CO481/CS467/PHYS467 Assignment 4

Due March 24, 2019, 3:00am

Instruction: Please submit your solutions to Crowdmark by the due date and time. Take special care to place the answer to each question in the right place. Questions are ordered according to the sequence of topics covered in class, and not by difficulty. Also, if you do not prove an earlier part of a question, you can still use the earlier part to answer a later part.

Question 1. Solving the collision problem using Grover search [4 marks]

Recall that the quantum search algorithm can find a marked item in a search space of size N using $O(\sqrt{N/M})$ queries, where M is the total number of marked items.

In the collision problem, you are given a black-box function $f\colon \{1,2,\dots,N\} \to S$ (for some set S) with the promise that f is two-to-one. In other words, for any $x \in \{1,2,\dots,N\}$, there is a unique $x' \in \{1,2,\dots,N\}$ such that $x \neq x'$ and $f(x) = f(x')$. The goal of the problem is to find any such pair (x, x') (called a collision).

(a) [2 marks] For any $K \in \{1,2,\dots,N\}$, consider the following quantum algorithm for the collision problem:

1. Query $f(1), f(2), \dots, f(K)$.
2. If a collision is found, output it.
3. Otherwise, search for a value $x \in \{K+1, K+2, \dots, N\}$ such that $f(x) = f(x')$ for some $x' \in \{1,2,\dots,K\}$.

How many quantum queries does this algorithm need to make in order to find a collision? Your answer should depend on N and K, and can be expressed using big-O notation.

(b) [2 marks] Make a good choice of K and show that $O(N^{1/3})$ queries are sufficient.

Question 2. Detecting eavesdropping in a preliminary version of BB84 [7 marks]

In this preliminary version of BB84, Alice wants to send a uniformly random secret key bit $s \in \{0,1\}$ to Bob. To protect the secret, Alice picks a second uniformly random bit $b \in \{0,1\}$. She stores the secret in system A_1, and the basis information in system A_2. She prepares the state $H^b|s\rangle$ in system C and sends C through an *insecure* quantum channel hoping that it will arrive at Bob's home safely. (In other words, Alice encodes her secret s either in the $\{|0\rangle, |1\rangle\}$ basis or the $\{|+\rangle, |-\rangle\}$ basis depending on whether $b = 0$ or 1 respectively.)

(a) [2 marks] What is the density matrix for the state on $A_1 A_2 C$? You can leave your answer in Dirac notation and there is no need to expand $|\pm\rangle$ for this part.

(b) [1 mark] An eavesdropper Eve cuts the optical fibre (the quantum channel) and receives the quantum state in system C instead! Eve knows the protocol that Alice uses, but knows nothing about s or b. Let ρ_0 and ρ_1 be the density matrices on system C conditioned on $s = 0$ and 1 respectively. Write down ρ_0 and ρ_1. You can leave the answer in Dirac notation and in terms of $|0\rangle, |1\rangle, |\pm\rangle$.

(c) [2 marks] Eve wants to determine the secret s. This is a state discrimination problem, in which with probability $\frac{1}{2}$, $s = 0$ and Eve receives from Alice ρ_0, and with probability $\frac{1}{2}$, $s = 1$ and Eve receives from Alice ρ_1. Suppose Eve's measurement outcome is e, stored in system E. The Helstrom-Holevo Theorem (topic 4) can be extended to the discrimination of mixed states and you can assume without proof that the measurement that maximizes the probability for $e = s$ is given by the projectors:

$$P_0 = \tfrac{1}{2}\left(I + \tfrac{Z+X}{\sqrt{2}}\right) \quad\text{and}\quad P_1 = \tfrac{1}{2}\left(I - \tfrac{Z+X}{\sqrt{2}}\right).$$

What is the probability for $e = s$?

