Short Course
Theory and Practice of Risk Measurement

Part 2
Axiomatic Theory of Monetary Risk Measures

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Part 2

- Monetary risk measures
- Acceptance sets and duality
- Coherent risk measures
- Convex risk measures
- Dual representation theorems

Axiomatic Theory of Risk Measures

In this part of the lectures, we study some properties of a "desirable" risk measure.

- Of course, desirability is very subjective. We stand mainly from a regulator's point of view to determine capital requirement for random losses.

Such properties are often called axioms in the literature. The main interest of study is

What characterizes the risk measures satisfying certain axioms?

- We assume $\mathcal{X} = L^\infty$ unless otherwise specified.
  - We allow random variables to take negative values - i.e. profits
- Recall that we use $(\Omega, \mathcal{F}, \mathbb{P})$ for an atomless probability space.

Monetary Risk Measures

Two basic properties

[CI] cash-invariance: $\rho(X + c) = \rho(X) + c$, $c \in \mathbb{R}$;

[M] monotonicity: $\rho(X) \leq \rho(Y)$ if $X \leq Y$.

These two properties are widely accepted.

- Here, risk-free interest rate is assumed to be 0 (or we can interpret everything as discounted).
- Recall that we treat a.s. equal random variables identical.

The property [CI]:

- By adding or subtracting a deterministic quantity c to a position leading to the loss X the capital requirement is altered by exactly that amount of c.
- Loss X with $\rho(X) > 0$: Adding the amount of capital $\rho(X)$ to the position leads to the adjusted loss $\tilde{X} = X - \rho(X)$, which is

$$\rho(\tilde{X}) = \rho(X - \rho(X)) = 0,$$

so that the position $\tilde{X}$ is acceptable without further injection of capital.

The property [M]:

- Positions that lead to higher losses with certainty require more risk capital.

Monetary Risk Measures

Definition

A monetary risk measure is a risk measure which is cash-invariant and monotone.

Monetary risk measures serve as the basic of any future study of risk measures.

Examples:

- VaR_p, $p \in (0,1)$ is monetary;
- ES_p, $p \in (0,1)$ is monetary;
- SD_b, $b > 0$ is cash-invariant, but not monotone*.

Lemma (*)

Any monetary risk measure is Lipschitz continuous with respect to $||\cdot||_\infty$:

$$|\rho(X)-\rho(Y)| \le ||X-Y||_\infty, \quad X, Y \in \mathcal{X}.$$

- L^∞ norm:

$$\begin{aligned} ||X||_\infty = \text{ess-sup}(|X|) &= \inf\{x \in \mathbb{R} : \mathbb{P}(|X| \le x) = 1\} \\ &= \mathrm{VaR}_1(|X|) = \mathrm{ES}_1(|X|). \end{aligned}$$

- $||\cdot||_\infty$-continuity is a basic property for all monetary risk measures

Acceptance Sets

Definition

The acceptance set of a risk measure ρ is defined as

$$\mathcal{A}_\rho := \{X \in \mathcal{X} : \rho(X) \le 0\}.$$

- Example: $\mathcal{A}_{\mathrm{VaR}_p} = \{X \in \mathcal{X} : \mathbb{P}(X \le 0) \ge p\}$.
- Example: $\mathcal{A}_{\mathrm{ES}_p} = \{X \in \mathcal{X} : \mathrm{ES}_p(X) \le 0\}$.
- Financial interpretation: the set of risks that are considered acceptable by a regulator or manager.
- A cash-invariant risk measure ρ is fully characterized by its acceptance set.

Proposition (*)

Suppose that ρ is a monetary risk measure with acceptance set $\mathcal{A}_\rho$. Then

(i) $\mathcal{A}_\rho$ *is not empty;*

(ii) $\mathcal{A}_\rho$ *is closed with respect to the L^∞ norm $||\cdot||_\infty$;*

(iii) $\sup\{m \in \mathbb{R} : m \in \mathcal{A}_\rho\} < \infty$;

(iv) $\mathcal{A}_\rho$ *is a lower-set: i.e.* $X \in \mathcal{A}_\rho$, $Y \in \mathcal{X}$ *and* $Y \leq X \implies Y \in \mathcal{A}_\rho$.

