CO481/CS467/PHYS467 Assignment 1

Due Sunday January 27, 2019, 3:00am

Instruction: Please submit your solutions to Crowdmark by the due date and time. Take special care to place the answer to each question in the right place.

Question 1. Exercise in Dirac notations (I) [8 marks]

We covered several representations of unitary operations in class.
For example, for σ_x, we have 3 representations:

- Action on a given basis: $|0\rangle \to |1\rangle$, $|1\rangle \to |0\rangle$
- Dirac notation: $|0\rangle\langle 1| + |1\rangle\langle 0|$
- Matrix representation: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

For a two-qubit (4-dim) system with basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, define two unitaries $U = \textsc{cnot}_{12}$ and $V = \textsc{swap}$ by their action on the basis:

- U: $|00\rangle \to |00\rangle$
  $|01\rangle \to |01\rangle$
  $|10\rangle \to |11\rangle$
  $|11\rangle \to |10\rangle$
- V: $|00\rangle \to |00\rangle$
  $|01\rangle \to |10\rangle$
  $|10\rangle \to |01\rangle$
  $|11\rangle \to |11\rangle$

(a) [2 mark] Write down the matrix representations and the Dirac notations for U and V.

(b) [3 mark] Consider the unitary $W = VUV^\dagger$, called $\textsc{cnot}_{21}$. Give all three representations for W.

(c) [3 mark] Consider the unitary on $\mathbb{C}^2$ call the Hadamard $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ given in the basis $\{|0\rangle, |1\rangle\}$.

What if H is applied to the first out of the two qubits? Give your answer as a unitary T in the matrix representation. What is UT (give the matrix representation and the action on the basis).

Question 2. Exercise in Dirac notations (II) – the transpose trick [5 marks]

Let $|\Phi\rangle = \frac{1}{\sqrt{d}}(|11\rangle + |22\rangle + \cdots + |dd\rangle)$. Show that, for any $d \times d$ matrix M, $(M \otimes I)|\Phi\rangle = (I \otimes M^T)|\Phi\rangle$ where M^T denotes the transpose of M in the computational basis $\{|i\rangle\}_{i=1}^d$.

Aside: $|\Phi\rangle$ is entangled over the two d-dim systems. The above question says that an operation M applied to the first system transform $|\Phi\rangle$ in the same way as a related operation M^T applied to the second system.

Question 3. No signalling in local measurements [10 marks]

Consider the two-qubit pure state $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$. The first qubit is held by Alice, the second qubit is held by Bob.

(a) [1 mark] Suppose a complete von Neumann measurement along the computational basis is applied to $|\psi\rangle$. What are the probabilities of the four possible outcomes?

(b) Instead, suppose Alice applies a computational basis measurement on her qubit on $|\psi\rangle$ (and Bob has not done anything yet).

(i) [1 mark] Write down the projectors for this measurement (as operators on the 4-dim Hilbert space).

(ii) [2 marks] What are Alice's probabilities of getting "0" and getting "1", and what are the respective *global* postmeasurement state?

(ii) [3 marks] *For each of the two measurement outcomes*, if Bob is then to measure along the computational basis, what are the conditional probabilities to obtain "0" and "1"? Infer from these answers the joint probabilities of the outcomes "00", "01", "10", "11". (Your answers for **(a)** and **(b)**(ii) should be consistent.)

(c) [1 mark] The case for Bob to measure his qubit first, followed by Alice measuring hers, can be analysed similarly to part **(b)**. But you do not need to repeat the similar analysis. You can already imply the conclusion from part **(b)**. Can you explain why (in just 1-2 sentences)?

(d) [2 marks] Briefly explain how to extend the answers for parts **(a)**-**(c)** to cover the case when the measurement bases of Alice and Bob are all arbitrary. (Hint: there is a very simple argument.)

Question 4. Quantum correlations without signalling. [10 marks]

(a) [2 marks] Let T be a hermitian operator (of general dimension d) with all eigenvalues being either 1 or -1 (with arbitrary multiplicities). Let $M_+ = \frac{I+T}{2}$, $M_- = \frac{I-T}{2}$. Show that $M_\pm$ are projectors onto the ± 1 eigenspaces of T. (You don't need to show that $M_\pm$ are projectors.)

(b) Let S_1 and S_2 be hermitian operators (of dimensions d_1 and d_2 respectively) with all eigenvalues being either 1 or -1.

(i) [1 mark] What are the eigenvalues of $S = S_1 \otimes S_2$?

(ii) [1 mark] If Alice measures the eigenvalue of S_1 in the first system, and Bob measures the eigenvalue of S_2 in the second system, write down the projectors that correspond to the four possible joint outcomes $(+1, +1)$, $(+1, -1)$, $(-1, +1)$, $(-1, -1)$.

(iii) [4 marks] Suppose the measurement is applied to a $+1$ eigenstate of S, i.e., the state $|\psi\rangle$ being measured satisfies $S|\psi\rangle = |\psi\rangle$, show that only two outcomes $(+1, +1)$ and $(-1, -1)$ have positive probabilities to occur. (So Alice's outcome and Bob's outcome always multiple to $+1$.)

(iv) [2 marks] Suppose the measurement is applied to a -1 eigenstate of S, i.e., the state $|\psi\rangle$ being measured satisfies $S|\psi\rangle = -|\psi\rangle$, show that only two outcomes $(+1, -1)$ and $(+1, -1)$ have positive probabilities to occur. (Note that in each case, the product of Alice's outcome and Bob's outcome is -1.)

Remark: We've derived that, when Alice measures S_1 and Bob measures S_2, the product of their outcomes is the same as if the operator $S = S_1 \otimes S_2$ is measured! Note that the actual measurement is *local* (Alice does her own and Bob does his own) with 4 outcomes, but the correlation in the measurement outcomes allows the outcome of a nonlocal measurement $S = S_1 \otimes S_2$ to be obtained. This is the underlying principle for non-local games (Bell inequalities), entanglement purification, quantum error correction codes, and quantum key distribution.