(d) [2 marks] Suppose Eve writes her measurement outcome on system E, and forwards the postmeasurement state (2-dim) in system C to Bob (without the measurement outcome). This is called an intercept-and-resend attack. Bob tells Alice he receives system C, and then Alice tells Bob the value of b. Bob measures in the $\{|0\rangle, |1\rangle\}$ basis if $b = 0$, and $\{|+\rangle, |-\rangle\}$ basis if $b = 1$. Let his outcome be t.

Conditioned on $s = 0$, $b = 0$, and $e = 0$, what is $\Pr(t \neq s)$?

Note that in the absence of Eve's attack, Bob's measurement outcome t should always be the secret s. As Eve learns (incomplete) information about the state in C, her measurement disturbs the state so that Bob's measurement no longer always gives $t = s$. If this experiment is repeated n times, and Alice and Bob cross-check "hashes" of the n-bit secret, they can detect Eve's attack with high probability. Quantum key distribution does not stop Eve from learning about the key, but to detect that she has tried to eavesdrop on the transmission as she disturbs the state.

Question 3. Decoherence, measurement, and the diagonalization map [9 marks]

(a) [2 marks] Consider the following function from 2×2 matrices to 2×2 matrices

$$\mathcal{D}_1(\rho) = \frac{1}{2}\left(e^{i\theta Z}\rho e^{-i\theta Z} + e^{-i\theta Z}\rho e^{i\theta Z}\right)$$

where Z is the Pauli-z matrix, and θ is a small positive real number. This describes a quantum operation in which the qubit system goes through a small Z rotation in a random direction. Show that:

$$\mathcal{D}_1\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & \cos 2\theta\, b \\ \cos 2\theta\, c & d \end{bmatrix}.$$

Note that the overall affect on the density matrix is the shrinking of the off-diagonal entries, which is called "decoherence".

(b) [1 mark] Consider the following function from 2×2 matrices to 2×2 matrices,

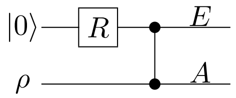
$$\mathcal{D}_2(\rho) = (1-p)\rho + pZ\rho Z.$$

where $0 < p < 1$ is a small positive real number. This describes a quantum operation in which the qubit system has a "phase error" Z with a small probability p. Show that:

$$\mathcal{D}_2\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & (1-2p)\, b \\ (1-2p)\, c & d \end{bmatrix}.$$

Note that $\mathcal{D}_2 = \mathcal{D}_1$ if we match the noise parameters $1 - 2p = \cos 2\theta$. This is an example of the non-uniqueness of the Kraus representation.

(c) [2 marks] Consider the following circuit:



where $R = \begin{bmatrix} \sqrt{1-p} & -\sqrt{p} \\ \sqrt{p} & \sqrt{1-p} \end{bmatrix}$, and the gate depicted with the vertical line ending with filled circles is the controlled-Z gate (diagonal with entries $1, 1, 1, -1$).

The circuit specifies a quantum operation $\mathcal{D}_3$ via a Stinespring dilation U from the second (bottom) qubit to EA, if the partial trace is applied to E. Derive the matrix representation for U (a 4×2 matrix) from the above circuit showing clearly the 2×2 blocks.

Your answer should show that each block gives a Kraus operator of $\mathcal{D}_2$ in part (b) so $\mathcal{D}_2 = \mathcal{D}_3$ and U is a Stinespring dilation of $\mathcal{D}_2$. (But you should not use this information to obtain U.)

(d) [4 marks] Let $1-2p = \cos 2\theta$ so $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3$. Find a unitary V acting only on E after the controlled-Z in the circuit, such that a subsequent partial trace of E in the computational basis gives $\mathcal{D}_1$ with the Kraus operators as given in part (a) (including the ordering which is the first/second Kraus operator).

This is a concrete example to see that the choice of basis used in the partial trace of E gives rise to the non-uniqueness of the Kraus representation.