Theorem (Duality*)

Let $\mathcal{A}$ be any lower-subset of $\mathcal{X}$ containing at least a constant. Then

$$\rho_{\mathcal{A}}(X) := \inf\{m \in \mathbb{R} : X - m \in \mathcal{A}\}$$

is a monetary risk measure. Moreover,

(i) *for any monetary risk measure ρ,*

$$\rho(X) = \rho_{\mathcal{A}_\rho}(X);$$

(ii) *$\mathcal{A}$ is a subset of $\mathcal{A}_{\rho_{\mathcal{A}}}$, and $\mathcal{A} = \mathcal{A}_{\rho_{\mathcal{A}}}$ if and only if $\mathcal{A}$ is $||\cdot||_\infty$-closed.*

About the duality:

- First version established in Artzner, Delbaen, Eber and Heath (ADEH, 1999, MF) in finite (discrete) probability spaces.
- Financial interpretation: $\rho_{\mathcal{A}}(X)$ is the least amount of money required to make X acceptable.
- It is indeed to model a regulator's mindset: s/he only need to consider whether to accept a risk rather than to evaluate a risk.

Relation to Mathematical Finance

Instead of a zero-interest bond, one may think about a general security S with $S_0 = 1$. Let $\mathcal{A}$ be a lower-subset of $\mathcal{X}$, and T be the time horizon of risks at consideration.

A risk measure can be defined as

$$\rho_{\mathcal{A}}(X) = \inf\{m \in \mathbb{R} : X - mS_T \in \mathcal{A}\}.$$

Relation to Mathematical Finance

We may have multiple securities in a financial market.

A risk measure can be defined as

$$\rho_{\mathcal{A}}(X) = \inf\{m : X - \pi_T \in \mathcal{A},\ \pi \in \Pi,\ \pi_0 = m\}.$$

where Π is the set of admissible self-financing portfolios.

- Example: $\mathcal{A} = \{X \in \mathcal{X} : X \leq 0\ \ \mathbb{P}\text{-a.s.}\}$.
  - This means the regulator only accepts profit, not any loss.
  - $\rho_{\mathcal{A}}(X)$ is the superhedging price of X.
  - In a complete market, it is the arbitrage-free price of X.
  - If only a zero-interest bond is available (original setting), then $\rho_{\mathcal{A}}(X) = \text{ess-sup}(X) = \mathrm{VaR}_1(X)$.

Subadditivity and Positive Homogeneity

Two additional properties:

[PH] positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$, $\lambda \in (0, \infty)$;

[SA] subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

Simple fact

If a risk measure ρ is positive homogeneous, then $\rho(0) = 0$.

Coherent Risk Measures

Definition (Coherent risk measures)

A coherent risk measure is a risk measure which is cash-invariant, monotone, positive homogeneous, and subadditive.

- Representation theory established in ADEH (1999), in finite probability spaces.
- Established in Delbaen (2000) on general probability spaces and $\mathcal{X} = L^\infty$.
- Generalization studies on $\mathcal{X} = L^p$, $p \geq 1$, can be found in Kaina and Rüschendorf (2009), Filipović and Svindland (2012).
    - Recall that L^p is the set of random variables with finite p-th moment

Diversification

Subadditivity is the closely related to the idea of diversification benefit: putting different risks in a portfolio to reduce the total risk.

- Very common practice in finance
- Example: standard deviation (volatility)

Subadditivity

Subadditivity advantages:

- diversification benefit - "*a merger does not create extra risk*";
- regulatory arbitrage: divide X into $Y+Z$ if $\rho(X) > \rho(Y) + \rho(Z)$;
- capturing "the tail risk";
- consistency with risk preference (second order stochastic dominance);
- convex optimization and capital allocation.

Subadditivity

Subadditivity is contested from different perspectives:

- aggregation penalty - *convex risk measures*;
- statistical inference - *estimation/robustness/backtesting*;
- financial practice - "*a merger creates extra risk*";
- legal consideration - "*an institution has limited liability*".

