of Insurance Supervisors Consultation Document (December 2014).

Open questions

Concerete mathematical questions:

- Full characterization of $\mathcal{D}_n$ and mixability
- Existence and determination of smallest $\prec_{\mathrm{cx}}$-element in $\mathcal{D}_n$
- General analytical formulas for $\overline{\mathrm{VaR}}_p$ ($\underline{\mathrm{VaR}}_p$) and $\underline{\mathrm{ES}}_p$
- Aggregation of random vectors

Practical questions:

- Capital calculation under uncertainty
- Robust decision making under uncertainty
- Regulation with uncertainty

Other directions

Some on-going directions on RADU

- Partial information on dependence[5]
- Connection with Extreme Value Theory
- Connection with martingale optimal transportation
- Both marginal and dependence uncertainty
- Computational solutions
- Other aggregation functionals

[5] Bignozzi-Puccetti-Rüschendorf (2015), Bernard-Rüschendorf-Vanduffel (2015+), Bernard-Vanduffel (2015), many more

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Danke

Thank you for your kind attention