Question 4. When is a quantum operation reversible? [3+3 marks]

We say that a quantum operation $\mathcal{E}$ taking system A to system B is *reversible* if there is another quantum operation $\mathcal{D}$ taking B to A such that the composition $\mathcal{D} \circ \mathcal{E}$ is the identity map $\mathcal{I}$ on A. For example, unitary operations are reversible, in that if $\mathcal{E}(\rho) = U\rho U^\dagger$, then, for $\mathcal{D}(\sigma) = U^\dagger \sigma U$, $\mathcal{D} \circ \mathcal{E} = \mathcal{I}$.

Let U be the Stinespring dilation of $\mathcal{E}$, and U takes vectors in A to BE. Let A be d-dimensional.

Fact: Let $\mathcal{N}$ map $d \times d$ matrices to $d \times d$ matrices, and R be a d-dimensional system. Let $\mathcal{I}$ be the identity map on R. Let $|\Phi\rangle$ be the maximally entangled state on RA, $|\Phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle|i\rangle$. If $(\mathcal{I} \otimes \mathcal{N})(|\Phi\rangle\langle\Phi|) = |\Phi\rangle\langle\Phi|$, then $\mathcal{N}$ is the identity map on A. You can use this fact without proof.

Let U be the Stinespring dilation of $\mathcal{E}$, so, $\mathcal{E}(M) = \operatorname{tr}_E UMU^\dagger$. Let $|\psi\rangle_{RBE} = (I_R \otimes U_A)|\Phi\rangle_{RA}$, where I_R is the identity matrix on R. Note that $|\psi\rangle$ is a tripartite state on RBE, and $\operatorname{tr}_{BE} |\psi\rangle\langle\psi| = (I/d)_R$.

Show that $\mathcal{E}$ is reversible if $\operatorname{tr}_B |\psi\rangle\langle\psi| = (I/d)_R \otimes \eta_E$ for some density matrix η on system E. (This condition is also necessary, due to the information gain implies disturbance principle, but you do not need to show necessity here.)

Hint: find two useful purifications of $(I/d)_R \otimes \eta_E$ and relate them.

Question 5. Practice with answer key, do not submit! [0 marks]

Consider the ensemble in which the state $|0\rangle$ occurs with probability $3/5$ and the state $(|0\rangle+|1\rangle)/\sqrt{2}$ occurs with probability $2/5$.

(a) What is the density matrix ρ of this ensemble?

(b) Write ρ in the form $\frac{1}{2}(I + r_x X + r_y Y + r_z Z)$, and plot ρ as a point in the Bloch sphere.

(c) Suppose we measure the state in the computational basis. What is the probability of getting the outcome 0? Compute this both by averaging over the ensemble of pure states and by computing $\mathrm{tr}(\rho|0\rangle\langle 0|)$, and show that the results are consistent.

(d) How does the density matrix change if we apply the Hadamard gate? Compute this both by applying the Hadamard gate to each pure state in the ensemble and finding the corresponding density matrix, and by computing $H\rho H^\dagger$.

Question 5. Practice with answer key, do not submit! [0 marks]

Consider the ensemble in which the state $|0\rangle$ occurs with probability 3/5 and the state $(|0\rangle+|1\rangle)/\sqrt{2}$ occurs with probability 2/5.

(a) What is the density matrix ρ of this ensemble?

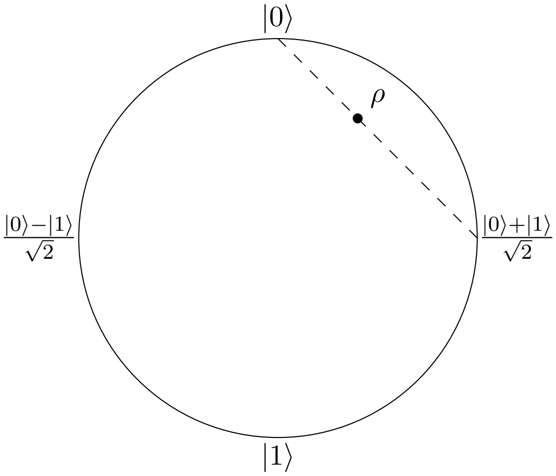
Answer:

$$\begin{aligned}\rho &= \frac{3}{5}|0\rangle\langle 0| + \frac{2}{5}\frac{|0\rangle+|1\rangle}{\sqrt{2}}\frac{\langle 0|+\langle 1|}{\sqrt{2}} \\ &= \frac{4}{5}|0\rangle\langle 0| + \frac{1}{5}|0\rangle\langle 1| + \frac{1}{5}|1\rangle\langle 0| + \frac{1}{5}|1\rangle\langle 1| \\ &= \frac{1}{5}\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}\end{aligned}$$