Positive Homogeneity

Positive homogeneity advantages:

- A change in units of measurement (e.g. currency) should not result in a change in capital requirements.
- Subadditivity implies that, a for positive integer n,

$$\rho(nX) = \rho(X + \cdots + X) \leq n\rho(X).$$

  Since there is no diversification between the losses in this portfolio, it is natural to require the equality to hold, which leads to the property of positive homogeneity.
- Easy to implement: almost all practical risk measures are positive homogeneous.

Positive Homogeneity

Positive homogeneity criticism:

PH does not account for liquidity risk: [PH] does not acknowledge that very large portfolios of risks might produce very high losses that in turn can make it difficult for the holder of the portfolio to raise enough cash in order to meet his obligations.

Coherent Risk Measures

With subadditivity and positive homogeneity, one may reduce cash-invariance and monotonicity to the following standardization and relevance properties:

[ST] standardization: $\rho(c) = c$ for all $c \in \mathbb{R}$.

[R] relevance: $\rho(X) \leq 0$ for all $X \leq 0$.

Proposition (*)

If a risk measure ρ is subadditive and positive homogeneous, then [ST] is equivalent to [CI], and [R] is equivalent to [M].

- In some literature, a coherent risk measure is (equivalently) defined as a risk measure satisfying [ST], [R], [SA] and [PH].

ES and VaR: subadditivity

The following two lemmas will be used repeatedly in the future.

Lemma (*)

For any random variable X, denote its distribution by F. There exists a $\mathrm{U}[0,1]$ random variable U_X such that $X = F^{-1}(U_X)$ a.s.

- Recall that $F^{-1}(t) = \mathrm{VaR}_t(X) = \inf\{x \in \mathbb{R} : F(x) \geq t\}$, $t \in (0,1]$.
- When F is continuous, one can take $U_X = F(X)$.
- One has $\mathrm{I}_{\{U_X \leq F(x)\}} = \mathrm{I}_{\{F^{-1}(U_X) \leq x\}}$ a.s.

Throughout the lectures, for any random variable X, let U_X be $\mathrm{U}[0,1]$-distributed such that $X = F^{-1}(U_X)$ a.s.

ES and VaR: subadditivity

Lemma (*)

For any random variable X, $p \in [0,1]$ and any random variable B with $0 \le B \le 1$ and $\mathbb{E}[B] = 1 - p$, we have $\mathbb{E}[X\mathrm{I}_{\{U_X \ge p\}}] \ge \mathbb{E}[XB]$.

Theorem (Subadditivity of ES*)

On any choice of set of risks $\mathcal{X} \supset L^\infty$, and any $p \in (0,1)$, VaR_p is not subadditive and ES_p is subadditive.

- VaR will have serious issues with aggregation and diversification.

ES and VaR: subadditivity

Theorem (VaR is subadditive for elliptical risks)

Suppose that (X, Y) follows a two-dimensional elliptical distribution. Then for $p \in [1/2, 1)$,
$\mathrm{VaR}_p(X + Y) \leq \mathrm{VaR}_p(X) + \mathrm{VaR}_p(Y)$.

- Elliptical distributions include Normal and t- distributions
- Reference on elliptical distributions: Fang-Kotz-Ng 1990

Examples

- SD_b, $b > 0$ is positive homogeneous;
- $\mathrm{Var}(\cdot)$ is not positive homogeneous;
- $\mathbb{E}[\cdot]$ is positive homogeneous;
- $\text{ess-sup}(\cdot)$ is positive homogeneous;
- VaR_p, $p \in (0,1)$ is positive homogeneous;
- ES_p, $p \in (0,1)$ is positive homogeneous.

Examples

- SD_b, $b > 0$ is subadditive;
- $\mathrm{Var}(\cdot)$ is not subadditive;
- $\mathbb{E}[\cdot]$ is subadditive (in fact, it is linear);
- $\text{ess-sup}(\cdot)$ is subadditive;
- VaR_p, $p \in (0,1)$ is not subadditive;
- ES_p, $p \in (0,1)$ is subadditive.

Examples

- $\mathbb{E}[\cdot]$ is a coherent risk measure;
- $\text{ess-sup}(\cdot)$ is a coherent risk measure;
- ES_p, $p \in [0,1]$, is a coherent risk measure; it includes the above two special cases;
- VaR_p, $p \in (0,1)$ is not a coherent risk measure .