(Either of the last two lines is an acceptable answer.)

(b) Write ρ in the form $\frac{1}{2}(I + r_x X + r_y Y + r_z Z)$, and plot ρ as a point in the Bloch sphere.

Answer: We have $r_x = \frac{2}{5}$, $r_y = 0$, and $\frac{1}{2}(1+r_z) = \frac{4}{5} \implies r_z = \frac{3}{5}$. So ρ is on the XZ plane, and located as follows:


$|0\rangle$
ρ
$\frac{|0\rangle-|1\rangle}{\sqrt{2}}$
$\frac{|0\rangle+|1\rangle}{\sqrt{2}}$
$|1\rangle$

Notice that it is 2/5 of the way along a line from the state $|0\rangle$ to the state $|+\rangle$.

(c) Suppose we measure the state in the computational basis. What is the probability of getting the outcome 0? Compute this both by averaging over the ensemble of pure states and by computing $\mathrm{tr}(\rho|0\rangle\langle 0|)$, and show that the results are consistent.

Answer:

First, we find the answer by averaging over the ensemble of pure states. With the state $|0\rangle$, $\Pr(0) = 1$, and with the state $|+\rangle$, $\Pr(0) = 1/2$. Averaging, we have a probability of $(3/5)(1) + (2/5)(1/2) = 4/5$.

Second, we find the answer by using the density matrix ρ from part (a).

$$\mathrm{tr}(\rho|0\rangle\langle 0|) = \mathrm{tr}\left(\left(\tfrac{4}{5}|0\rangle\langle 0| + \tfrac{1}{5}|0\rangle\langle 1| + \tfrac{1}{5}|1\rangle\langle 0| + \tfrac{1}{5}|1\rangle\langle 1|\right)(|0\rangle\langle 0|)\right) = \frac{4}{5}, \text{ or}$$

$$\mathrm{tr}(\rho|0\rangle\langle 0|) = \mathrm{tr}\left(\frac{1}{5}\begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \frac{4}{5}.$$

(d) How does the density matrix change if we apply the Hadamard gate? Compute this both by applying the Hadamard gate to each pure state in the ensemble and finding the corresponding density matrix, and by computing $H\rho H^\dagger$.

Answer: We have $H|0\rangle = |+\rangle$ and $H|+\rangle = |0\rangle$, so applying the gate to each element of the ensemble gives the density matrix

$$\begin{aligned}\frac{3}{5}|+\rangle\langle+| + \frac{2}{5}|0\rangle\langle 0| &= \frac{3}{5}\frac{|0\rangle + |1\rangle}{\sqrt{2}}\frac{\langle 0| + \langle 1|}{\sqrt{2}} + \frac{2}{5}|0\rangle\langle 0| \\ &= \frac{3}{10}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) + \frac{2}{5}|0\rangle\langle 0| \\ &= \frac{7}{10}|0\rangle\langle 0| + \frac{3}{10}(|0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|).\end{aligned}$$

On the other hand, conjugating ρ by H gives

$$\begin{aligned}H\rho H &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\frac{1}{5}\begin{pmatrix}4 & 1\\ 1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \\ &= \frac{1}{10}\begin{pmatrix}5 & 2\\ 3 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \\ &= \frac{1}{10}\begin{pmatrix}7 & 3\\ 3 & 3\end{pmatrix},\end{aligned}$$

in agreement with the previous calculation.