Convexity

Convexity is introduced to relax [SA] and [PH] which can reflect liquidity risk. It also represents diversification benefit.

[CX] Convexity: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$, $\lambda \in [0,1]$.

Proposition (*)

If $\rho(0) = 0$, then [CX] implies

$$\rho(tX) \geq t\rho(X) \quad \text{for } t \geq 1,$$

and

$$\rho(tX) \leq t\rho(X) \quad \text{for } t < 1.$$

This reflects aggregation penalty (liquidity risk).

Convexity

What is a proper diversification benefit? There are three basic ideas:

- To compare $\rho(X+Y)$ with $\rho(X)+\rho(Y)$: this leads to subadditivity
- To compare $\rho(\lambda X+(1-\lambda)Y)$ with $\lambda\rho(X)+(1-\lambda)\rho(Y)$: this leads to convexity
- To compare $\rho(X+Y)$ with $\rho(X^c+Y^c)$ where X^c and Y^c are "non-diversified version of X and Y": this leads to convex-order consistency (we will study this later)

Interesting connection:

Proposition (*)

If a monetary risk measure satisfies two of [CX], [PH] and [SA], then it satisfies all of them.

- This result dates back to Deprez and Gerber (1985)
- Coherent risk measures are convex.

Definition (Convex risk measures)

A convex risk measure is a risk measure which is cash-invariant, monotone, and convex.

- Representation theory established in Föllmer and Schied (2002, FS) and Fritteli and Rossaza Gianin (2002, JBF). Both papers acknowledged that the idea came from a talk given by David Heath in 2000.
- More examples of coherent and convex risk measures will be studied in Part III of the lectures.

Examples

- SD_b, $b > 0$ is convex;
- $\mathrm{Var}(\cdot)$ is convex*;
- $\mathbb{E}[\cdot]$ is convex;
- $\text{ess-sup}(\cdot)$ is convex;
- VaR_p, $p \in (0,1)$ is not convex;
- ES_p, $p \in (0,1)$ is convex.

Simple fact

If ρ is a convex risk measure, then $\hat{\rho}(\cdot) = \rho(\cdot) - c$ for some $c \in \mathbb{R}$ is also a convex risk measure. In particular, $\hat{\rho}(\cdot) = \rho(\cdot) - \rho(0)$ is a risk measure satisfying $\hat{\rho}(0) = 0$.

- With out loss of generality, we can assume that $\rho(0) = 0$ when we study convex risk measures.

Acceptance Sets of Coherent and Convex Risk Measures

Theorem: ADEH, 1999*

A monetary risk measure is coherent if and only if its acceptance set is a convex cone.

Theorem: Föllmer and Schied, 2002*

A monetary risk measure is convex if and only if its acceptance set is convex.

In the next we will look at representation theorems of risk measures:

Given that a risk measure ρ satisfies some properties, what should ρ look like?

Linearity and Sublinearity

We review some basic facts about linearity and sublinearity.

- A functional ρ is sublinear if it satisfies [SA] and [PH].
- A functional ρ is linear if it satisfies $\rho(X+Y) = \rho(X) + \rho(Y)$ and [PH].

Linearity is too restrictive for risk measures:

- we are not promoting the use of a linear risk measure - it serves as a technical tool to understand coherent risk measures
- in the next we will see that basically only expectations satisfy linearity: $\mathbb{E}^Q[\cdot]$ for some (finitely additive measure) Q

Finitely Additive Measures

Recall some definitions:

- A set function $Q : \mathcal{F} \to [0, \infty)$ is a finitely additive measure if
  - (i) $Q(\varnothing) = 0$
  - (ii) for all finite collection $\{E_i\}_{i=1,\dots,n}$ of disjoint sets, it holds that

$$Q\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} Q(E_i).$$

- A finitely additive measure Q is a probability measure if
  - (iii) $Q(\Omega) = 1$
  - (iv) for all countable collection $\{E_i\}_{i\in\mathbb{N}}$ of disjoint sets, it holds that

$$Q\left(\bigcup_{i\in\mathbb{N}} E_i\right) = \sum_{i\in\mathbb{N}} Q(E_i).$$

Linearity and Sublinearity

Proposition (*)

If a monetary risk measure ρ on L^∞ is linear, then there exists a finitely additive measure Q absolutely continuous w.r.t. $\mathbb{P}$, $Q(\Omega) = 1$, such that

$$\rho(X) = \mathbb{E}^Q[X], \quad X \in \mathcal{X}.$$

- In most cases, one takes Q as a probability measure
- if Ω is finite, then $\mathcal{F}$ is finite, and hence Q is a probability measure

Finitely Additive Measures

Finitely additive measures that are not σ-additive are very rare

- explicit example of finitely additive measure on a σ-field is impossible to construct; see Lauwers (2010)
- an (implicit) example is given on page 507, Example A.53 of Föllmer and Schied (2011).

Linearity and Sublinearity

Simple fact

A monetary risk measure is linear if and only if its acceptance set is half-space.

Proposition (*)

- *A sublinear functional satisfies* $\rho(\lambda X) \geq \lambda\rho(X)$ *for* $\lambda < 0$, $X \in \mathcal{X}$.
- *A linear functional satisfies* $\rho(\lambda X) = \lambda\rho(X)$ *for* $\lambda \in \mathbb{R}$, $X \in \mathcal{X}$.
- *If* $\rho_\theta,\ \theta \in \Theta$ *are linear functionals, then* $\rho = \sup_{\theta\in\Theta}\rho_\theta$ *is a sublinear functional.*

Representation Theorem of Coherent Risk Measures

Now suppose that Ω is a finite set and $\mathcal{X}$ consists of all random variables in this probability space.

Theorem: ADEH, 1999; Huber, 1980

A risk ρ is a coherent risk measure if and only if it has the following representation:

$$\rho(X) = \sup_{Q \in \mathcal{R}} \mathbb{E}^Q[X], \ X \in \mathcal{X}$$

where $\mathcal{R}$ is a collection of probability measures absolutely continuous w.r.t. $\mathbb{P}$.

- The finiteness of Ω serves to unify finitely additive measures and probability measures.

Representation Theorem of Coherent Risk Measures

Now suppose Ω is general and $\mathcal{X} = L^\infty$ (throughout the rest of this lecture). Let $\mathcal{M}$ be the set of finitely additive measures $Q : \mathcal{F} \to [0,1]$ with $Q(\Omega) = 1$, $Q \in \mathcal{M}$.

Theorem: Delbaen, 2000*

A coherent risk measure ρ has the following representation:

$$\rho(X) = \sup_{Q \in \mathcal{R}} \mathbb{E}^Q[X], \ X \in \mathcal{X}$$

where $\mathcal{R}$ is a subset of $\mathcal{M}$.

- $\mathcal{M} = \{Q \in \mathbf{Ba} : Q(\Omega) = 1, \ Q \geq 0\}$, where **Ba** is the dual space of L^∞ (equipped with the norm $||\cdot||_\infty$)
- The same form appears in non-linear expectations: see Peng (2010).

The proof of the previous theorem involves the very well-known Hahn-Banach Theorem, which has many versions. We use the following version on real linear space.

Hahn-Banach Theorem

Let V be a real linear space and U be a subspace of V. Suppose that $F : V \to \mathbb{R}$ is a sublinear functional, $g : U \to \mathbb{R}$ is a linear functional, and $g \leq F$ on U. Then there exists a linear functional $G : V \to \mathbb{R}$ such that $G \leq F$ on V and $G = g$ on U.

Representation of Expected Shortfall*

For $p \in (0,1)$,

$$\mathrm{ES}_p(X) = \sup_{Q \in \mathcal{R}} \mathbb{E}^Q[X], \ X \in \mathcal{X},$$

where $\mathcal{R} = \{Q \text{ is a probability measure} : \mathrm{d}Q/\mathrm{d}\mathbb{P} \le 1/(1-p)\}$.

Mean and Essential Supremum

Representations of Mean and Essential Supremum

The following representations hold:

$$\mathbb{E}[X] = \sup_{Q \in \{\mathbb{P}\}} \mathbb{E}^Q[X], \ X \in \mathcal{X},$$

and

$$\text{ess-sup}(X) = \sup_{Q \in \mathcal{R}} \mathbb{E}^Q[X], \ X \in \mathcal{X},$$

where

$\mathcal{R} = \{$all probability measures absolutely continuous w.r.t. $\mathbb{P}\}$.

- Both the mean and the essential supremum are special cases of ES ($p = 0$ and $p = 1$).

Representation Theorem of Coherent Risk Measures

To interpret the representation

$$\rho(X) = \sup_{Q \in \mathcal{R}} \mathbb{E}^Q[X], \; X \in \mathcal{X}:$$

- not only the distribution of X under $\mathbb{P}$ matters - the nature of the risk may not be described by its statistical performance
- $\mathcal{R}$ can be seen as a set of scenarios, called a generalized scenario - this is in line with risk management practice; for instance Chicago Mercantile Exchange (CME) uses a scenario-based approach
  - it provides complementary information to measures based on statistics of the loss distribution
  - choice of the set of scenarios is a complicated task

Representation Theorem of Coherent Risk Measures

Quoting ADEH (1999):

"Model risk can be taken into account by including in the set $\mathcal{R}$ a family of distributions for the future prices, possibly arising from other models."

Quoting ADEH (1999):

"It is important to distinguish between a point mass scenario and a simulation trial: the first is chosen by the investor or the supervisor, the second is chosen randomly according to a distribution they have prescribed beforehand."

Representation Theorem of Coherent Risk Measures

Quoting ADEH (1999):

"Any coherent risk measure appears therefore as given by a worst case method in a framework of generalized scenarios. At this point we emphasize that scenarios should be announced to all traders within the firm (by the manager) or to all firms (by the regulator). [...] As for the regulation case we allow ourselves to interpret a sentence from Stulz (1996): 'regulators like Value at Risk, because they can regulate it' as pointing to the formidable task of building and announcing a reasonable set of scenarios."

Representation Theorem of Coherent Risk Measures

The expectation under the worst scenario becomes the risk measure

- $\mathbb{E}[\cdot]$ only concerns the actual probability measure $\mathbb{P}$
- $\operatorname{ess-sup}(\cdot)$ concerns the all possibility measures
- ES_p, $p \in (0,1)$ concerns the scenarios which are not "too unrealistic" with respect to $\mathbb{P}$

Uncertainty at two levels

- epistemological uncertainty: what if $\mathbb{P}$ is inaccurate? addressed
- ontological uncertainty: what if $\mathbb{P}$ is wrong? not addressed

Remark

The supremum over supremum is still a supremum: for instance, one may think about

$$\rho(X) = \sup_{Q \in \mathcal{R}} \mathrm{ES}_p^Q(X)$$

where $\mathrm{ES}_p^Q(X)$ is the p-Expected Shortfall of X under Q. This ρ is still a coherent risk measure and admits the same type of dual representation.

In the representation

$$\rho(X) = \sup_{Q \in \mathcal{R}} \mathbb{E}^Q[X], \; X \in \mathcal{X},$$

one naturally asks "when can we make sure that $\mathcal{R}$ is as subset of the set of probability measures, like in the finite-Ω case?"

This requires some continuity on the risk measure ρ.

Continuity of Risk Measures

When one considers continuity of risk measures, one may look for continuity with respect to weak, $\mathbb{P}$- or a.s. convergence. However, this is not possible for convex risk measures.

- Toy example: Let $X_n = n^2 \mathrm{I}_{\{U \geq 1/n\}}$ for some U[0,1] random variable U. Then $X_n \to 0$ a.s. but X_n is getting more "dangerous" in many senses. If a risk measure preserves this kind of convergence, it might not be a good thing.

Proposition (*)

There is no coherent or convex risk measure that is continuous w.r.t. a.s. convergence in L^∞.

Fatou Property

The Fatou property is an alternative requirement of continuity/regularity for risk measures

Fatou property

(FP) Fatou property: If $X, X_1, X_2, \dots \in \mathcal{X} = L^\infty$, $\sup_{k\in\mathbb{N}} ||X_k||_\infty < \infty$ and $X_k \to X$ a.s., then

$$\liminf_{k\to\infty} \rho(X_k) \geq \rho(X).$$

- Think about Fatou's Lemma: if $\{X_k,\ k = 1, \dots\}$ is bounded below, then

$$\liminf_{k\to\infty} \mathbb{E}[X_k] \geq \mathbb{E}[\liminf_{k\to\infty} X].$$

Fatou Property

Lemma (*)

For a monetary risk measure ρ, the following statements are equivalent:

(i) ρ *has the Fatou property;*

(ii) ρ *is continuous from below (a.s. or $\mathbb{P}$ convergence):*

$$X_k \uparrow X \;\Rightarrow\; \rho(X_k) \uparrow \rho(X);$$

(iii) $\mathcal{A}_\rho$ *is closed under the weak* topology* $\sigma(L^\infty, L^1)$.

- The proof of part (iii) will not be covered in this lecture

Examples

- SD_b, $b > 0$ has the Fatou property;
- $\mathrm{Var}(\cdot)$ has the Fatou property;
- $\mathbb{E}[\cdot]$ has the Fatou property;
- $\text{ess-sup}(\cdot)$ has the Fatou property;
- VaR_p, $p \in (0,1)$ has the Fatou property;
- ES_p, $p \in (0,1)$ has the Fatou property.

Later on in this lecture, we will not verify the Fatou property, while all risk measures we will encounter have this property.

Representation Theorem of Coherent Risk Measures

The most popular result on coherent risk measures:

Theorem: Delbaen, 2000

A coherent risk measure ρ with the Fatou property has the following representation:

$$\rho(X) = \sup_{Q \in \mathcal{R}} \mathbb{E}^Q[X], \; X \in \mathcal{X}$$

where $\mathcal{R}$ is a collection of probability measures absolutely continuous w.r.t. $\mathbb{P}$.

- The proof is skipped in this lecture

Representation Theorem of Convex Risk Measures

On the representation of a coherent risk measure:

- all scenarios are considered equally, and then take the supremum
- in practice, scenarios are not equally likely; some should have less weights
- in economic decision theory (Maccheroni-Marinacci-Rustichini 2006 Econometrika), the study of a decision Y with ambiguity often concerns the robust utility

$$\inf_{Q\in\mathcal{P}}\{v^Q(Y)+\alpha(Q)\}$$

where v^Q is a utility function under scenario Q (often taken as $\mathbb{E}^Q[u(Y)]$ for a real utility function u), and α is a function measuring the likelihood of a scenario Q.

Representation Theorem of Convex Risk Measures

Let $\mathcal{P}$ be the set of probability measures absolutely continuous w.r.t. $\mathbb{P}$.

A natural idea is to distinguish different scenarios putting penalty on (un)likelihood or (un)desirability of the sceanrio.

- This leads to a risk measure of the following kind:

$$\rho(X) = \sup_{Q\in\mathcal{P}} \{\mathbb{E}^Q[X] - \alpha(Q)\}, \ X \in \mathcal{X},$$

  where $\alpha : \mathcal{P} \to (-\infty, \infty]$ is a function.

- It is easy to verify that the above ρ is a convex risk measure (with the Fatou property).

Representation Theorem of Convex Risk Measures

Theorem: Föllmer-Schied 2002; Frittelli-Rosazza Gianin 2002*

A convex risk measure ρ with the Fatou property has the following representation:

$$\rho(X) = \sup_{Q \in \mathcal{P}} \{\mathbb{E}^Q[X] - \alpha(Q)\}, \ X \in \mathcal{X}$$

where $\alpha : \mathcal{P} \to (-\infty, \infty]$ is called a penalty function.

- Interpretation: put a penalty on each scenario - robust generalized scenario.

Representation Theorem of Convex Risk Measures

The penalty function $\alpha : \mathcal{P} \to (-\infty, \infty]$ can be taken as the minimum penalty:

$$\alpha^{\min}(Q) = \sup\{\mathbb{E}^Q[X] - \rho(X) : X \in \mathcal{X}\}$$
$$= \sup\{\mathbb{E}^Q[X] : X \in \mathcal{A}_\rho\}.$$

($\alpha^{\min}$ is the *Fenchel-Legendre transform* of ρ.)

- For any α in the representation, it holds that $\alpha \geq \alpha^{\min}$.
- $\rho(0) = 0$ is equivalent to $\inf\{\alpha(Q) : Q \in \mathcal{P}\} = 0$.